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DIRICHLET SEMIGROUPS ON BOUNDED DOMAINS

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ABSTRACT. Let Γ be a simply connected domain in $\mathbf{R}^n$ with smooth boundary and let $d\mu = \rho(x)dx$ be a probability measure on Γ such that $0 < \varepsilon \leqq \rho(x) \leqq R < \infty$ a.e. on Γ. Let $a(x)$ be an $n \times n$ matrix valued function on Γ which is uniformly bounded and which is uniformly bounded below by $\lambda > 0$. It is shown that the maximal and minimal Dirichlet forms associated with $a(x)$ and $\rho(x)$ are represented by self-adjoint operators, each of which generates a hypercontractive semigroup.

1. Introduction. Suppose that $d\mu = \rho(x)dx$ is a probability measure on a domain $\Gamma \subset \mathbf{R}^n$ and that for $x \in \Gamma$, $a(x)$ is a positive definite $n \times n$ matrix. We can define a sesquilinear form h on $L^2(\Gamma, d\mu)$ by specifying

$$h(u, v) = \int_\Gamma \nabla u \cdot (a\nabla \bar{v})dx = \sum_{i,j=1}^n \int_\Gamma u_{x_i}\bar{v}_{x_j}a_{ij}dx$$

for some suitably chosen domain of h. If the boundary of Γ is sufficiently regular, if a and ρ satisfy certain boundedness conditions, and if the domain of h is chosen properly, then h will be a Dirichlet (i.e. closed Markov) form, as defined by Beurling and Deny **[2]** and by Fukushima **[5]**. h is then represented by a positive self-adjoint operator A which generates a sub-Markov semigroup of operators $\{e^{-tA}\}$ on $L^2(\Gamma, d\mu)$. It follows that $\{e^{-tA}\}$ is an L^p-contractive semigroup as will be seen in §3.

In the case $\Gamma = \mathbf{R}^n$ and $a(x) = \rho(x)I$, there has been a great deal of interest in the relationship between properties of the density ρ and hypercontractivity of the semigroup e^{-tA} (see, e.g., **[3]**, **[4]**, **[6]**, **[7]**, **[11]**, **[16]** and **[17]**). Conditions on ρ which have been shown to imply hypercontractivity fall generally into two classes: (i) conditions which stipulate the decay of $\rho(x)$ for large $|x|$; (ii) conditions pertaining to the regularity of ρ.

For the semigroup e^{-tA} to be hypercontractive, it is in fact necessary that the density $\rho(x)$ decay in some uniform sense like $e^{-|x|^2}$. The regularity

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conditions, on the other hand, have arisen merely as a consequence of the former methods of proof.

This article basically concerns itself with establishing hypercontractivity of a semigroup corresponding to a Dirichlet form under the assumption that the domain Γ is bounded. With the absence of a need for decay conditions, a more direct proof of hypercontractivity has been developed without the necessity of regularity conditions on ρ. The theory is valid for a fairly general class of strongly elliptic matrix valued functions $a(x)$.

The standard means for establishing hypercontractivity of a Dirichlet semigroup is to show that the logarithmic Sobolev inequality

$$\int_\Gamma |u|^2 \ln|u| d\mu \leqq c\, h(u, u) + \|u\|_2^2 \ln\|u\|_2 + \delta\|u\|_2^2$$

holds for all elements u of the form domain. It is shown in §5 that this logarithmic Sobolev inequality holds if and only if the form domain can be continously imbedded in the Orlicz space $L^2\ln\, L(\Gamma, d\mu)$.

It is then observed that for a proper choice of form core, and with certain conditions on Γ, μ and $a(x)$, the form domain is contained in the Sobolev space $W^{1,2}(\Gamma, d\mu)$. If Γ is bounded, then using the classical Sobolev imbedding theorem, it follows that $W^{1,2}(\Gamma, d\mu)$ can be imbedded continuously into $L^p(\Gamma, d\mu)$ for some $p > 2$; in turn $L^p(\Gamma, d\mu)$ can be imbedded into $L^2\ln L(\Gamma, d\mu)$. Thus hypercontractivity of the Dirichlet semigroup is proved without any reference to the regularity conditions which were previously essential. This suggests that, for a probability measure μ on an unbounded domain Γ, a possible means of obtaining hypercontractivity results for a corresponding Dirichlet semigroup might consist of establishing a relationship between properties of μ and the existence of an imbedding $W^{1,2}(\Gamma, d\mu) \to L^2\ln\, L(\Gamma, d\mu)$. One would expect the rate of decay of $d\mu$ for large $|x|$ to be of central importance.

In proving hypercontractivity without requiring any regularity of the density $\rho(x)$, it has been necessary to modify significantly the theory which was set forth in §4 of [7]; the modified theory appears in §4 of this paper. It is interesting to note that it has been possible to replace the former proofs, which relied on a careful determination of the domain of the Dirichlet operator, with arguments requiring only a determination of the form domain.

2. The Dirichlet form. Throughout this discourse we shall assume that Γ is a simply connected domain in $\mathbf{R}^n$. We define a measure μ on Γ by specifying $d\mu = \rho(x)\, dx$ where $\rho \in L^\infty(\Gamma)$ and ρ is locally bounded away from zero on $\bar{\Gamma}$, where $\bar{\Gamma}$ denotes the closure of Γ (i.e., for any compact set $K \subset \bar{\Gamma}$, $\rho \geqq \varepsilon_K > 0$ a.e. on K).

We denote by $C_0^\infty(\Gamma)$ the set of all infinitely differentiable functions with compact support contained in Γ, and by $C_0^\infty(\bar{\Gamma})$ the set of all functions $\varphi\chi_\Gamma$ where $\varphi \in C_0^\infty(\mathbf{R}^n)$ and χ_Γ is the characteristic function of the set Γ. $\mathscr{D}'(\Gamma)$ denotes the set of distributions on Γ.

For a non-negative integer k, and for $1 \leqq p < \infty$, the Sobolev space $W^{k,p}(\Gamma, d\mu)$ is the set of all functions whose distributional derivatives of order less than or equal to k are in $L^p(\Gamma, d\mu)$. $W^{k,p}(\Gamma, d\mu)$ is a Banach space with respect to the norm

$$\|u\|_{k,p,d\mu} = \Big(\sum_{|\alpha|\leqq k} \|D^\alpha u\|_{p,d\mu}^p\Big)^{1/p}$$

where $\|\cdot\|_{0,p,d\mu} = \|\cdot\|_{p,d\mu}$ is the usual L^p norm with respect to μ. We will use the notation $\|\cdot\|_{k,p} = \|\cdot\|_{k,p,d\mu}$ when the meaning is clear. Note that $C_0^\infty(\bar{\Gamma})$ is contained in $W^{k,p}(\Gamma, d\mu)$.

The Sobolev space $W_0^{k,p}(\Gamma, d\mu)$ is defined to be the closure of $C_0^\infty(\Gamma)$ with respect to the norm $\|\cdot\|_{k,p,d\mu}$. In general, $W_0^{k,p}(\Gamma, d\mu)$ is a proper subspace of $W^{k,p}(\Gamma, d\mu)$ for $k \geqq 1$.

We define $W_{\text{loc}}^{k,p}(\Gamma, d\mu)$ to be the set of all functions u such that $u \in W^{k,p}(\Gamma_1, d\mu)$ for all open sets $\Gamma_1 \subset \Gamma$ such that $\bar{\Gamma}_1$ is compact.

Remark. If Γ is a bounded domain, then the norms $\|\cdot\|_{k,p,d\mu}$ and $\|\cdot\|_{k,p,dx}$ are equivalent since we have $0 < \varepsilon_{\bar{\Gamma}} \leqq \rho \leqq \|\rho\|_\infty < \infty$ a.e. on $\bar{\Gamma}$. In this case, the spaces $W^{k,p}(\Gamma, d\mu)$ and $W^{k,p}(\Gamma, dx)$ coincide; likewise the spaces $W_0^{k,p}(\Gamma, d\mu)$ and $W_0^{k,p}(\Gamma, dx)$ coincide. In the same manner, for all domains Γ we have $W_{\text{loc}}^{k,p}(\Gamma, d\mu) = W_{\text{loc}}^{k,p}(\Gamma, dx)$; hence we will use the notation $W_{\text{loc}}^{k,p} = W_{\text{loc}}^{k,p}(\Gamma, d\mu)$.

We let now $a(x) = (a_{ij}(x))$ be an $n \times n$ matrix valued function defined for $x \in \Gamma$ which is locally strongly elliptic: i.e., for compact sets $K \subset \bar{\Gamma}$, there exists $\delta_K > 0$ such that for any n dimensional vector ξ,

$$\xi \cdot a(x)\bar{\xi} = \sum_{i,j=1}^n \xi_i\bar{\xi}_j a_{ij}(x) \geqq \delta_K |\xi|^2 \tag{1}$$

for almost all $x \in K$. We then define a sesquilinear form

$$\tilde{h}(u, v) = \int_\Gamma \nabla u \cdot (a\overline{\nabla v})\, dx = \sum_{i,j=1}^n \int_\Gamma u_{x_i}\bar{v}_{x_j} a_{ij}\, dx \tag{2}$$

for all $u, v \in C_0^\infty(\bar{\Gamma})$. We consider $\tilde{h}$ as an unbounded form on $\mathscr{H} = L^2(\Gamma, d\mu)$ and note that $\tilde{h}$ is positive, i.e., $\tilde{h}(u) = \tilde{h}(u, u) \geqq 0$ for all $u \in C_0^\infty(\bar{\Gamma})$. Moreover, we have the following result.

Proposition 2.1. *$\tilde{h}$ is closable. Denoting the closure of $\tilde{h}$ by h_1, we have the domain of h_1, denoted* Dom(h_1), *contained in* $W_{\text{loc}}^{1,2}$.

We define h_1 to be the maximal Dirichlet form on Γ corresponding to a

and μ. This definition and the proof of this proposition are derived from the exposition on Markov symmetric forms given by M. Fukushima in [5].

PROOF OF PROPOSITION 2.1. Suppose $u_k \in C_0^\infty(\bar{\Gamma})$, $u_k \to 0$ in $\mathscr{H}$ and $\tilde{h}(u_k - u_m) \to 0$. We first show that $\tilde{h}(u_k) \to 0$, thereby exhibiting that $\tilde{h}$ is closable.

For a compact set $K \subset \bar{\Gamma}$, it follows from (1) that

$$\sum_{i,j=1}^{n} \int_K (u_k - u_m)_{x_i} \overline{(u_k - u_m)_{x_j}} a_{ij}(x)\, dx$$

$$\geqq \delta_K \sum_{i=1}^{n} \int_K |(u_k - u_m)_{x_i}|^2\, dx.$$

Hence $(u_m)_{x_i}$ converges in $L^2_{\mathrm{loc}}(\Gamma)$ for $i = 1, \ldots, n$. Since $u_m \to 0$ in $\mathscr{H}$, it follows then that $u_m \to 0$ in $W^{1,2}_{\mathrm{loc}}(\Gamma)$. Now it is not hard to show that a subsequence of $\{u_m\}$, which we will denote by $\{u_{m_\ell}\}$, exists such that $(u_{m_\ell})_{x_i} \to 0$ a.e. on Γ for $i = 1, \ldots, n$. Hence by Fatou's lemma, we have

$$\tilde{h}(u_k) = \int_\Gamma \lim_{\ell \to \infty} \left(\sum_{i,j=1}^{n} (u_k - u_{m_\ell})_{x_i} \overline{(u_k - u_{m_\ell})_{x_j}} \right) a_{ij}(x)\, dx$$

$$\leqq \liminf_{\ell \to \infty} \tilde{h}(u_k - u_{m_\ell}).$$

It follows that $\tilde{h}(u_k) \to 0$ as $k \to \infty$, and $\tilde{h}$ is closable.

Now, for $u \in \mathrm{Dom}(h_1)$ there exists a sequence $u_k \in C_0^\infty(\bar{\Gamma})$ such that $u_k \to u$ in $\mathscr{H}$ and $h_1(u_k - u_m) \to 0$. It then follows as above that u_k converges to u in $W^{1,2}_{\mathrm{loc}}(\Gamma)$.

REMARK. By placing more restrictive conditions on $a(x)$ and $\partial\Gamma$ (the boundary of Γ), we can obtain a more precise determination of h_1. For example, if $\partial\Gamma$ satisfies the segment property, defined in §4, and if there exist constants $0 < \lambda < \Lambda < \infty$ such that $\lambda\rho(x)\,|\xi|^2 \leqq \xi \cdot a(x)\bar{\xi} \leqq \Lambda\rho(x)\,|\xi|^2$ for all n dimensional vectors ξ and almost all $x \in \Gamma$, then it can be shown that $\mathrm{Dom}(h_1) = W^{1,2}(\Gamma, d\mu)$ and that for $u, v \in \mathrm{Dom}(h_1)$, $h_1(u, v) = \int_\Gamma \nabla u \cdot a\overline{\nabla v}\, dx$. This follows from the fact that if $\partial\Gamma$ satisfies the segment property, then $C_0^\infty(\bar{\Gamma})$ is dense in $W^{1,2}(\Gamma, d\mu)$ (see, e.g., Theorem 3.18 in [1]).

We next consider $\tilde{h}$ defined by equation (2) with domain $\mathscr{C}$ such that $C_0^\infty(\Gamma) \subset \mathscr{C} \subset C_0^\infty(\bar{\Gamma})$. $\tilde{h}$, considered as an unbounded form on $\mathscr{H}$, is closable: h_1 is a closed extension of $\tilde{h}$. The closure, denoted by h, will be called the Dirichlet form on Γ corresponding to $a(x)$, μ and $\mathscr{C}$. For the remainder of this paper, we shall use the term Dirichlet form exclusively to refer to just such a form h. In the particular case that $\mathscr{C} = C_0^\infty(\Gamma)$, we denote the corresponding form by h_0, and call it the *minimal Dirichlet form corresponding to a and* μ.

As in [5], we will say that a symmetric form ε is *Markov* if for any $\delta > 0$

there exists a non-decreasing function $\varphi_\delta(t)$, $t \in \mathbf{R}$, which satisfies the following conditions.

(i) $\varphi_\delta(t) = t$ for $0 \leqq t \leqq 1$. $|\varphi_\delta(t)| \leqq t$ and $-\delta \leqq \varphi(t) \leqq 1 + \delta$ for all $t \in \mathbf{R}$,

(ii) If $u \in \mathrm{Dom}(\varepsilon)$, u real valued, then $\varphi_\delta(u) \in \mathrm{Dom}(\varepsilon)$ and $\varepsilon(\varphi_\delta(u)) \leqq \varepsilon(u)$.

Theorem 3.3. in [5] states that if a symmetric form ε is Markov and closable, then the closure is Markov. Hence by considering for $\delta > 0$, *a* C^∞ function $\varphi_\delta(t)$ satisfying (i) and also the property $0 \leqq \varphi_\delta'(t) \leqq 1$ for all $t \in \mathbf{R}$, it follows readily that each Dirichlet form defined above is Markov (a similar statement is made in Example 1 of [5]).

The fact that a Dirichlet form is Markov will turn out to be very useful in the L^p-contractivity proof of the next section.

3. The Dirichlet Operator Semigroup. Given a Dirichlet form h corresponding to $a(x)$, μ and $\mathscr{C}$, the representation theorem for closed positive forms provides us with a positive self-adjoint operator A on $\mathscr{H} = L^2(\Gamma, d\mu)$ which represents h. $\mathrm{Dom}(A) = \{u \in \mathrm{Dom}(h)$ such that there exists $w \in \mathscr{H}$ satisfying $\langle w, v\rangle_{d\mu} = h(u, v)$ for all $v \in \mathrm{Dom}(h)\}$, where $\langle\cdot, \cdot\rangle_{d\mu}$ represents the inner product on $\mathscr{H}$. For such u, Au is defined to be the corresponding w. We define A to be the *Dirichlet operator associated with h.* In this section, we will show that A is the generator of an L^p-contractive semigroup. We begin with the following determination of A.

PROPOSITION 3.1. $\mathrm{Dom}(A) \subset \{u \in \mathrm{Dom}(h)$ *such that* $(1/\rho)\sum_{i,j=1}^n (u_{x_i} a_{ij})_{x_j} \in \mathscr{H}\}$. *For such u, we have*

$$Au = -(1/\rho) \sum_{i,j=1}^n (u_{x_i} a_{ij})_{x_j}.$$

PROOF. Define, for all $u \in \mathrm{Dom}(A)$, $Tu = -\sum_{i,j=1}^n (u_{x_i} a_{ij})_{x_j}$. Hence T maps $\mathrm{Dom}(A)$ into $\mathscr{D}'(\Gamma)$, and for $\varphi \in C_0^\infty(\Gamma)$, we have $\varphi, \bar{\varphi} \in \mathrm{Dom}(h)$, and

$$\begin{aligned}\langle Tu, \varphi\rangle_{\mathscr{D}'} &= \sum_{i,j=1}^n \int_\Gamma u_{x_i} \varphi_{x_j} a_{ij}\, dx = h(u, \bar{\varphi}) \\ &= \langle Au, \varphi\rangle_{d\mu} = \int_\Gamma Au\, \varphi\rho\, dx\end{aligned}$$

where $\langle\cdot, \cdot\rangle_{\mathscr{D}'}$ represents the pairing between a distribution and an element of $C_0^\infty(\Gamma)$. Hence $Tu = \rho Au$ in $\mathscr{D}'$ so that Tu is a locally integrable function. The proposition then follows.

REMARK. Note that for $u \in \mathrm{Dom}(A)$, we have not determined that each of the terms $(1/\rho)\,(u_{x_i} a_{ij})_{x_j}$ is in $\mathscr{H}$, but only that their sum is.

REMARK. By placing more restrictive conditions on $\partial\Gamma$, $a(x)$ and $\rho(x)$, we can impose a greater degree of regularity on elements of Dom(A). For example, if $\partial\Gamma$ is smooth, $a(x) \in C^1(\Gamma)$ and $\rho(x) \in C(\bar{\Gamma})$, then we can deduce that $\mathrm{Dom}(A) \subset W^{2,2}_{\mathrm{loc}}(\Gamma)$ using standard regularity theorems for elliptic equations (see, e.g., **[12]**). It should be noted however that we are not in a position to determine a core for A in the tenor of **[4]** or [7].

REMARK. The different choices of the form core $\mathscr{C}$ generally result in entirely different Dirichlet operators A. For example, if $\partial\Gamma$, $a(x)$ and $\rho(x)$ are sufficiently regular, it is evident from the definition of A that elements of the domain of the operator associated with the maximal form must satisfy Neumann boundary conditions on $\partial\Gamma$; likewise, elements of the domain of the operator associated with the minimal form must satisfy Dirichlet conditions on $\partial\Gamma$.

Since the operator A is positive, it generates a contraction semigroup of operators $\{e^{-tA}\}_{t\in[0,\infty)}$ on $\mathscr{H}$. It follows from Theorem 3.2 in **[5]** that e^{-tA} is weakly Markov for each $t > 0$. I.e., for $u \in \mathscr{H}$ such that $0 \leqq u \leqq 1$ a.e. on Γ, we have $0 \leqq e^{-tA}u \leqq 1$ a.e. on Γ. This is a stronger result than the positivity preserving result in [7], which states that $e^{-tA}u \geqq 0$ a.e. on Γ whenever $u \geqq 0$ a.e. on Γ.

We will assume for the remainder of this paper that $\int_\Gamma \rho(x)\, dx = 1$ so that μ is a probability measure on Γ.

We say that $\{e^{-tA}\}_{t\in[0,\infty)}$ is *L^p-contractive*, or that *A generates an L^p-contractive semigroup*, if the restriction to $L^p(\Gamma, d\mu)$ of $\{e^{-tA}\}$ is a contraction semigroup for $p > 2$ and the closure in $L^p(\Gamma, d\mu)$ of $\{e^{-tA}\}$ is a contraction semigroup for $p < 2$.

THEOREM 3.2. *Let A be the operator associated with a Dirichlet form h. Then $\{e^{-tA}\}_{t\in[0,\infty)}$ is an L^p-contractive semigroup which is strongly continuous for $1 \leqq p < \infty$.*

PROOF. We need only make minor revisions in the proof of Theorem X.55 in **[13]**, which states that if S is a positive self-adjoint operator on $L^2(\mathbf{R}^n, d\mu)$, μ a probability measure, and if e^{-tS} is positivity preserving with $e^{-tS}(1) = 1$, then e^{-tS} is L^p-contractive and moreover, the semigroup is strongly continuous for $1 \leqq p < \infty$.

The condition $e^{-tA}(1) = 1$ is not generally satisfied, but the Markov condition satisfied by e^{-tA} is an adequate substitute. If we take $u \in L^2(\Gamma, d\mu)$ such that $u \geqq 0$ a.e. on Γ, then

$$\begin{aligned}\|e^{-tA}u\|_1 &= \langle 1, e^{-tA}u\rangle_{d\mu} = \langle e^{-tA}1, u\rangle_{d\mu}\\ &\leqq \langle 1, u\rangle_{d\mu} = \|u\|_1.\end{aligned}$$

The inequality follows from the fact that e^{-tA} is Markov. The rest of this proof is exactly like the proof of Theorem X.55 in **[13]**.

EXAMPLE. Suppose that $\Gamma = \mathbf{R}^n$, $a(x) = \rho(x)I$, and $\mathscr{C} = C_0^\infty(\mathbf{R}^n)$ so that $h(u, v) = \int_{R^n} \nabla u \cdot \overline{\nabla v} \rho dx$ with $\mathrm{Dom}(h) = W^{1,2}(\mathbf{R}^n, d\mu)$. In this case, Theorem 3.2 generalizes Theorem 3.5 in [7] which provides the same result, assuming that ρ is continuous and that A has a predetermined core.

4. Hypercontractivity. Suppose that h is either the minimal or maximal Dirichlet form on a bounded domain Γ. We will show in this section that if the boundary $\partial\Gamma$ is sufficiently regular, and if the function $a(x)$ satisfies appropriate boundedness conditions, then the corresponding semigroup of operators is hypercontractive if h satisfies a certain logarithmic Sobolev inequality. We will say that an L^p-contractive semigroup $\{e^{-tB}\}_{t\in[0,\infty)}$ is hypercontractive if there exist $t > 0$ and $p > 2$ such that $\|e^{-tB}\|_{2,p} < \infty$, where $\|\cdot\|_{q,r}$ denotes the norm of an operator from L^q to L^r (see, e.g., [13] for equivalent definitions of hypercontractivity).

We will assume in this section that Γ satisfies the segment property: For every point $x \in \partial\Gamma$ there exists an open set U and a nonzero vector y so that $x \in U$ and if $z \in \bar{\Gamma} \cap U$, then $z + ty \in \Gamma$ for $0 < t < 1$. It is then the case that $C_0^\infty(\bar{\Gamma})$ is dense in $W^{k,p}(\Gamma, d\mu)$ for $k \in Z^+$, $1 \leqq p < \infty$ (a similar remark was made in §2).

We assume that μ is a probability measure on Γ where $d\mu = \rho(x)dx$, ρ as specified in §2. The following preliminary calculation is valid for a bounded or unbounded domain Γ.

For $2 \leqq p < \infty$, we define $q = p^* = p/(p - 1)$ (so that $(1/p) + (1/q) = 1$). For $u \in L^p(\Gamma, d\mu)$, we define

$$u_p = (\operatorname{sgn} u)|u|^{p-1}\ u = \begin{cases} u|u|^{p-2} & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}$$

LEMMA 4.1. *The map $\zeta(u) = u_p$ is a continuous bijection of $W^{1,p}(\Gamma, d\mu)$ into $W^{1,q}(\Gamma, d\mu)$. For $i = 1, \ldots, n$ and $u \in W^{1,p}(\Gamma, d\mu)$, we have*

$$(3) \qquad (u_p)_{x_i} = \begin{cases} (p - 1)|u|^{p-2} u_{x_i}, & u(x) \neq 0 \\ \\ 0 & , u(x) = 0 \end{cases} \qquad \text{a.e. on } \Gamma.$$

Moreover, ζ maps $W_0^{1,p}(\Gamma, d\mu)$ into $W_0^{1,q}(\Gamma, d\mu)$.

PROOF. Since ζ is a homeomorphism from $L^p(\Gamma, d\mu)$ to $L^q(\Gamma, d\mu)$, then ζ: $W^{1,p} \to W^{1,q}$ is one-to-one. Also, that (3) holds for $u \in C_0^\infty(\bar{\Gamma})$ was established in the proof of Lemma 4.2 of [7].

We now establish (3) for arbitrary $u \in W^{1,p}(\Gamma, d\mu)$. For such a u, there exists $\{u_k\} \subset C_0^\infty(\bar{\Gamma})$ such that $\|u_k - u\|_{1,p} \to 0$, as mentioned above. Since ξ is a homeomorphism from $L^p(\Gamma, d\mu)$ to $L^q(\Gamma, d\mu)$, we have $\|(u_k)_p - u_p\|_q \to 0$. Moreover, for $i = 1, \ldots, n$,

$$\begin{aligned}\|((u_k)_p)_{x_i} - (p-1)\,|u|^{p-2}\,u_{x_i}\|_q &\le (p-1)\|\,|u_k|^{p-2}((u_k)_{x_i} - u_{x_i})\|_q \\ &+ (p-1)\|(|u_k|^{p-2} - |u|^{p-2})u_{x_i}\|_q.\end{aligned}$$

But

$$\|\,|u_k|^{p-2}\,((u_k)_{x_i} - u_{x_i})\|_q^q \le \left(\int |u_k|^p\,d\mu\right)^{p-2/p-1}\left(\int |(u_k)_{x_i} - u_{x_i}|^p\,d\mu\right)^{1/p-1}$$

$\to 0$ as $k \to \infty$ since $u_k \to u$ in $W^{1,p}$. Also,

$$\|(|u_k|^{p-2} - |u|^{p-2})|u_{x_i}|\,\|_q^q \le \left(\int |u_{x_i}|^p\,d\mu\right)^{1/p-1}\left(\int |u_k^{p-2} - u^{p-2}|^{p/p-2}\,d\mu\right)^{p-2/p-1}.$$

It is a routine exercise now to show that a subsequence $u_{k_j}^{p-2}$ converges to u^{p-2} in $L^{p/(p-2)}$(see, e.g., p. 76 of **[15]**). Using a standard distribution theory argument, then, we obtain (3) for $i = 1, \ldots, n$.

The continuity of ζ follows from the closed graph theorem. And if $u \in C_0^\infty(\Gamma)$, then $u_p \in C_0^1(\Gamma)$(a similar calculation appears in the proof of Lemma 4.2 of [7]). It follows by continuity that ζ maps $W_0^{1,p}(\Gamma, d\mu)$ into $W_0^{1,q}(\Gamma, d\mu)$. Hence the proof of Lemma 4.1 is complete.

In order to prove the main hypercontractivity results of this article, it will be necessary to require that the domain Γ be bounded. It will be convenient to meet that requirement at this point. We assume for the remainder of this paper that Γ is a bounded domain.

We now have $d\mu = \rho(x)dx$ where $0 < \varepsilon \leqq \rho(x) \leqq R < \infty$ a.e. on Γ. As we remarked earlier, we then have $W^{k,p}(\Gamma, d\mu) = W^{k,p}(\Gamma, dx)$, both of which we will denote simply by $W^{k,p}$; similarly we have $W_0^{k,p}(\Gamma, d\mu) = W_0^{k,p}(\Gamma, dx)$, both of which will be denoted $W_0^{k,p}$.

We also have $a(x)$ strongly elliptic on Γ. I.e., there exists $\lambda > 0$ such that $\xi \cdot a(x)\,\bar{\xi} \geqq \lambda|\xi|^2$ for all n-dimensional vectors ξ and almost all $x \in \Gamma$. We assume in addition that for some $\Lambda < \infty$, $\xi \cdot a(x)\,\bar{\xi} \leqq \Lambda|\xi|^2$ for all vectors ξ and for almost all $x \in \Gamma$. It follows then that $a_{ij}(x) \in L^\infty(\Gamma)$ for $i, j = 1, \ldots, n$.

Finally, for the remainder of this section we assume that either $\mathscr{C} = C_0^\infty(\Gamma)$ or $\mathscr{C} = C_0^\infty(\bar{\Gamma})$, denoting by h_0 the Dirichlet form corresponding to $C_0^\infty(\Gamma)$ and by h_1 the Dirichlet form corresponding to $C_0^\infty(\bar{\Gamma})$.

With the new restrictions on $a(x)$, the following two conditions on a sequence $\{u_k\} \subset C_0^\infty(\bar{\Gamma})$ are equivalent:

(i) $u_k \to u$ in L^2 and $h_1(u_k - u) \to 0$

(ii) $u_k \to u$ in $W^{1,2}$.

In particular, we have $\mathrm{Dom}(h_1) = W^{1,2}$ and $\mathrm{Dom}(h_0) = W_0^{1,2}$.

It has been shown by L. Gross (Example 2 in [**6**]) that if the operator corresponding to a Dirichlet form h generates a hypercontractive semigroup, then h satisfies the following logarithmic Sobolev inequality: there exist constants $c > 0$ and δ such that

$$\int_\Gamma |u|^2 \ln |u| \, d\mu \leqq ch(u) + \|u\|_2^2 \ln \|u\| + \delta \|u\|_2^2 \tag{4}$$

for all $u \in \mathrm{Dom}(h)$. Our aim for the remainder of this section is to prove the converse in the case that h is either the minimal or maximal dirichlet form.

We now let h stand for either h_0 or h_1; correspondingly $\mathscr{C} = C_0^\infty(\Gamma)$ or $C_0^\infty(\bar{\Gamma})$. We suppose that there exist $c > 0$ and δ such that (4) holds for all $u \in \mathscr{C}$, and note that (4) extends immediately to all $u \in \mathrm{Dom}(h)$, the proof being similar to proof of Lemma 4.3 in [**7**]. Moreover, we have the following result.

LEMMA 4.2. *For all real valued* $u \in \mathrm{Dom}(h) \cap W^{1,p}$, $p > 2$,

$$\int_\Gamma |u|^p \ln |u| \, d\mu \leqq \frac{c}{2} \frac{p}{p-1} \sum_{i,j=1}^n \int_\Gamma u_{x_i} (u_p)_{x_j} a_{ij} dx + \|u\|_p^p \ln \|u\|_p + \frac{2\delta}{p} \|u\|_p^p. \tag{5}$$

PROOF. For real valued $u \in \mathscr{C}$, (5) follows by substituting $|u|^{p/2}$ into (4) and noting that

$$(|u|^{p/2})_{x_i} (|u|^{p/2})_{x_j} = p^2/(4(p-1)) u_{x_i} (u_p)_{x_j} \text{ a.e. on } \Gamma$$

(a similar calculation appears in the proof of Lemma 4.2 of [**7**]).

$\mathscr{C}$ is dense in $\mathrm{Dom}(h) \cap W^{1,p}$ in the norm $\|\cdot\|_{1,p}$ since $\mathscr{C}$ is dense in $\mathrm{Dom}(h)$ in the norm $\|\cdot\|_{1,2}$ and since the imbedding $W^{1,p} \to W^{1,2}$ is continuous. Hence for real valued $u \in \mathrm{Dom}(h) \cap W^{1,p}$ there exists $\{u_k\} \subset \mathscr{C}$, u_k real valued, such that $\|u_k - u\|_{1,p} \to 0$. According to Lemma 4.1, then $\|(u_k)_p - u_p\|_{1,q} \to 0$. And since $a_{ij} \in L^\infty(\Gamma)$ for $i, j = 1, \ldots, n$, we have

$$\sum_{i,j=1}^n \int_\Gamma (u_k)_{x_i} ((u_k)_p)_{x_j} a_{ij} dx \to \sum_{i,j=1}^n \int_\Gamma u_{x_i} (u_p)_{x_j} a_{ij} dx \text{ as } k \to \infty,$$

after which the lemma follows as in the proof of Lemma 4.4 of [**7**].

We will derive the main hypercontractivity result of this section as a corollary to the following theorem, which is a direct consequence of Theorem 1 and Corollary 2.1 of [**6**].

THEOREM 4.3. *Suppose that B generates a positivity preserving L^p-contractive semigroup on the spaces $L^p(M, d\nu)$, $1 \leqq p \leqq \infty$, where ν is a probability measure on M. Suppose that $c(p)$ and $\gamma(p)$ are continuous functions on $[2, p_1)$ and that $c(p) > 0$ on $[2, p_1)$ for some $p_1 > 2$. If for all p such that $2 \leqq p < p_1$ we have*

$$\int_M |u|^p \ln |u| \, d\nu \leqq c(p) \int_M Buu_p \, d\nu + \|u\|_p^p \ln \|u\|_p + \gamma(p) \, \|u\|_p^p$$

for all non-negative $u \in \mathrm{Dom}(B) \cap L^p(M, d\nu)$ such that $Bu \in L^p(M, d\nu)$, then $\{e^{-tB}\}_{t \in [0,\infty)}$ is hypercontractive.

REMARK. It can routinely be shown that, for $p > 2$, $\{u \in \mathrm{Dom}(B) \cap L^p(M, d\nu)$ such that $Bu \in L^p(M, d\nu)\}$ is the domain of the generator of the restriction of $\{e^{-tB}\}$ to $L^p(M, d\nu)$.

PROOF OF THEOREM 4.3. This proof is a simple variation of the proof of Theorem 4.5 in [7]. We let $p(t)$ be the solution of the initial value problem $c(p)\,(dp/dt) = p$, $p(0) = 2$. There exists some interval $[0, t_0]$ on which p is an increasing, continuously differentiable function and such that $2 < p(t_0) \leqq p_1$. We set $p_0 = p(t_0)$. We then have $P(t_0) = \int_0^{t_0} \gamma(p(s))\, ds < \infty$, and it follows from Theorem 1 and Corollary 2.1 of **[6]** that $\|e^{-t_0 B}\|_{2,p_0} \leqq e^{p(t_0)} < \infty$. Hence $\{e^{-tB}\}$ is hypercontractive.

For the proof of the next theorem, it will be necessary to assume that the boundary of Γ is of class C^1: i.e., we assume that $\partial\Gamma$ can be covered by a finite number of n-dimensional neighborhoods U_i such that each U_i can be mapped in a one-to-one way onto $\{y \in \mathbf{R}^n$ such that $|y| < 1\}$, with $U_i \cap \Gamma$ mapping onto $\{y \in \mathbf{R}^n$ such that $|y| < 1$, $y_1 > 0\}$ and $U_i \cap \partial\Gamma$ mapping onto $\{y \in \mathbf{R}^n$ such that $|y| < 1$, $y_1 = 0\}$, by a mapping Φ_i which together with its inverse has continuous derivatives of order less than or equal to one.

THEOREM 4.4. *Suppose that Γ is a bounded simply connected domain in $\mathbf{R}^n$ which is of class C^1. Suppose that $d\mu = \rho(x)\, dx$ is a probability measure on Γ such that $0 < \varepsilon < \rho(x) < R < \infty$ a.e. on Γ, and that $a(x)$ is an $n \times n$ matrix valued function such that $\lambda |\xi|^2 \leqq \xi \cdot a(x)\, \xi \leqq \Lambda |\xi|^2$ for all n-dimensional vectors ξ and almost all $x \in \Gamma$, where $0 < \lambda$, $\Lambda < \infty$. Suppose that either $\mathscr{C} = C_0^\infty(\Gamma)$ or $\mathscr{C} = C_0^\infty(\bar{\Gamma})$ and that h is the Dirichlet form on Γ corresponding to μ, a, and $\mathscr{C}$. If h satisfies the logarithmic Sobolev inequality (4) for all $u \in \mathscr{C}$, then the operator A associated with h generates a hypercontractive semigroup.*

PROOF. For $p > 2$, we denote by A_p the generator of the restriction of the semigroup $\{e^{-tA}\}$ to L^p. To prove Theorem 4.4, it suffices, in view of

Lemma 4.2 and Theorem 4.3, to exhibit an index $p_1 > 2$ such that for $2 \leqq p < p_1$,

$$\int_\Gamma Au\, \overline{u_p}\, d\mu = \sum_{i,j=1}^n \int_\Gamma u_{x_i}\, \overline{(u_p)_{x_j}}\, a_{ij} dx \tag{6}$$

for all $u \in \mathrm{Dom}(A_p)$.

Given the conditions on Γ, $\rho(x)$ and $a(x)$, it follows directly from Theorem 1 in **[10]** that there exists $p_1 > 2$ such that for $2 \leqq p < p_1$ and $w \in L^p$, there exists a unique $\psi \in W_0^{1,p}$ such that

$$\int_\Gamma w\bar{v} d\mu = \sum_{i,j=1}^n \int_\Gamma \psi_{x_i} \overline{v_{x_j}} a_{ij} dx \tag{7}$$

for all $v \in W_0^{1,p^*}$, where $p^* = p/(p-1)$. If $\mathscr{C} = C_0^\infty(\Gamma)$, we let $w = Au$ for $u \in \mathrm{Dom}(A_p)$. Since (7) holds in particular for all $v \in W_0^{1,2}$, we must have $\psi = u$ by definition of the operator representation A. Then (6) follows by substituting $v = u_p \in W_0^{1,p^*}$ into (7).

It has similarly been shown that, given the conditions on Γ, $\rho(x)$ and $a(x)$, there exists $p_1 > 2$ such that for $2 \leqq p < p_1$ and $w \in L^p$, there exists a unique $\psi \in W^{1,p}$ such that

$$\int_\Gamma w\bar{v} d\mu = \sum_{i,j=1}^n \int_\Gamma \psi_{x_i} \overline{v_{x_j}} a_{ij} dx + \int_\Gamma \psi\bar{v} d\mu \tag{8}$$

for all $v \in W^{1,p^*}$ (see **[8]**). If $\mathscr{C} = C_0^\infty(\bar{\Gamma})$, we let $w = Au + u$ for $u \in \mathrm{Dom}(A_p)$. Since (8) holds for all $v \in W^{1,2}$ since A represents h, and since

$$\langle f, g \rangle = \sum_{i,j=1}^n \int_\Gamma f_{x_i} \overline{g_{x_j}} a_{ij} dx + \int_\Gamma f\bar{g} d\mu$$

defines an inner product on $W^{1,2}$, we must have $\psi = u$. Then (6) follows by substituting u_p into (8); hence Theorem 4.4 follows.

REMARK. Though Theorem 4.4 is adequate for the purposes of this paper, it seems as if a similar result should hold for an unbounded domain Γ with sufficiently regular boundary. It can in fact be shown that Lemma 4.2 is valid for such a domain Γ if there exist $0 < \lambda < \Lambda < \infty$ such that $\lambda\rho(x)\, |\xi|^2 \leqq \xi \cdot a(x)\bar{\xi} \leqq \Lambda\rho(x)\xi^2$ for all vectors ξ and almost all $x \in \Gamma$. Hence Theorem 4.4 can be generalized by producing duality theorems for unbounded domains which are similar to the theorems for bounded domains utilized in the proof of Theorem 4.4.

5. An Orlicz space imbedding. If the logarithmic Sobolev inequality (4) holds for all u in the domain of a Dirichlet form h, h as qualified in the previous section, then we have an imbedding of the form domain into the

Orlicz space $L^2 \ln L(\Gamma, d\mu)$. A converse to this statement will provide a means for establishing hypercontractivity of the Dirichlet semigroup.

To define the Orlicz space, we first note that there exists a function $B \in C^2(0, \infty)$ with the following properties.

(i) $B(x) = x^2 \ln x$ for $x^2 \ln x \geqq 1$,

(ii) $B(x) \geqq |x^2 \ln x|$ on $[0, \infty)$,

(iii) $B(0) = B'(0) = 0$, and

(iv) $B''(0) \geqq 0$ on $(0, \infty)$,

If ν is a probability measure on M, then the set of all measurable functions u on M such that $\int_M B(|u(x)|)\, d\nu < \infty$ is a Banach space with respect to the norm $\|u\|_B = \inf\{r > 0$ such that $\int_M B(|u(x)|/r)\, d\nu \leqq 1\}$. This space is an Orlicz space (see, e.g., [1] for a general discussion of Orlicz spaces) which will be denoted by $L^2 \ln L(M, d\nu)$. We shall also use the somewhat more appealing notation $\|\cdot\|_{L^2 \ln L} = \|\cdot\|_B$.

REMARK. Given a second function B_1 satisfying (i)–(iv), the norms $\|\cdot\|_B$ and $\|\cdot\|_{B_1}$ are equivalent; hence the space $L^2 \ln L(M, d\nu)$ is well defined.

THEOREM 5.1. *Suppose that ν is a probability measure on M. Suppose that ε is a closed positive form on $L^2(M, d\nu)$ with domain denoted* Dom(ε). *Then there exist constants $c > 0$ and δ such that*

$$\int_M |u|^2 \ln |u|\, d\nu \leqq c\varepsilon(u) + \|u\|_2^2 \ln \|u\|_2 + \delta \|u\|_2^2 \tag{9}$$

for all $u \in$ Dom(ε) *if and only if* Dom(ε) *can be imbedded continuously in $L^2 \ln L(M, d\nu)$, where* Dom(ε) *is viewed as a Banach space with respect to the norm $\|u\|_\varepsilon = (\|u\|_2^2 + \varepsilon(u))^{1/2}$.*

PROOF. If $u \in$ Dom(ε) and u satisfies (9), then certainly $u \in L^2 \ln L(M, d\nu)$; hence the imbedding exists. Continuity of the imbedding is an immediate consequence of the closed graph theorem.

Suppose, conversely, that the imbedding, denoted by i, exists and is continuous, with $\|i\|_{\mathrm{Dom}(\varepsilon), L^2 \ln L} = R < \infty$. (9) certainly holds for $u \equiv 0$ for any contants c and δ.

We consider then an element u of Dom(ε) such that $\|u\|_2 = 1$ and observe that $\|u\|_{\mathrm{Dom}(\varepsilon)} \geqq 1$ and $\|u\|_{L^2 \ln L} < (R+1)\|u\|_{\mathrm{Dom}(\varepsilon)}$. We choose $\alpha \geqq 1$ so that $\|u\|_{L^2 \ln L} < \alpha < (R+1)\, \|u\|_{\mathrm{Dom}(\varepsilon)}$. Hence $\int_M B(|u|/\alpha) d\nu \leqq 1$ Also

$$\int_M B\left(\frac{|u|}{\alpha}\right) d\nu = \int_{\frac{|u|}{\alpha} < 1} B\left(\frac{|u|}{\alpha}\right) d\nu + \int_{\frac{|u|}{\alpha} \geqq 1} \left(\frac{|u|}{\alpha}\right)^2 \ln\left(\frac{|u|}{\alpha}\right) d\nu$$

so that

$$\int_{\frac{|u|}{\alpha}<1} \left(\frac{|u|}{\alpha}\right)^2 \ln\left(\frac{|u|}{\alpha}\right) d\nu \leqq 1.$$

On the other hand,

$$\int_{\frac{|u|}{\alpha}<1} \left(\frac{|u|}{\alpha}\right)^2 \ln\left(\frac{|u|}{\alpha}\right) d\nu \leqq \int_{\frac{|u|}{\alpha}<1} B\left(\frac{|u|}{\alpha}\right) d\nu \leqq 1$$

where the first inqeuality follows from condition (ii) listed above. It follows that

$$\int_M |u|^2 \ln |u|\, d\nu = \alpha^2 \int_M \left(\frac{|u|}{\alpha}\right)^2 \ln\left(\frac{|u|}{\alpha}\right) d\nu$$

$$+ \int_M |u|^2 \ln \alpha d\nu \leqq 2\alpha^2 + \ln \alpha \leqq 3\alpha^2.$$

Hence we have

$$\int_M |u|^2 \ln |u|\, d\nu \leqq 3\alpha^2 \leqq 3(R+1)^2 \|u\|^2_{\mathrm{Dom}(\varepsilon)}$$

$$= 3(R+1)^2[\varepsilon(u) + \|u\|_2^2]$$

from whence it follows that (9) holds with $c = \delta = 3(R+1)^2$ for $u \in \mathrm{Dom}(\varepsilon)$ such that $\|u\|_2 = 1$. For $u \in \mathrm{Dom}(\varepsilon)$ such that $\|u\|_2 = \beta > 0$, we insert u/β into the inequality (9) to obtain (9) for u with the same constants c and δ. Thus Theorem 5.1 follows.

THEOREM 5.2. *Suppose that Γ, μ and $\alpha(x)$ satisfy the conditions of Theorem* 4.4 *and that either $\mathscr{C} = C_0^\infty(\Gamma)$ or $\mathscr{C} = C_0^\infty(\bar{\Gamma})$. If h is the Dirichlet form on Γ corresponding to μ, $\alpha(x)$ and $\mathscr{C}$, then the operator A associated with h generates a hypercontractive semigroup.*

PROOF. The classical Sobolev imbedding theorem for a bounded domain with smooth boundary provides us with a continuous imbedding of $W^{1,2}$ into L^p for some $p > 2$ (see, e.g., [1]). Then since μ is a finite measure, we have a continuous imbedding of L^p into the larger space $L^2 \ln L$.

Since in any case Dom(h) $\subset W^{1,2}$, we have by composition a continuous imbedding of Dom(h) into $L^2 \ln L$. Thus the conditions of Theorem 5.1 are satisfied, so that the conditions of Theorem 4.4 are satisfied, from whence Theorem 5.2 follows.

REMARK. In the case $\Gamma = \mathbf{R}^n$, there is a great deal of evidence relating hypercontractivity of a Dirichlet semigroup, existence of a logarithmic Sobolev inequality (4), and decay of the corresponding measure $d\mu = \rho(x)\, dx$ for large $|x|$. One should therefore expect to be able to establish a direct relationship between continuity of the imbedding $W^{1,2}(\Gamma, d\mu) \to L^2 \ln L(\Gamma, d\mu)$ and decay of the measure $d\mu$.

As a first step in this direction, this author has recently shown [9] that in order for the imbedding $W^{1,2}(\mathbf{R}^n, \rho(x)dx) \to L^2(\mathbf{R}^n, \rho(x)dx)$ to be compact, it is necessary that $\rho(x)$ decay faster than $e^{-\alpha|x|}$ for all α, and sufficient for $\rho(x)$ to decay in a certain uniform sense like $e^{-|x|^\alpha}$ for some $\alpha > 1$ (local regularity of $\rho(x)$ is not involved). This is significant in light of the fact that, since μ is a finite measure, continuity of the imbedding $W^{1,2}(\Gamma, \rho(x)dx) \to L^2 \ln L(\Gamma, \rho(x)dx)$ implies compactness of the imbedding $W^{1,2}(\Gamma, \rho(x)dx) \to L^2(\Gamma, \rho(x)dx)$ (see, e.g., [1]). One would expect, in light of the evidence, to find a relationship between continuity of the imbedding $W^{1,2}(\Gamma, \rho(x)dx) \to L^2 \ln L(\Gamma, \rho(x)dx)$ and decay of $\rho(x)$ in some sense like $e^{-|x|^2}$.

REMARK. An important consequence of the relationship between hypercontractivity of a Dirichlet semigroup and compactness of the imbedding $W^{1,2} \to L^2$ is the following. Suppose that A represents a Dirichlet form h on $L^2(\Gamma, d\mu)$ and that $\mathrm{Dom}(h) \subset W^{1,2}(\Gamma, d\mu)$. Suppose that $\{e^{-tA}\}$ is hypercontractive. It follows from Theorem XIII. 64 in [14] that, since the imbedding $\mathrm{Dom}(h) \to L^2(\Gamma, d\mu)$ is compact, then $L^2(\Gamma, d\mu)$ has a complete orthonormal basis $\{\varphi_n\} \subset \mathrm{Dom}(A)$ such that $A\varphi_n = \mu_n\varphi_n$ with $\mu_1 \leqq \mu_2 \leqq \cdots$ and $\mu_n \to \infty$ as $n \to \infty$.

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