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CONJUGATE TYPE BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction and preliminaries. Two-point boundary value problems (BVP's) for delay differential equations have been studied extensively, beginning with the work of G. A. Kamenskiĭ, S. B. Norkin and others (see [5], [7]) which was motivated by variational problems and problems in oscillation theory. L. J. Grimm and K. Schmitt [4] and Ju. I. Kovač and L. I. Savčenko [6] employed solutions of various differential inequalities for the study of two-point problems with retarded argument. In this paper, we show how a bilateral iteration procedure can be developed to yield existence and inclusion theorems for multipoint boundary value problems of conjugate type for nonlinear functional-differential equations.

Let $n > 1, I = [a, b]$ be a real compact interval, let $a = x_1 < x_2 < \cdots < x_k = b$, let $p_1(x), p_2(x), \ldots, p_n(x)$ be continuous on I, and define the linear differential operator L by

$$Ly = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y. \tag{1.1}$$

A Ju. Levin (see Coppel [1]) has obtained the following result which will play a central role in our work.

THEOREM 1.1. *Let L and I be as above, and suppose that L is disconjugate on I. Then the Green's function $G(x, s)$ for the k-point conjugate type boundary value problem*

$$Ly = 0, \tag{1.2}$$

$$y^{(i)}(x_j) = 0, i = 0, \ldots, n_j - 1, j = 1, \ldots, k, \tag{1.3}$$

where $\sum_{j=1}^{k} n_j = n$, satisfies the inequality

$$G(x, s)(x - x_1)^{n_1}(x - x_2)^{n_2} \cdots (x-x_k)^{n_k} \geqq 0, x_1 < s < x_k. \tag{1.4}$$

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2. Multipoint problems. Let I be as above, with L defined by (1.1) and disconjugate on I; let f: $I \times \mathbf{R}^2 \to \mathbf{R}$ and g: $I \to \mathbf{R}$ be continuous, and let c_{ij}, $i = 0, \ldots, n_j - 1$, $j = 1, \ldots, k$, be real constants, where $\sum_{j=1}^{k} n_j = n$. Define $\alpha = \min(\min_{x \in I} g(x), a)$, $\beta = \max(\max_{x \in I} g(x), b)$, $J_1 = [\alpha, a]$, and $J_2 = [b, \beta]$.

Consider the conjugate type BVP

$$(2.1) \qquad Ly(x) = f(x, y(x), y(g(x))),$$

$$(2.2) \qquad \begin{aligned} &y^{(i)}(x_j) = c_{ij},\ 0 \leqq i \leqq n_j - 1, j = 1, \ldots, k, \\ &y(x) \equiv \phi_\ell(x),\ x \in J_\ell,\ \ell = 1, 2, \end{aligned}$$

where $\phi_\ell(x)$ is continuous on J_ℓ and $\phi_1(a) = c_{01}$, $\phi_2(b) = c_{0k}$. We shall denote (2.1) by

$$(2.3) \qquad Ly = f[x, y],$$

and the boundary conditions (2.2) by

$$(2.4) \qquad Ty = \begin{Bmatrix} c \\ \phi \end{Bmatrix}.$$

Assume that f satisfies the uniform Lipschitz condition

$$(2.5) \qquad |f(x, y_1, z_1) - f(x, y_2, z_2)| \leqq P(|y_1 - y_2| + |z_1 - z_2|)$$

for all (x, y_1, z_1), (x, y_2, z_2) in $I \times \mathbf{R}^2$, where P is a constant. Suppose there exist functions $v_1(x)$ and $w_1(x)$ continuous on $J_1 \cup I \cup J_2$ and n times continuously differentiable on I, such that

$$Tv_1 = Tw_1 = \begin{Bmatrix} c \\ \phi \end{Bmatrix},$$

and such that, for $x \in I$,

$$(2.6) \qquad \begin{aligned} &Lv_1 - f[x, v_1] + A_1(x) \leqq 0, \\ &Lw_1 - f[x, w_1] - A_1(x) \geqq 0, \end{aligned}$$

where $A_1(x) \equiv P(|v_1(x) - w_1(x)| + |v_1(g(x)) - w_1(g(x))|)$. Let $l_c(x)$ denote the unique solution of the problem $Lu = 0$, $u^{(i)}(x_j) = c_{ij}$, $i = 0, \ldots, n_j - 1$, $j = 1, \ldots, k$, and construct sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ as follows:

$$(2.7) \qquad \begin{aligned} v_{m+1}(x) &= \begin{cases} \phi_1(x), \ x \in J_1, \\ \ell_c(x) + \int_I G(x, s)(f[s, v_m] - A_m(s))ds, \ x \in I, \\ \phi_2(x), \ x \in J_2; \end{cases} \\ w_{m+1}(x) &= \begin{cases} \phi_1(x), \ x \in J_1, \\ \ell_c(x) + \int_I G(x, s)(f[s, w_m] + A_m(s))ds, \ x \in I, \\ \phi_2(x), \ x \in J_2, \end{cases} \end{aligned}$$

where

(2.8) $$A_m(x) = P(|v_m(x) - w_m(x)| + |v_m(g(x)) - w_m(g(x))|),\ m \geqq 1.$$

THEOREM 2.1. *Let L be given by* (1.1) *and be disconjugate on $I = [a, b]$. Let $f: I \times \mathbf{R}^2 \to \mathbf{R}$ and $g: I \times \mathbf{R} \to \mathbf{R}$ be continuous and let f satisfy* (2.5). *Suppose there exist functions $v_1(x)$ and $w_1(x)$ which satisfy* (2.4) *and* (2.6), *and define the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ by* (2.7). *Then the* BVP (2.1)–(2.2) *has a solution $y(x)$ such that, for each $m \geqq 1$,*

(2.9) $$\begin{gathered} v_m(x) \geqq v_{m+1}(x) \geqq y(x) \geqq w_{m+1}(x) \geqq w_m(x),\ x \in I_1, \\ v_m(x) \leqq v_{m+1}(x) \leqq y(x) \leqq w_{m+1}(x) \leqq w_m(x),\ x \in I_2, \end{gathered}$$

where $I_1 = \{x \in I: G(x, s) \leqq 0\}$ and $I_2 = \{x \in I: G(x, s) \geqq 0\}$.

PROOF. Set $u_m(x) = v_m(x) - w_m(x)$, $m \geqq 1$. By (2.6), $Lu_1 \leqq 0$ for $x \in I$, and

$$Tu_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix};$$

thus, $u_1(x) = \int_I G(x, s)Lu_1(s)ds$ has sign opposite to that of $G(x, s)$ for $x \in I$. Similarly, for each $m > 1$,

$$u_{m+1}(x) = \int_I G(x, s)(f[s, v_m] - f[s, w_m] - 2A_m(s))ds,$$

for each $x \in I$. Noting that $f[x, v_n] - f[x, w_m] - 2A_m(x) \leqq 0$ for $x \in I$, it follows that, for each $m \geqq 1$,

(2.10) $$v_m(x) \geqq w_m(x),\ x \in I_1;\ v_m(x) \leqq w_m(x),\ x \in I_2.$$

We now show the monotonicity of the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ on I_1 and on I_2. From (2.6), note that $L(v_1 - v_2) \leqq 0$ and $L(w_1 - w_2) \geqq 0$ for $x \in I$. Since

$$T(v_1 - v_2) = T(w_1 - w_2) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$(v_1 - v_2)(x) \geqq 0 \geqq (w_1 - w_2)(x)$, $x \in I_1$ and $(v_1 - v_2)(x) \leqq 0 \leqq (w_1 - w_2)(x)$, $x \in I_2$. For each $m \geqq 2$,

$$\begin{aligned} L(v_m - v_{m+1}) &= f[x, v_{m-1}] - f[x, v_m] - A_{m-1}(x) + A_m(x) \\ &= \begin{cases} f[x, v_{m-1}] - f[x, v_m] - P(v_{m-1}(x) - v_m(x)) \\ \quad + P(w_{m-1}(x) - w_m(x)) - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad + P|v_m(g(x)) - w_m(g(x))|,\ x \in I_1; \\ f[x, v_{m-1}] - f[x, v_m] + P(v_{m-1}(x) - v_m(x)) \\ \quad - P(w_{m-1}(x) - w_m(x)) - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad + P|v_m(g(x)) - w_m(g(x))|,\ x \in I_2. \end{cases} \end{aligned}$$

(2.11)

$$L(w_m - w_{m+1}) = f[x, w_{m-1}] - f[x, w_m] + A_{m-1}(x) - A_m(x)$$

$$= \begin{cases} f[x, w_{m-1}] - f[x, w_m] - P(w_{m-1}(x) - w_m(x)) \\ + P(v_{m-1}(x) - v_m(x)) + P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ - P|v_m(g(x)) - w_m(g(x))|, \ x \in I_1; \\ f[x, w_{m-1}] - f[x, w_m] + P(w_{m-1}(x) - w_m(x)) \\ - P(v_{m-1}(x) - v_m(x)) + P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ - P|v_m(g(x)) - w_m(g(x))|, \ x \in I_2. \end{cases}$$

Assume now, as induction hypothesis, that for $m > 1$,

$$(v_{m-1} - v_m)(x) \geqq 0 \geqq (w_{m-1} - w_m)(x), \ x \in I_1,$$
$$(v_{m-1} - v_m)(x) \leqq 0 \leqq (w_{m-1} - w_m)(x), \ x \in I_2.$$

Consider $Lv_m - Lv_{m+1}$, for $x \in I$. Suppose first that $x \in I_1$. From (2.11), it follows that

$$\begin{aligned} Lv_m - Lv_{m+1} \leqq {} & P|v_{m-1}(g(x)) - v_m(g(x))| + P(w_{m-1}(x) - w_m(x)) \\ & - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| + P|v_m(g(x)) - w_m(g(x))|. \end{aligned}$$

If $g(x)$ is in J_1 or J_2, then

$$Lv_m - Lv_{m+1} \leqq P(w_{m-1}(x) - w_m(x)) \leqq 0.$$

If $g(x)$ is in I_1, then

$$Lv_m - Lv_{m+1} \leqq P(w_{m-1}(x) - w_m(x)) + P(w_{m-1}(g(x)) - w_m(g(x))) \leqq 0.$$

If $g(x)$ is in I_2, then

$$Lv_m - Lv_{m+1} \leqq P(w_{m-1}(x) - w_m(x)) + P(w_m(g(x)) - w_{m-1}(g(x))) \leqq 0.$$

Thus, for $x \in I_1$, $Lv_m - Lv_{m+1} \leqq 0$. Similarly, for $x \in I_2$, $Lv_m - Lv_{m+1} \leqq 0$. Since

$$T(v_m - v_{m+1}) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$v_m - v_{m+1} \geqq 0$, $x \in I_1$ and $v_m - v_{m+1} \leqq 0$, $x \in I_2$. Analogously, we find that $Lw_m - Lw_{m+1} \geqq 0$ on I and that $w_m - w_{m+1} \leqq 0, x \in I_1$ and $w_m - w_{m+1} \geqq 0$, $x \in I_2$. Hence,

$$v_m(x) \geqq v_{m+1}(x) \geqq w_{m+1}(x) \geqq w_m(x), \ x \in I_1,$$
$$v_m(x) \leqq v_{m+1}(x) \leqq w_{m+1}(x) \leqq w_m(x), \ x \in I_2, \ m \geqq 1.$$

It remains to show that there is a solution $y(x)$ of (2.1)–(2.2) which satisfies (2.9). Note that, on I_1, I_2, J_1 and J_2, the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ are monotonic, bounded, and equicontinuous. By Ascoli's theorem, they have uniform limits $v(x)$ and $w(x)$ with $v(x) \geqq w(x), x \in I_1$, $v(x) \leqq w(x)$, $x \in I_2$, and $v(x) \equiv w(x) \equiv \phi_\ell(x)$ on J_ℓ, $\ell = 1, 2$. It follows from (2.7) that, for $x \in I$,

$$\begin{aligned} Lv(x) &= f[x, v] - A(x), \\ Lw(x) &= f[x, w] + A(x), \end{aligned}$$

where $A(x) = P(|v(x) - w(x)| + |v(g(x)) - w(g(x))|)$, and that

$$Tv = Tw = \begin{Bmatrix} c \\ \phi \end{Bmatrix}.$$

Now, for each function $y(x) \in C(J_1 \cup I \cup J_2)$, define $\bar{y}$ by

$$\bar{y}(x) = \begin{cases} \phi_1(x), & \text{if } x \in J_1, \\ \left.\begin{array}{ll} v(x), & \text{if } y(x) > v(x), \\ y(x), & \text{if } v(x) \geqq y(x) \geqq w(x), \\ w(x), & \text{if } y(x) < w(x), \end{array}\right\}, & x \in I_1, \\ \left.\begin{array}{ll} v(x), & \text{if } y(x) < v(x), \\ y(x), & \text{if } v(x) \leqq y(x) \leqq w(x), \\ w(x), & \text{if } y(x) > w(x), \end{array}\right\}, & x \in I_2, \\ \phi_2(x), & \text{if } x \in J_2, \end{cases}$$

and define $F(x, y(x), y(g(x))) = f(x, \bar{y}(x), \bar{y}(g(x)))$. The function F is continuous and bounded on $I \times \mathbf{R}^2$ and it follows from the Schauder Fixed Point Theorem that the problem

$$\begin{aligned} Ly &= F(x, y(x), y(g(x))), \\ Ty &= \begin{Bmatrix} c \\ \phi \end{Bmatrix} \end{aligned}$$

has a solution $y(x)$. We now show that $y(x)$ satisfies

(2.12) $\quad w(x) \leqq y(x) \leqq v(x), x \in I_1, w(x) \geqq y(x) \geqq v(x), x \in I_2,$

and hence that $y(x)$ is a solution of (2.1)–(2.2) which satisfies (2.9). Consider $w(x) - y(x)$. Using the definition of $\bar{y}$, we find that

$$\begin{aligned} Lw - Ly = {} & f[x, w] + P(|v(x) - w(x)| + |v(g(x)) - w(g(x))|) \\ & - f(x, \bar{y}(x), \bar{y}(g(x))) \geqq 0, \end{aligned}$$

and

$$T(w - y) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Thus, $w(x) \leqq y(x)$, $x \in I_1$, $w(x) \geqq y(x)$, $x \in I_2$. Similarly, $v(x) \geqq y(x)$, $x \in I_1$, $v(x) \leqq y(x)$, $x \in I_2$. Hence, $y(x)$ satisfies (2.12) and the proof is complete.

REMARKS. (a) The procedure developed here can be applied to additional kinds of boundary value problems, including k-focal problems with retarded argument, see [2]. We obtained an analogous result for k-focal problems for ordinary differential equations in an earlier paper [3]. The computations in the two-point k-focal case are simpler because the Green's function is of constant sign on the entire interval.

(b) If $G = \max_{x \in I} |\int_I G(x, s) ds|$ and if $2PG < 1$, a contraction mapping argument may be used to prove the existence and uniqueness of a solution of (2.1)–(2.2). If, in fact, $6PG < 1$, then $A_m(x)$, defined by (2.8), tends to zero as $m \to \infty$. Thus, $v(x) = w(x)$ is the unique solution of (2.1)–(2.2).

(c) If G is as in (b), $2PG < 1$, and $|f(x, y, z)|$ is bounded by a constant B for all $(x, y, z) \in I \times \mathbf{R}^2$, the functions $v_1(x)$ and $w_1(x)$ can be chosen as

$$v_1(x) = \begin{cases} \phi_1(x), \ x \in J_1, \\ \ell_c(x) - \dfrac{B}{1-2PG} \displaystyle\int_I G(x, s) ds \\ \phi_2(x), \ x \in J_2, \end{cases}$$

$$w_1(x) = \begin{cases} \phi_1(x), \ x \in J_1, \\ \ell_c(x) + \dfrac{B}{1-2PG} \displaystyle\int_I G(x, s) ds \\ \phi_2(x), \ x \in J_2. \end{cases}$$

(d) The requirement that $v_1(x)$ and $w_1(x)$ satisfy the boundary conditions (2.2) can be relaxed somewhat. If v_1 and w_1 satisfy conditions analogous to the conditions (3.1)–(3.4) of Theorem 3.1 of [8], a modification of the iteration procedure leads to the conclusion of Theorem 2.1.

(e) As an example, consider the BVP

$$y''' = 1 - xy(x) + y(2x - 1), \tag{2.13}$$

$$\begin{aligned} &y(x) \equiv -x, \ x \in J_1 = [-1, 0], \\ &y(0) = y(1) = y(2) = 0, \\ &y(x) \equiv x - 2, \ x \in J_2 = [2, 3]. \end{aligned} \tag{2.14}$$

For this problem, $P = 2$. Let $w_1(x) = x(x-1)(x-2)$, $v_1(x) = -w_1$, for $x \in I$. Then it is easy to see that

$$Lv_1 - f[x, v_1] + A_1(x) = -6 - f[x, v_1] + A_1(x) \leqq 0;$$
$$Lw_1 - f[x, w_1] - A_1(x) \geqq 0, \ x \in I.$$

Hence the problem (2.13)–(2.14) has a solution $y(x)$ between v_1 and w_1.

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