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MODERN PERSPECTIVES ON CLASSICAL FUNCTION THEORY

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Not long ago, I came across an article by a famous French mathematician, one of the foremost exponents of the Bourbakist school of mathematics, purporting to describe the present direction of mathematical research. By the end of the third paragraph he had managed to dismiss the theory of analytic functions of a complex variable as having cut itself off from the "main stream" of mathematics by "indulging in overly specialized questions." Well, what can one reasonably expect from someone who asserts that "the invention of functors is one of the main goals of modern mathematicians"? Perhaps benign neglect by those who favor the Grand Design is not such a bad thing; it enables complex analysts to work in peace. Not yet, at least, has anyone laid function theory on the Procrustean bed of his own ideology and tried to trim it, head, limbs, and all, to the specifications of his own taste, whim, or fancy. If function theory is to be dubbed a "living fossil" (like the Jews, in Toynbee's scheme of history), so be it.

Actually, the situation is not so bad. A discipline which can boast contemporary exponents of the caliber of Nevanlinna, Ahlfors, Beurling, and Schiffer (not to mention the bright stars of several younger generations) is surely far from played out. To tell the truth, few sensible people ever thought it was. I had to look long and hard for an unfavorable comment, and in the process I encountered numerous unsolicited encomia from men of such high sensibility and diverse interests as Eugene Wigner, Felix Browder, Georg Kreisel, and Clifford Truesdell (references available on request). For such individuals, impervious to the fad of the hour, complex variables has a permanent value.

And yet, there is something in Professor Dieudonné's assessment [**4**] that strikes a nerve. Function theory is a little bit like Euclid. All of us have had to learn some, and the basic theory is so coherent, so all of a piece, hangs together so well with no loose ends, that there is the ever present temptation to conclude that one has learned it all.

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It isn't so. A hundred and fifty years young, function theory is going strong, and if its greedy and ungrateful nephews and nieces are gathering for a deathwatch and the reading of the will, they are in for a long wait. Still, when you are 150 years old, you need, if not all the help you can get, at least an occasional tonic. The prescription, for there is one, is a remedy tried and true; it is the constant illumination and re-examination of the subject and its problems both in the clear light of history and with the laser beam provided by parallel progress in other branches of mathematics. And, in this latter connection, while it hardly seems necessary to justify the use of methods from other branches of mathematics in complex analysis (or any other discipline), it is appropriate to note that function theory has given mathematics so much (topology and algebraic geometry, for instance) that it seems only fair that it get something in return.

So that is the sermon I would like to preach to this audience, which I take to be sympathetic, today. It isn't hard to illustrate. It would be enough (if I may be permitted my own unsolicited encomia) for me to direct you to Albert Baernstein's brilliant application of ideas from real variables to a broad spectrum of function-theoretic problems or to Carl FitzGerald's penetrating and ingenious use of classical techniques in the theory of univalent functions and cognate areas.

Since it is I who am speaking today, not they, I have decided instead to talk about the work I know best, my own. And I would like to use it as a kind of peg on which to hang a pet thesis of mine, which is, quite simply, that even the basic theorems of complex analysis, the classical corpus covered in a first-year course in function theory, can afford an ample arena for interesting and worthwhile research. I'd like to illustrate this thesis by a number of specific examples taken from the most basic function theory, things like the theorems of Cauchy and Morera, the mean-value-theorem for harmonic functions, the theory of normal families, and the Picard theorems. In each case, surprising new insights have been obtained and unrealized connections uncovered, whether by a close re-examination of old avenues of thought or by applying new lines of reasoning made available by developments elsewhere in mathematics. Thus, to crib a line from Felix Klein, this talk might reasonably be entitled "Elementary Complex Analysis from an Advanced Point of View."

Now, after that lengthy preamble, let me begin.

1. The first result I'd like to discuss is also the most advanced, the Little Theorem of Picard, proved almost exactly 100 years ago. It is not too much to say that this theorem, which asserts that a nonconstant entire function takes on every complex value with at most exception, caused a revolution in mathematics, at least in function theory. For almost 20 years after it first was proved, mathematicians labored to find an "elementary" proof of this result. (Finally Borel succeeded; his reasoning has long since been discarded in favor of more agreeable arguments.) Even today the result is regarded as deep. In a first graduate course in function theory one typically proves it either (following Picard's original line of reasoning) using a combination of the modular function and the monodromy theorem or, via the "elementary" route, by means of Bloch's theorem or Schottky's theorem.

Let me try to "debunk"—the use of the word in such a connection is due to Littlewood—Picard's theorem. I want to show you that, at least for a class of entire functions large enough to include all functions ever encountered in practice, Picard's theorem can be proved by finite induction! So far as I know, this is a new observation; at least, I haven't met anyone who will own to having seen the argument before **[12]**.

The starting point is to observe that Picard's theorem is, in a natural sense, a generalization of the Fundamental Theorem of Algebra (which says that a nonconstant polynomial takes on every finite value). Now polynomials are characterized by the relation

$$\limsup_{r\to\infty} \frac{\log M(r)}{\log r} < \infty ,$$

where $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ is the maximum modulus of the function in question. It seems reasonable to explore the situation for functions which satisfy

$$\limsup_{r\to\infty} \frac{\log_n M(r)}{\log r} < \infty , \tag{1}$$

where $\log_n x = \log(\log(\cdots (\log x) \cdots))$ (n times). These are functions which, roughly, grow no faster than $\exp(\exp(\cdots (\exp z^k)\cdots))$ ($n - 1$ times). While the collection of all functions which satisfy (1) for some n does not exhaust the set of all entire functions, it does include all functions ever normally encountered (and much more). For instance, functions which satisfy (1) for $n = 2$ constitute the much-studied class of functions of finite order.

Before proving Picard's theorem for functions which satisfy (1), it will be convenient to isolate a useful fact relating the growth of the maximum modulus of an analytic function to the growth of the maximum of its real part. To this end, let

$$A(r) = A(r, f) = \max_{|z|=r} \operatorname{Re} f(z)$$

What we need is an inequality of the form

$$M(r) < K_1 A(K_2 r) \qquad (r \text{ large}) \tag{2}$$

for some constants K_1, $K_2 > 0$. Such an estimate is immediate from the classical lemma of Borel-Carathéodory, which gives

$$M(r) \leqq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)| \qquad (r < R).$$

The point is that this fact is completely elementary—it can be proved using nothing more involved than the power series expansion for an analytic function—so there is no dirt being swept under the rug.

To prove Picard's theorem for functions satisfying (1), suppose it has been shown to be true for $n = N$. Let f satisfy (1) with $n = N + 1$ and suppose further that f fails to taken on some complex value w_0. Since $f - w_0$ then omits 0 and again satisfies (1), we may as well assume that $w_0 = 0$. Then $f = e^g$, where g is again entire. Since

$$\begin{aligned}\log_N A(r, g) &= \log_{N+1} e^{A(r,g)} \\ &= \log_{N+1} M(r, e^g) = \log_{N+1} M(r, f),\end{aligned}$$

we have

$$\limsup_{r\to\infty} \frac{\log_N A(r, g)}{\log r} = \limsup_{r\to\infty} \frac{\log_{N+1} M(r, f)}{\log r} < \infty .$$

This, together with (2) yields

$$\limsup_{r\to\infty} \frac{\log_N M(r, g)}{\log r} < \infty .$$

It follows from the induction hypothesis that g takes on every complex value, with at most one exception. Thus, for fixed $w \in \mathbf{C}$, g takes on all but one of the values $w + 2\pi in, n = 0, \pm 1, \pm 2, \ldots$; and so $f = e^g$ takes on the value e^w infinitely often. This completes the proof. We observe that we have shown slightly more than we claimed; namely, either f takes on every complex value or it omits a single value and takes on every other value infinitely often.

When I've shown this argument to people, the usual response is something like, "Well, of course the whole point of Picard's theorem is that it works for all entire functions." I have no desire to debate the point (though it is worth noting that Borel himself argued passionately against precisely such a point of view [3]). But if that is so, it seems to me that it becomes clear only in the light of an argument such as the one given above; one comes to understand the power of a technique only by seeing what one can prove without it. If Picard's theorem is a deep result, it is because of those inaccessible functions the induction argument cannot reach.

That brings me to a related point. The obvious question—and it is always asked—is whether the argument can be completed to give the result for all entire functions, say by transfinite induction. I don't know.

Whether and why one would want to prove Picard's theorem in such a fashion is, of course, an entirely different question.

Before turning to a slightly different topic, I'd like to make one further point. The inductive proof of Picard's theorem has, I think, a substantial pedagogical value. While almost every graduate student in mathematics has to take some complex function theory, there are many institutions (and, I am sorry to say, the University of Maryland is one of them) where it is entirely possible to satisfy the requirement in analysis by taking a year of real variables and only a semester of complex analysis. Many students do just that. If they're lucky, they get to see the Riemann Mapping Theorem at the very end of the semester; but it would be a rare lecturer indeed who could manage to cover Picard's theorem in the first semester. The proof given above is so short, so elementary, fits so naturally into the development of the basic theory, that it offers the instructor who is so inclined the opportunity to hit one of the genuine high points of the subject relatively early in the first semester.

2. It has always seemed to me a bit mysterious how much easier it is to prove Picard's Little Theorem than to prove the Great Theorem. According to the latter, a function analytic in a (punctured) neighborhood of an essential singularity takes on every value, with at most one exception, infinitely often. Since one may shrink the neighborhood at will, it is clearly enough to show that in each such neighborhood all values (except one) are taken on at least once. This formulation lays bare the essential difference between the two results; in the Little Theorem, the function is defined for all complex values, the singularity lies at infinity, and a global assertion is made; in the Big Theorem, the function need be defined only locally, in some neighborhood of the singularity (which may be taken to be the point at infinity, if one so desires), and the assertion on the distribution of values of the function is localized to the neighborhood in question.

Actually, there is no particular profit in distinguishing the value infinity from other values; both theorems can be (and are most naturally) formulated as assertions about meromorphic functions. The Little Theorem then says that a nonconstant meromorphic function on $\mathbf{C}$ takes on every value in the extended complex plane $\hat{\mathbf{C}}$ with at most two exceptions (since we now allow one more value, ∞), and the Big Theorem states that a function meromorphic in a punctured neighborhood of an essential singularity takes on all but at most two values in $\hat{\mathbf{C}}$ on that neighborhood. In the sequel we shall adopt these formulations.

One of the most agreeable proofs of the Big Theorem is via the theory of normal families. More precisely, one uses Montel's theorem, which states that a family of meromorphic functions, all of which omit (the same) 3 values $a, b, c \in \hat{\mathbf{C}}$ on their common domain, is normal. The deduction of

Picard's Big Theorem from Montel's theorem is very easy and by now quite familiar, so I shall not trouble you with it. Usually one proves Montel's theorem either via the modular function or by means of Schottky's theorem. Either route leads, in fact, to the desired destination without a detour through Montel's theorem, though of course the extra work can be made to yield an additional payoff (Julia's theorem).

Now Montel's theorem bears a striking family resemblance to Picard's Little Theorem. Indeed, Picard's Little Theorem says that a meromorphic function omitting three values in $\hat{\mathbf{C}}$ is a constant, and Montel's theorem deduces from essentially the same hypothesis the normality of a family of meromorphic functions. Is this anything more than coincidence?

Speculation in this direction goes back over fifty years to André Bloch, a brilliantly original mathematician who was also a maniacal mass murderer.[1] He laid down the heuristic principle that if a property forces an entire (or a globally defined meromorphic) function to be constant, then any family of analytic (meromorphic) functions on a domain which enjoy that same property must be a normal family.

I first learned Bloch's principle as a graduate student reading the second volume of Hille's *Analytic Function Theory*, and I promptly forgot about it. More recently, I came across it again, in the late Abraham Robinson's splendid retiring address to the Association for Symbolic Logic [**8**]. In that address, Robinson listed the explication of Bloch's principle as one of twelve problems worthy of the attention of logicians (and, by extension, mathematicians). As luck would have it, at just that time I was sharing an office with Christian Pommerenke, whose work on the boundary behavior of analytic functions turned out to provide a better "explication" [**11**] than even Robinson anticipated.

Before I proceed any further, let me pause to point out to you that Bloch's principle is false. Worse yet, it cannot possibly be true. Let me explain. The property of omitting three distinct values forces a meromorphic function to be constant but does not force a family of functions to be normal, (a counterexample is the family of functions $\{f_n(z)\}, f_n(z) = nz$, on the unit disc); the functions in the family must all omit the same three values. Yet for a single function these two properties coincide! Reasonable men will see in this example neither untoward nit-picking nor the counsel of despair; it shows, rather, the necessity for formulating precisely appropriate restrictions (which one hopes will be minimal) on the properties to be considered. Robinson himself recognized this need and formulated a restricted version of the principle, which he hoped would be amenable to proof by nonstandard analysis. In fact, the nonstandard analysis has proved to be something of a red herring; as for the restrictions, it turns out that one can actually get by with much less.

It happens to be convenient to adopt a rather pedantic notation for functions and properties if one intends to prove a version of Bloch's principle. Since I intend to do nothing of the sort here today, let me instead describe the result in question in an informal fashion and make a plea for good will.

Let P be a property which is

(i) hereditary (i.e., stable under restriction),

(ii) continuous (i.e., stable under convergence), and

(iii) invariant (i.e., stable under linear substitution).

Suppose that the only meromorphic [entire] functions on $\mathbf{C}$, which have P are constants. Then, for any domain $D \subset \mathbf{C}$, the collection of all functions meromorphic [analytic] on D which have P is a normal family.

It is important to note that the theorem holds equally for analytic and meromorphic functions.

How does one prove this? By using the power of negative thinking and giving a necessary and sufficient condition for a family of functions not to be normal.[2] More precisely, if a non-normal family of meromorphic functions has a property P which satisfies (i), (ii), and (iii), one can construct a nonconstant meromorphic function on $\mathbf{C}$ which has P; and this is a contradiction. Such a construction occurs in the work of Lohwater and Pommerenke [**6**] on the asymptotic embedding of parabolic Riemann surfaces in the cluster sets of certain functions which are called, appropriately enough, non-normal; it is easily modified to handle the present situation. By one of those delicious ironies of history, there is nothing in the argument that was not already available to Bloch fifty years earlier. In fact, the proof is entirely elementary, and the only point that can be said to be at all delicate is handled by a simple device popularized by Landau in his proof of Bloch's famous generalization of the one-quarter theorem. The argument also has the distinction of affording one of the very few instances of a nontrivial application of the well-known criterion (due to Felix Marty) for normality in terms of the spherical derivative.

Where does all this lead? In case the property under consideration depends only on the values taken on by the function, our theorem tells pretty much the full story. Thus, to prove Montel's theorem, take for P the property "either f omits the (fixed) values a, b, $c \in \hat{\mathbf{C}}$ or f is constant". Conditions (i) and (iii) are automatically satisfied and (ii) follows easily from Hurwitz's theorem. We drop the Little Picard Theorem in the hopper, turn the crank, and out comes Montel's theorem.

Actually, it is no harder to prove more involved variations on the same theme. For instance, take for P the property that the function f omits three values a, b, c (allowed to depend on f), the product of whose chordal distances $\chi(a, b)\chi(b, c)\chi(c, a)$ is bounded away from zero by some (small) fixed constant $\varepsilon > 0$. Then, as before, P satisfies (i), (ii) and (iii), so we can conclude that any family of functions which satisfy P is normal. Or, for a somewhat more recondite example, suppose that f is meromorphic on $\mathbf{C}$, and that all its poles have multiplicity $\geqq \ell$, all its zeros multiplicity $\geqq m$, and all zeros of $f(z) - 1$ multiplicity $\geqq n$. Nevanlinna's Second Fundamental Theorem [7, p. 280] implies that if $\ell^{-1} + m^{-1} + n^{-1} < 1$, then f is constant. If follows just as before that any family of meromorphic functions, all of which satisfy the above multiplicity conditions on some given domain, is a normal family [5, p. 238].

None of these theorems is new, but in each case Bloch's principle puts the work where it belongs; it puts the work into proving a theorem about a single, globally defined function. In some cases, at least, the savings in labor and clarity involved are very considerable. Thus, at the very least, the principle has the value of a systematic approach.

Does it also lead to new theorems? No, at least, not yet.[3] Let me explain why, since the answer suggests a number of interesting open problems. The crux occurs at (iii), where one least expects it. Typically, properties of analytic functions involve not just the values of the functions themselves, but also the values of their derivatives. No such condition is likely to be linearly invariant. For instance, it has long been known, and is relatively easy to show, that an entire function which satisfies

$$f(z) \neq 0 \quad \text{and} \quad f'(z) \neq 1 \tag{3}$$

must be constant. It is also true (Miranda) that a family of functions (on the disc, say) which satisfy (3) is normal. Since (3) does not continue to hold when f is replaced by $g(z) = f(az + b)$, the hypothesis of linear invariance is not satisfied; so we cannot obtain Miranda's theorem as a special case of our version of Bloch's principle.

To make a virtue of adversity, let me note that this failure admits an optimistic interpretation. It is a relatively recent (and highly nontrivial) result of Clunie that an entire function with the property

$$f(z)f'(z) \neq 1 \tag{4}$$

must be constant. Property (4) also fails to satisfy linear invariance, and it remains an open quesion whether it forces a family of analytic functions to be normal. Now there is every reasonable expectation that an approach on lines similar to those limned above can be developed to handle the easy case (3). Since the difficulties encountered in applying Bloch's principle to (4) seem rather similar to those which arise in dealing with (3), one can at least hope that a successful approach to (3) will also enable one to handle (4). (Naturally, pessimists can argue in exactly the opposite direction.) Of course, it is entirely possible that (4) does not force a family to be normal, in which case our formulation of Bloch's principle would provide a kind of rough explanation for this failure. At the present stage, all this is only wishful thinking, but the question is well worth investigating.

I should also like to mention two further, much vaguer and chancier, directions for possible research. Let P be a property of functions defined on open subsets of the plane. Associate to each open set $U \subset \mathbf{C}$ the collection of meromorphic functions on U which have P. In case P is hereditary, the aggregate of all such collections (which we may identity with P) has the structure of a pre-sheaf of sets, a fact pointed out to me by Mike Razar. While it seems most unlikely that this structure could be exploited to function-theoretic advantage, one cannot rule out the possibility altogether, and someone may wish to examine it.

A second direction of possible research, suggested to me by Yakar Kannai, is connected with the logical properties of the formulation of various properties. It seems well within the realm of possibility that new elaborations of Bloch's principle, in which attention is focused on such logical properties, await discovery. Unfortunately, my limited competence in logic renders any detailed speculation in this direction out of the question.

Thus, for the moment at least, the principal interest of the circle of ideas I've been discussing (beyond whatever philosophic interest attaches to the transformation of a purely heuristic device into a bona fide theorem) is pedagogic; it provides a new and fairly direct route to Montel's theorem and thence to Picard's Big Theorem, Julia's Theorem, the theorems of Schottky and Landau, and points beyond.

3. Thus far, I have focused on results, proved by old methods, whose interest lies mainly in the direction of casting new light on familiar phenomena. Now I should like to make a sharp turn and talk instead about some results which open altogether new vistas. Surprisingly (or, on reflection, not-so-surprisingly) these come from re-examination of some extremely elementary aspects of complex analysis.

Let me begin with a subject very close to my heart, Morera's theorem. One version of this result, due to Carleman, goes as follows. Let D be a domain in $\mathbf{C}$, and let f be continuous on D. Suppose that

$$\int_{\Gamma} f(z)dz = 0 \tag{5}$$

for all circles Γ contained, with their interior, in D. Then f is analytic on D. Since this is not the usual version of the theorem, let me also remind you of the proof. Suppose $f \in C^1(D)$. Fix $z_0 \in D$ and let Γ_r be the disc of radius r centered at z_0. Then, for r sufficiently small, one has from Green's theorem

$$0 = \int_{\Gamma_r} f(z)dz = 2i \iint_{\Delta_r} \frac{\partial f}{\partial \bar{z}}\, dx\, dy,$$

where Δ_r is the disc bounded by Γ_r. It follows that

$$0 = \lim_{r \to 0} \frac{1}{\pi r^2} \iint_{\Delta_r} \frac{\partial f}{\partial \bar{z}}\, dx\, dy = \frac{\partial f}{\partial \bar{z}}(z_0).$$

Since z_0 was arbitrary, $\partial f/\partial \bar{z} \equiv 0$ in D, so f is analytic there. The general case, in which f is not assumed to be C^1, follows from a simple approximation argument (convolution with a smooth "mollifier").

Successful mathematicians learn early on in their careers to ask the following question: What else can I prove with this argument? Not often enough do they ask the opposite question: What can I prove without it? Let us, therefore, pause briefly to try to understand what makes the previous proof go, so that we may give it up.

Since analyticity is a local property, it is quite clear that it is sufficient to assume that (5) holds only for small circles (actually, the passage to nonsmooth functions requires a certain uniformity, a fairly delicate point to which we shall return later). It is equally clear that the argument stands or falls on the possibility of sending r to zero. Adopting the philosophy of negative thinking espoused above, we may ask what is the situation if (5) holds, but only for large circles, i.e., for circles whose radii are bounded away from zero by some positive number.

At the outset, at least, it is natural to restrict attention to functions defined in the whole plane, since otherwise not every point in D will be the center of an appropriate circle which lies entirely in D. Several years ago I proved **[9]** the following theorem.

THEOREM. *Let $f \in C(\mathbf{R}^2)$ and suppose that*

$$\int_\Gamma f(z)dz = 0 \tag{6}$$

for all circles Γ having radius r_1 or r_2. Then f is an entire function so long as r_1/r_2 is not a quotient of zeros of the Bessel function $J_1(z)$. In case r_1/r_2 is such a quotient, f need not be analytic anywhere.

This is, I think, a fairly surprising result; at least, it surprised me. It is also a highly unstable result, in that the (countable) set of quotients of zeros of $J_1(z)$ is dense on the real line, so the slightest perturbation of r_1 or r_2 may lead from a positive result to no result at all (or vice versa).

Where do the Bessel functions come from? At one time, I thought I knew the answer to that question; now I'm not so sure. There are by now a number of different approaches to proving the theorem, and in each approach the Bessel functions arise in a slightly different connection, whether as Fourier transforms, as solutions of a certain ODE, or as eigenfunctions of the Laplacian. The truest answer, perhaps, is that they are simply a part of nature (like Kronecker's natural numbers), and how they arise is more a function of how we view the problem that anything else.

I don't want to enter into the details of the proof here. The main idea is to view (6) as a pair of convolution equations (one for each value of r) and then to use the theory of distributions and some Fourier analysis in just the right way. Naturally, a deep result is involved, the Fundamental Theorem of Mean Periodic Functions of a Single Variable, due to Laurent Schwartz and itself proved (over 30 years ago) by means of complex function theory. (In a certain sense, therefore, we are dealing here with a successful case of self-fertilization.)

To continue our submotif of missed opportunities, we should note that there is no good reason why our version of Morera's theorem was not proved 30 years ago. There are, of course, bad reasons. For one thing, it is easy—too easy—to see that there is no one-radius theorem, and that seems to end the matter. I was fortunate enough not to see the easy proof of this fact, and so I was led to a much more elaborate counterexample than was really necessary, one which, as luck would have it, suggested what should be true. There is also another, more technical, reason. Once one recognizes (6) as a pair of convolution equations, it is natural enough to try to apply Schwartz's theorem. That result, however, applies to functions of a single variable, while the Fourier transforms associated with (6) are functions of two variables, x_1, x_2. Anyone who takes the time to compute the transforms explicitly, however, will see that they actually depend only on the single (new) variable $(x_1^2 + x_2^2)^{1/2}$, so one can apply Schwartz's theorem in the desired fashion after all. There is an obvious moral in all this.

I'd like to mention two further function-theoretic results one can obtain using similar techniques. Suppose f is continuous on the plane. For each $z \in \mathbf{C}$, the restriction of f to the circle of radius one centered at z will have a Fourier expansion

$$f(z + e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} a_n(z) e^{in\theta},$$

where

$$a_n(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + e^{i\theta}) e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1} f(z + \zeta)\, \zeta^{-n-1}\, d\zeta$$

It follows that if f is an entire function, then

$$(7) \qquad a_n(z) \equiv 0, \qquad n = -1, \ -2, \ -3, \ \ldots .$$

Conversely, it is easy to see that if (7) holds for a particuar z, then f can be extended continuously from the circle $\Gamma(z) = \{\zeta : |\zeta - z| = 1\}$ to the disc $\Delta(z) = \{\zeta : |\zeta - z| \leqq 1\}$ as a function analytic in the interior of $\Delta(z)$.

Suppose that (7) holds for all $z \in \mathbf{C}$. Must f be entire? Certainly, f admits an analytic extension from each circle $\Gamma(z)$ to its interior, but it is by no means obvious that the extensions agree on overlapping discs. Nonetheless, even more is true.

THEOREM. [10]. *Let $f \in C(\mathbf{R}^2)$ and let $n > 1$ be fixed. Suppose*

$$\int_0^{2\pi} f(z + e^{i\theta}) \begin{Bmatrix} e^{i\theta} \\ e^{in\theta} \end{Bmatrix} d\theta = 0$$

for all $z \in \mathbf{C}$. Then f is an entire function.

Thus, the vanishing of a single negative Fourier coefficient in addition to the first is sufficient to imply analyticity. There is no unpleasant exceptional set in this theorem. This last fact depends ultimately on the following deep result of Siegel's from the theory of transcendental numbers: the non-zero zeros of Bessel functions of rational index are transcendental. And, as you may surmise, the proof of that result is again function theory.

The other result I want to mention is an analogue of our version of Morera's theorem for the hyperbolic plane, i.e., the open unit disc with the non-Euclidean geometry induced by the Poincaré metric $ds = 2|dz|/(1 - |z|^2)$. Suppose f is continuous in the disc and that (6) holds for all circles having radii r_1 or r_2 (when measured in the hyperbolic geometry). Then f is analytic so long as the equations

$$P_z^{-1}(\cosh r_j) = 0, \qquad j = 1, 2,$$

have no common solution $z \in \mathbf{C}$. Here the role of the Bessel function is played by the associated Legendre function P_z^{-1}. These functions are connected by the limiting relation

$$\lim_{\alpha \to 0} \frac{P_{-iz/\alpha}^{-1}(\cosh \alpha r)}{\sinh \alpha r} = \frac{J_1(rz)}{rz},$$

so one can actually get our previous theorem (at least formally) as a kind of limiting case of the present result.

The tools employed in proving such results are so general and so flexible that it should come as no great surprise that one can prove similar theorems characterizing the solutions of equations much more general than $\partial f/\partial \bar{z} = 0$. In fact, for any homogeneous polynomial $P_n(\xi_1, \xi_2, \ldots, \xi_n)$, the global solutions of $P(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)u = 0$ can be characterized [10] by an appropriate two radius condition. Similarly, extensions to more general spaces are also possible. Such extensions, to spaces of constant curvature and, more generally, to rank one symmetric spaces are carried out in [1] and [2]. The results are neither less precise nor less explicit than those I have just discussed. And so, a new development in function theory leads to a number of parallel developments in other areas of mathematics.

4. The final topic I should like to discuss today connects rather directly with some of the aforementioned generalizations. It is a product of my attempt to understand the precise relationship between the theorems of Cauchy and Morera, on the one hand, and Gauss's mean value theorem for harmonic functions and Koebe's converse to it, on the other.

Let μ be a finite complex Borel measure supported on the closed unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$. Let D be a domain in $\mathbf{R}^n$. *A* function $u \in C(D)$ has the generalized mean value property (GMVP) with respect to μ if

$$\int u(x + rt)d\mu(t) = 0 \tag{8}$$

whenever $x \in D$ and $0 < r <$ dist $(x, \partial D)$.

The GMVP is abstracted from the conditions of the Gauss-Koebe and Cauchy-Morera theorems mentioned above. To make the connection with mean values explicit, fix n and choose $d\mu = d\Omega - \delta_0$, where $d\Omega = d\Omega_{n-1}$ is the uniform mass distribution on the unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ of total measure one and δ_0 is the point mass at the origin. Then (8) becomes

$$\int u(x + rt)d\Omega(t) = u(x), \qquad x \in D, \quad 0 < r < \text{dist}(y, \partial D), \tag{9}$$

the classical mean value property. On the other hand, taking $n = 2$ and choosing for $d\mu$ the restriction of $d\zeta$ to the unit circle, we obtain from (8)

$$\int_{|\zeta|=1} f(z + r\zeta\,)d\zeta = 0, \qquad z \in D, \quad 0 < r < \text{dist}(z, \partial D),$$

or, what is the same thing,

$$\int_{|w-z|=r} f(w)dw = 0, \qquad z \in D, \quad 0 < r < \text{dist}(z, \partial D), \tag{10}$$

which provides the connection with Cauchy's theorem.

What one would like would be some kind of grandfather theorem which includes as special cases both of the previous examples. Ideally, this would take the form of necessary and sufficient conditions (harmonicity for (9), analyticity for (10)) on a function $u \in \mathrm{C}(D)$ for it to satisfy the GMVP with respect to a given measure μ. This seems almost too much to hope for; and, indeed, while the study of conditions like (8) is quite old, it had always been under rather special assumptions on the nature of μ.

Nonetheless, such conditions can be found and described explicitly. They are not even particularly difficult to state. With the notation established above, let

$$F(z) = \int e^{-i(z\cdot t)}\, d\mu(t) \qquad (z\cdot t = z_1 t_1 + \cdots + z_n t_n)$$

be the Fourier transform of the measure μ. Since μ has compact support, F is an entire function on $\mathbf{C}^n$ and has an expansion

$$F(z) = \sum_{n=0}^{\infty} Q_n(z) \tag{11}$$

in homogeneous polynomials.

THEOREM. **[10]**. *A necessary and sufficient condition that $u \in C(D)$ satisfy the* GMVP (8) *is that u be a weak (distributional) solution to the system of linear partial differential equations*

$$Q_n(D)u = 0, \qquad n = 0, 1, 2, \ldots . \tag{12}$$

Here $Q_n(D)$ is the differential operator obtained by replacing the variable $z = (z_1, \cdots, z_n)$ with the symbolic vector $D = (-i\partial/\partial x_1, \cdots, -i\partial/\partial x_n)$.

Actually, it follows from Hilbert's Basis Theorem that the infinite system (12) is always equivalent to some finite subsystem, though it is not clear whether this fact is of any practical value. So far as cases of concrete interest are concerned, (12) seems quite satisfactory, and there is certainly no problem in obtaining the expansion (11).

The connection between (8) and (12) is provided by an operational identity I call the generalized Pizzetti Formula:

$$\int u(x + rt)d\mu(t) = [F(-rD)u](x). \tag{13}$$

Here u is assumed to be real-analytic and r is sufficiently small. In the special case $d\mu = d\Omega_1$ $(=(1/2\pi)\, d\theta$ on the unit circle), we have

$$F(z) = F(z_1, z_2) = J_0((z_1^2 + z_2^2)^{1/2}),$$

so that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta &= J_0([(-r)^2((-\,i\partial/\partial x)^2 + (-\,i\partial/\partial y)^2)]^{1/2})u(z) \\ &= J_0(r\sqrt{-\Delta})u(z) \\ &= \sum_{n=0}^{\infty} \frac{r^{2n}}{(n!)^2 2^{2n}} \Delta^n u(z). \end{aligned} \tag{14}$$

This last formula is known as Pizzetti's formula and goes back to the early years of the present century. Analogues for the surface measures $d\Omega_n$ introduced above are familiar and involve higher Bessel functions, but it seems very remarkable that the general relation (13) was not recognized earlier.

To illustrate the theorem, take $n = 3$, $d\mu = d\Omega_1 - \delta_0$. Then

$$F(z) = J_0((z_1^2 + z_2^2)^{1/2}) - 1$$

and $Q_n(D) = 0$ for $n = 0, 1, 3, 5, 7, \ldots$, while $Q_n(D) = c_n\Delta^{n/2}$, $c_n \neq 0$, for $n = 2, 4, 6, \ldots$. Thus the system (12) reduces to $\Delta^n u = 0$ $n = 1, 2, 3, \ldots$, which is equivalent to the single equation $\Delta u = 0$. Similarly, if we choose $d\mu = dz$ on $|z| = 1$, $Q_{2n}(D) = 0$ while

$$Q_{2n+1}(D) = c_n \frac{\partial}{\partial \bar{z}} \Delta^n,$$

$c_n \neq 0$. Thus $Q_n(D)u = 0$ is equivalent to the single (Cauchy-Riemann) equation $\partial u/\partial \bar{z} = 0$.

A somewhat less familiar mean value condition can be obtained by taking $d\mu = \cos 2\theta \, d\theta$ on the unit circle. Then (8) becomes

$$\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\cos 2\theta \, d\theta = 0,$$

which turns out to be equivalent to $\Box u \equiv u_{xx} - u_{yy} = 0$. You might like to try to prove this equivalence directly.

So far as (8) is concerned, the preceding results tell the whole story. It should be remembered, however, that (8) was itself abstracted from the classical mean-value property of harmonic functions. Since abstractions rarely exhibit all essentials of the original situation, there may be some profit in taking a backward glance at the classical situation.

Koebe's converse to the mean-value theorem for harmonic functions holds under weaker conditions than (9). In fact, it is sufficient that (9) hold only for all sufficiently small r, i.e., for $0 < r < \varepsilon(x)$, where no assumption is made on ε other than $\varepsilon(x) > 0$. In other words, one can dispense entirely with any assumption of uniformity in condition (9). On the other hand, while one can weaken the condition $0 < r < \operatorname{dist}(x, \partial D)$ for the GMVP considerably—for instance, to $0 < r < \varepsilon(x)$, where $\varepsilon(x)$ is bounded away from zero on each compact subset of D—some uniformity is essential to allow application of the smoothing techniques used in the proof of the theorem. This is more than a defect of method. Below we shall give a simple example for which

$$\int u(x + rt)\,d\mu(t) = 0, \qquad x \in D,\ 0 < r < \varepsilon(x), \tag{15}$$

but u is not a solution of the associated system $Q_n(D)u = 0$.

In addition to casting new light on the converse to the mean-value theorem, this raises such questions as characterizing the functions which satisfy (15) for a fixed μ or characterizing those measures for which (15) is equivalent to (8). A case of particular interest arises from the choice $d\mu = dz$ (on the unit circle). Does

(16) $$\int_{|\zeta - z| = r} f(\zeta)d\zeta = 0, \qquad z \in D, \quad 0 < r < \varepsilon(z),$$

imply that f is analytic? For smooth functions the answer is, of course, yes; but, so far as I know, the general case remains unsettled.

If one allows limiting processes, the situation becomes even more interesting. Blaschke showed in 1916 that if u is continuous and for $z \in D$

(17) $$\lim_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int u(z + re^{i\theta})d\theta - u(z) \right\} = 0,$$

then u is harmonic on D. (This is obvious from Pizzetti's formula (14), but the point here is that no regularity is assumed for u beyond the stated hypothesis of continuity.) We can reformulate the condition (17) for general measures as follows.

Suppose that μ is a finite complex Borel measure on $\mathbf{B}^n$ and let k be the largest integer such that μ is orthogonal to all polynomials of total degree less than k. Let $u \in C(D)$ $(D \subset \mathbf{R}^n)$. Then (17) is just the special case of

(18) $$\lim_{r \to 0} \frac{1}{r^k} \int u(x + rt)d\mu(t) = 0$$

obtained by choosing $d\mu = (1/2\pi)d\theta - \delta_0$. What are necessary and sufficient conditions (on u) for (18) to hold? In case $u \in C^k(D)$, it is easy to see that the required condition is that $Q_k(D)u = 0$, where Q_k is, as before, the homogeneous polynomial of degree k occurring in the Taylor expansion of the Fourier transform of μ. For the proof, simply expand the integrand in a k^{th}-degree Taylor expansion about x and integrate term by term. If u has less smoothness, this argument fails (obviously); and, indeed, for particular choices of μ a solution to (18) need not satisfy the associated differential equation.

Special cases of (18) are familiar from real variables. For instance, let $n = 1$ and take $\mu = \delta_1 - \delta_{-1}$. Then $k = 1$ and $Q_1(D) = 2\,d/dx$. Condition (18) becomes

$$\lim_{r \to 0} \frac{u(x + r) - u(x - r)}{t} = 0, \qquad x \in D,$$

the requirement that the symmetric derivative of u vanish identically on the interval D. A theorem of Khinchin asserts that such a function must be constant (i.e., must satisfy $Q_1(D)u = 0$). Similarly, the choice $\mu = \delta_1 - 2\delta_0 + \delta_{-1}$ yields $k = 2$ and leads to the familiar condition of Schwarz

$$\lim_{r \to 0} \frac{u(x + r) - 2u(x) + u(x - r)}{r^2} = 0,$$

which is equivalent to $Q_2(D)u \equiv d^2u/dx^2 = 0$ (i.e., u is linear).

A slight modification of this last example yields a negative result. Take $\mu = \delta_2 - 2\delta_1 + \delta_0$; again one has $k = 2$, so the expected condition is that u be linear. However, any function which is only piecewise linear will satisfy

$$\lim_{r \to 0} \frac{u(x + 2r) - 2u(x + r) + u(x)}{r^2} = 0.$$

In fact, for such a function one actually has

$$u(x + 2r) - 2u(x + r) + u(x) = 0, \qquad 0 < r < \varepsilon(x).$$

Now piecewise linear functions are dense in all continuous functions, so we are very far indeed from the anticipated conclusion of linearity. On the other hand, such functions do satisfy the pointwise equation $d^2u/dx^2 = 0$ on a large set. There is reason to believe that this behavior is typical; a solution to (18) should satisfy $Q_k(D)u = 0$ on a dense open set in D.

A number of natural questions suggest themselves at once. How bad can the exceptional set (on which the differential equation is not satisfied) be? For which measures μ does (18) imply $Q_k(D)u = 0$? (The answer to this question, at least for discrete measures, is most likely bound up with the stability theory for finite difference approximations to differential equations.) Must a function of class C^{k-1} which satisfies (18) be a solution of the associated differential equation? All indications point in this direction (translation: we know of no counterexample), but we do not have so much as a hint toward the proof. Even the question of whether

$$\lim_{r \to 0} \frac{1}{r^2} \int_{|\zeta - z| = r} f(\zeta)\, d\zeta = 0, \qquad z \in D,$$

(the natural weakening of (16)) implies analyticity remains open. A positive answer would provide a new variation on the classical Looman-Menchoff theorem and would be of considerable interest.

It's time for me to stop. I hope that the ideas discussed, the examples mentioned, and the connections drawn with such areas as logic and number theory, harmonic analysis and PDE, special functions and real variables give ample enough indication of the continued vitality of this venerable yet vital, old but ever-new subject. I've said it once, I'll say it again: complex analysis is alive and well.

Footnotes

[1]The heady blend of mathematics and crime is hardly original with Bloch; it persists to the present. It will be recalled that the ineffable Professor Moriarty, Sherlock Holmes's bête noire, had written a treatise on the binomial formula. In our own day, a celebrated automata theorist has been brought to trial for masterminding a bizarre kidnapping-

decapitation across international borders. And the villain in a recent adventure film starring Charles Bronson owns up at one point to long years of research in functional analysis and combinatorics, leaving to the viewer the unhappy choice of whether to put this down as just one more crime or, rather, as an appropriate punishment.

[2]My friend Benjy Weiss remarked after seeing the proof that we do not emphasize often enough to our students the value of obtaining necessary and sufficient conditions for something not to hold.

[3]These words, it transpires, were out of date even before they were written, at least so far as the reasoning used to establish Bloch's principle is concerned. This has found application in the theory of complex manifolds, most notably to prove Brody's Theorem, which says a compact complex manifold which contains no complex lines is hyperbolic. (For the differential-geometric notion of hyperbolicity see S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.) The result in question had been conjectured by Griffiths and attracted a considerable amount of attention before finally being proved by Robert Brody in his Harvard thesis, "Intrinsic metrics and measures on compact complex manifolds" (1975). According to H. Wu (*Some theorems on projective hyperbolicity*, J. Math. Soc. Japan **33** (1981)), "with a trivial change in terminology, Zalcman's arguments would have proved Brody's theorem."

In a somewhat different direction, Dr. Ruth Minowitz ("Normal families of quasiregular mappings," University of Maryland Technical Report 78–73) has extended Bloch's principle to quasiregular functions in space and used it to prove analogues of the Big Picard Theorem and Julia's Theorem for functions belonging to that class.

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