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ON AUTOMORPHIC FORMS FOR THE GENERAL LINEAR GROUP

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ABSTRACT. Properties of Hecke operators for $GL(3, \mathbf{Z})$ are investigated as well as the analytic continuation of Eisenstein series for $GL(n, \mathbf{Z})$. The results described arose in the investigation of harmonic analysis of $GL(n, \mathbf{Z})$-invariant functions on the space of positive $n \times n$ real matrices.

1. Introduction. Automorphic forms for the general linear group, $GL(n, \mathbf{Z})$ of $n \times n$ integer matrices of determinant ± 1, can be viewed as relatives of the trigonometric functions—relatives which play a role in harmonic analysis on the Minkowski fundamental domain of positive definite $n \times n$ real matrices $\mathscr{P}_n$ modulo $GL(n, \mathbf{Z})$. Automorphic forms for $GL(n, \mathbf{Z})$ can also be considered to be related to Siegel modular forms which appear in the study of abelian integrals. The present paper is intended to be an expository discussion of some of the results needed to derive harmonic analysis on the fundamental domain $\mathscr{M}_n = \mathscr{P}_n/GL(n, \mathbf{Z})$ from that on $\mathscr{M}_{n-1}$. In order to remain at an expository level it will be considered legal to restrict to the case $n = 3$ (the case $n = 2$ being assumed known). For similar reasons the adelic interpretation will not be considered. §1 gives the properties of Hecke operators T_m for $GL(3)$ mostly. This includes the analytic continuation and Euler product of the L-function corresponding to an automorphic form for $GL(3, \mathbf{Z})$ which is an eigenfunction for all the Hecke operators. The analytic continuation of the L-function works for $GL(n)$, for all n (not just $n = 2, 3$), and is just a re-interpretation of work of Maass and Selberg. §2 studies the explicit analytic continuation of the L-functions and Eisenstein series for $GL(3)$ by a method which differs from that of Maass and Selberg in that it does without the differential operators introduced by Selberg. The method is close to one used by Arakawa, as well as to the adelic ideas of Jacquet and Shalika which appear in Inventiones Math. **38** (1976), 1–16. Helen Strassberg has also obtained such analytic continuations by adelic means. in **[20]**. §1 and §2 are closely related, since the Hecke operators for $GL(n)$ relate the two basic types of Eisenstein series for $GL(n)$. There is also a connection between

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the method of analytic continuation of Eisenstein series for $GL(3)$ and Fourier expansions of automorphic forms for $GL(n)$. The arguments presented here involve nothing more than matrix multiplications and some simple integral formulas for the symmetric space $\mathscr{P}_n$.

A number theorist is motivated to study harmonic analysis on $\mathscr{P}_n$ in order, for example, to generalize Hecke's correspondence between modular forms and Dirichlet series to Siegel modular forms (i.e., modular forms for the symplectic group $Sp(n, \mathbf{Z})$). Recently Kaori Imai achieved this generalization in **[9]**. A brief introduction to her work can be found in **[10]**. One hopes that the generalization of this result to congruence subgroups of the Siegel modular group will help to unravel the behavior of some of the more mysterious number-theoretic L-functions. And there are recent results of Winnie Li in **[14]**, which suggest how to proceed with congruence subgroups in a way which most nearly resembles the method used for $Sp(n, \mathbf{Z})$ itself.

Before proceeding further, one needs some definitions. More details on harmonic analysis on symmetric spaces can be found in **[22]**. The symmetric space under consideration here is $\mathscr{P}_n$, which can be identified with $O(n)\backslash GL(n, \mathbf{R})$ via the mapping

$$\begin{aligned} O(n)\backslash GL(n, \mathbf{R}) &\to \mathscr{P}_n \\ O(n)g &\to {}^t gg, \end{aligned} \tag{1.1}$$

where ${}^tg$ is the transpose of $g \in GL(n, \mathbf{R})$. And the action of A in $GL(n, \mathbf{R})$ on Y in $\mathscr{P}_n$ will be denoted

$$Y[A] = {}^tAYA. \tag{1.2}$$

It will also be useful to define the symmetric space

$$\mathscr{SP}_n = \{W \in \mathscr{P}_n \mid \det W = 1\} \cong SO(n)\backslash SL(n, \mathbf{R}). \tag{1.3}$$

We will use $|Y|$ to denote the determinant of the matrix Y in $\mathscr{P}_n$.

The $GL(n, \mathbf{R})$-invariant integral on $\mathscr{P}_n$ is

$$\int f(Y)\,|Y|^{-(n+1)/2}\, dy, \tag{1.4}$$

where dY is Lebesgue measure on $\mathbf{R}^{n(n+1)/2}$. And measures can be normalized so that the $SL(n, \mathbf{R})$-invariant measure dW on $\mathscr{SP}_n$ is given by setting

$$Y = t^{1/n}W,\ Y \in \mathscr{P}_n,\ t > 0,\ W \in \mathscr{SP}_n,\ |Y|^{-(n+1)/2}\, dY = t^{-1}\, dt\, dW. \tag{1.5}$$

Thus analysis on $\mathscr{P}_n$ is essentially a product of analysis on the multiplicative group of positive reals times analysis on $\mathscr{SP}_n$.

When $n = 2$, one can identify $\mathscr{SP}_2$ with the non-Euclidean, Poincaré or Lobatchevsky upper half plane $\mathscr{H} = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ via the map:

$$\mathscr{H} \longrightarrow \mathscr{S}\mathscr{P}_2$$

$$\begin{matrix} x + iy \\ y > 0 \end{matrix} \longmapsto \begin{pmatrix} 1/y & 0 \\ 0 & y \end{pmatrix}\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}.$$

Harmonic analysis on $\mathscr{H}/SL(2, \mathbf{Z})$ means the expansion of $f \in L^2(\mathscr{H}/SL(2, \mathbf{Z}))$ in a mixed Fourier series/integral of eigenfunctions of the noneuclidean Laplacian $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. This result was obtained by Roelcke and Selberg in the 1950's (cf. **[12]**, **[17]**, **[22]**).

Harmonic analysis on $\mathscr{P}_n/GL(n, \mathbf{Z})$ for n larger than 2, is still in its infancy. References include the work of Langlands, Harish-Chandra, Selberg, Maass, Arthur, Venkov, and the author (cf. **[13]**, **[5]**, **[17]**, **[4]**, **[22]**, **[25]**). The Laplacian in the case of $\mathscr{H}$ is replaced by the polynomial algebra in n indeterminates of $GL(n, \mathbf{R})$-invariant differential operators L on $\mathscr{P}_n$. To say that L is G-invariant is to say that L commutes with the action of the group G on functions on the symmetric space. An example of *a* $GL(n, \mathbf{R})$-invariant differential operator L on $\mathscr{P}_n$ is

$$L = |Y|\left|\frac{\partial}{\partial Y}\right|, \text{ where } \frac{\partial}{\partial Y} = \left(\frac{1}{2}(1 + \delta_{ij})\frac{\partial}{\partial y_{ij}}\right)_{1 \leq i, j \leq n}. \tag{1.6}$$

Here δ_{ij} is Kronecker's delta and $|\partial/\partial Y|$ is the determinant of the matrix $\partial/\partial Y$ of differential operators. Selberg gives an explicit basis of the polynomial algebra of $GL(n, \mathbf{R})$-invariant differential operators on $\mathscr{P}_n$ in **[17]**.

The space $\mathscr{P}_n$ is a symmetric space because it is a Riemannian manifold with a geodesic-reversing isometry at the identity given by $u(Y) = Y^{-1}$ (cf. **[22]**). There will be a use for the operator L^u, where L is *a* $GL(n, \mathbf{R})$-invariant differential operator on $\mathscr{P}_n$, with L^u defined by

$$L^u f(Y) = L(f \circ u)(u^{-1}(Y)). \tag{1.7}$$

It is shown in **[15]** and **[22]** that L^u is the conjugate-adjoint $\bar{L}^*$ of L, with respect to the inner product of functions $f, g \colon \mathscr{P}_n \to \mathbf{C}$

$$(f, g) = \int_{\mathscr{P}_n} f(Y)\overline{g(Y)}\, |Y|^{-(n+1)/2}\, dy. \tag{1.8}$$

That is $(Lf, g) = (f, L^*g)$.

An *automorphic form* v for $\Gamma = SL(n, \mathbf{Z})$ or $GL(n, \mathbf{Z})$ is a function $v\colon \mathscr{S}\mathscr{P}_n \to \mathbf{C}$ such that the following three properties hold:

(1) v is an eigenfunction of all the $SL(n, \mathbf{R})$-invariant differential

(1.9) operators L on $\mathscr{S}\mathscr{P}_n$, $Lv = \lambda(L)v$, for some $\lambda(L) \in \mathbf{C} = 0$,

(2) $v(Y[A]) = v(Y)$ for all Y in $\mathscr{S}\mathscr{P}_n$ and A in Γ,

(3) v satisfies some growth condition which will not be specified.

For the present purposes (3) might as well be taken to be that v must lie in $L^1(\mathscr{S}\mathscr{P}_n/\Gamma)$. If one could say more about Fourier expansions of automorphic forms v for $GL(n, \mathbf{Z})$, the third condition ought to be a weaker one—excluding exponentially growing Fourier coefficients at infinity (cf. **[22]**). In the case that $n = 2$, for example, condition (3) requires that $|v(x + iy)| \leqq C\, y^p$ for some positive constant C and some power p, uniformly in x as y goes to infinity. Automorphic forms as in (1.9) were first discussed in the case $n = 2$ by H. Maass during the late 1940's. Maass considers the general case in **[15**, §10], using the name grössencharacter, rather than automorphic form for $SL(n, \mathbf{Z})$. The name grössencharacter is justified by the fact that the Hecke grössencharacters play the same role in inverting Hecke's correspondence between Hilbert modular forms and Dirichlet series that automorphic forms for $GL(n, \mathbf{Z})$ play in inverting Hecke's correspondence between Siegel's modular forms and Dirichlet series. The Hilbert case has been discussed by Weil, Jacquet, Langlands, and Stark. The Siegel case has been discussed by Imai in **[9]** and **[10]**.

A *cusp form* is defined to be an automorphic form v as in (1.9) with vanishing constant term in all of its Fourier expansions with respect to the maximal parabolic subgroups $P_{k,n-k}$ defined for $1 \leqq k \leqq n - 1$ by

$$(1.10)\quad P_{k,n-k} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \middle| A \in GL(k, \mathbf{Z}),\ D \in GL(n-k, \mathbf{Z}),\ B \in \mathbf{Z}^{k\times(n-k)} \right\}.$$

This makes sense, because one can use the partial Iwasawa decomposition of Y in $\mathscr{P}_n$

$$(1.11)\qquad Y = \begin{pmatrix} T & 0 \\ 0 & V \end{pmatrix}\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}, \text{ with } T \in \mathscr{P}_k,\ V \in \mathscr{P}_{n-k},\ Q \in \mathbf{R}^{k\times(n-k)},$$

and see that $v(Y)$ must be a periodic function of Q if v is invariant under $P_{k,n-k}$.

The main example of an automorphic form for $GL(n, \mathbf{Z})$ is the Eisenstein series, which comes in several guises, and is not a cusp form. The *first type of Eisenstein series* is defined by

$$(1.12)\qquad e_n(Y, s) = \sum_{A \in \Gamma/P} p_s(Y[A]),$$

for $Y \in \mathscr{P}_n$, $s \in \mathbf{C}^n$, $\operatorname{Re} s_j > 1$, $1 \leqq j \leqq n - 1$. Here $p_s(Y)$ denotes the *power function*, defined for $Y \in \mathscr{P}_n$, $s \in \mathbf{C}^n$, by

$$(1.13)\qquad p_s(Y) = \prod_{j=1}^{n} |Y_j|^{s_j}, \text{ if } Y = \begin{pmatrix} Y_j & * \\ * & * \end{pmatrix},\ Y_j \in \mathscr{P}_j,$$

and the sum in (1.12) is over representatives A of $GL(n, \mathbf{Z})$ modulo P equal to the minimal parabolic subgroup of upper triangular matrices. Since the power functions are eigenfunctions of the $GL(n, \mathbf{R})$-invariant differential operators on $\mathscr{P}_n$, the Eisenstein series automatically satisfies (1) and (2) of definition (1.9).

It is shown in **[22]**, **[24]**, **[15]** that the series (1.1) converges for $\operatorname{Re} s_j > 1$,

$j = 1, \ldots, n-1$. It is also possible to obtain the analytic continuation of $e_n(Y, s)$ to a meromorphic function of the n complex variables $s \in \mathbf{C}^n$ (cf. [15], [22]). The case $n = 2$ is Epstein's zeta function divided by $\zeta(2s)$. It gives the continuous part of the Roelcke-Selberg spectral decomposition of the Laplacian on $\mathscr{H}/SL(2, \mathbf{Z})$. The discrete part of this spectral decomposition comes from the residue at $s = 1$ and the mysterious cusp forms (cf. [7], [22], [12]). Note that $e_2(Y, (s, 0)) = e_2(Y, s)$ will only be in $L^1(\mathscr{S}\mathscr{P}_2/SL(2, \mathbf{Z}))$ if $0 < \operatorname{Re} s < 1$. The cusp forms and constants are also in $L^1(\mathscr{S}\mathscr{P}_2/SL(2, \mathbf{Z}))$, since both are bounded on the fundamental domain $\mathscr{S}\mathscr{P}_2/SL(2, \mathbf{Z})$, which has finite volume.

Harmonic analysis on the *fundamental domain* $\mathscr{M}_n = \mathscr{P}_n/GL(n, \mathbf{Z})$ or $\mathscr{S}\mathscr{M}_n = \mathscr{S}\mathscr{P}_n/GL(n, \mathbf{Z})$ is probably best done inductively. This leads one to define the second type of Eisenstein series. Let Σ_m denote the set of eigenvalues $\Lambda(L)$ of $SL(n, \mathbf{R})$-invariant operators L on $\mathscr{S}\mathscr{P}_m$ corresponding to a total orthonormal set of automorphic forms v_λ necessary for harmonic analysis of functions f in $L^2(\mathscr{S}\mathscr{M}_n)$. By this we mean that

$$(1.14) \qquad f(W) = \int_{\lambda \in \Sigma_m} (f, v_\lambda)\, v_\lambda(W)\, d\lambda, \quad Lv_\lambda = \lambda(L) v_\lambda.$$

Here (f, g) denotes the usual L^2-inner product on $\mathscr{S}\mathscr{M}_n$ using the invariant volume, where $d\lambda$ is the spectal measure, which is some mixture of continuous and point measures. For example, the work of Roelcke and Selberg shows that

$$\Sigma_2 = \{\lambda_n \mid n \in \mathbf{Z}, n \geqq 0\} \cup \{s(s-1) \mid \operatorname{Re} s = 1/2\},$$

where the discrete part of the spectrum consists of 0 and the eigenvalues of the cusp forms, some of which have been tabulated by Hejhal in [7]. And the continuous part of Σ_2 consists of eigenvalues of Δ for the Eisenstein series $e_2(Y, s)$. The measure $d\lambda$ is $(1/4\pi)dt$ on the continuous part, if $\lambda = s(s-1)$ and $s = 1/2 + \text{it}$. Considering this as well as the description of harmonic analysis on $\mathscr{P}_m$ itself which is contained in the work of Harish-Chandra and Helgason (cf. [22]), one expects that Σ_m arises from a subset of $\mathbf{C}^{m-1}$ in the same sort of way. That is, one believes that the eigenvalues λ in Σ_m arise as specializations of those of the power functions $p_s(Y)$, $s \in \mathbf{C}^m$, $Y \in \mathscr{P}_m$, with the Eisenstein series $e_m(Y, s)$ representing the highest dimensional part of the spectrum. The present work was motivated by a search for an elementary discussion of these matters.

The *second type of Eisenstein series* is defined for λ in Σ_m, s in $\mathbf{C}$ with $\operatorname{Re} s > n/2$, $Y \in \mathscr{P}_n$ by

$$(1.15) \qquad E_{s,\lambda}(Y) = \sum_{\substack{A=(A_1 *) \in GL(n,\mathbf{Z})/P(m,n-m) \\ A_1 \in \mathbf{Z}^{n\times m}}} |Y[A_1]|^{-s}\, v_\lambda((Y[A_1])_0),$$

if $1 \leqq m \leqq n-1$, where v_λ, $\lambda \in \Sigma_m$, denotes a complete orthornormal set of automorphic forms for $SL(m, \mathbf{Z})$, as in (1.14), the *parabolic subgroup* $P(m, n - m)$ is as in (1.10), and $Y^0 = |Y|^{-1/m} Y \in \mathcal{S}\mathcal{P}_m$, for Y in $\mathcal{P}_m$. One would conjecture that the $E_{s,\lambda}$ for $m = 2$, their residues at poles in s, and the cusp forms for $SL(3, \mathbf{Z})$ should give a complete orthonormal decomposition of the invariant differential operators on $\mathcal{S}\mathcal{P}_3/SL(3, \mathbf{Z})$, using methods analogous to those in Kubota's book **[12]** in the case $n = 2$.

Applying the integral test from **[24]** to study the convergence of $E_{s,\lambda}$, one sees that the sum (1.15) can be compared with the integral

$$\int_{\substack{\mathcal{P}_m/GL(m,\mathbf{Z}) \\ |Y| \geq 1}} v_\lambda(Y^0)\, |Y|^{-s-(n+1)/2}\, dY = c \int_{W \in \mathcal{S}\mathcal{M}_m} v_\lambda(W) dW \int_{t=1}^{\infty} t^{-s-1+n/2}\, dt,$$

for some positive constant c. Assuming that v_λ is in $L^1(\mathcal{S}\mathcal{M}_m)$, it follows that $E_{s,\lambda}$ converges for $\operatorname{Re} s > n/2$. There is another way to prove this when v_λ is bounded, since then $E_{s,\lambda}$ can be bounded by

$$E_{s,0}(Y) = \sum_{\substack{B \in GL(n,\mathbf{Z})/P(m,n-m) \\ B = (B_1 *)}} |Y[B_1]|^{-s}, \; B_1 \in \mathbf{Z}^{m \times (n-m)}.$$

This is essentially Koecher's zeta function studied in **[11]** and shown to converge for $\operatorname{Re} s > n/2$.

When $m = 2$ and λ lies in the continuous part of the spectrum Σ_2 of Δ on $\mathcal{S}\mathcal{M}_2$, then $v_\lambda(W) = e_2(W, u)$, with $\lambda = u(u - 1)$, and e_2 edfined by (1.12). Then it is easily seen that the two types of Eisenstein series coincide in this special case, since

$$(1.16) \qquad \begin{aligned} E_{s,\lambda}(Y) &= e_3(Y, (u, s - u/2, 0)), \qquad \text{if } n = 3, m = 2, \\ e_2(W, u) &= v_\lambda(W), \; \lambda = u(u - 1). \end{aligned}$$

In §2, the relation between the L-functions associated to Hecke operators and the Eisenstein series will be explored. In §3, it will be shown that Riemann's method of theta functions leads to a method of analytic continuation for the Eisenstein series $E_{s,\lambda}(Y)$ defined by (1.15) which bares the connection between the integrals involving lower dimensional terms of theta and the constant terms in the Fourier expansions of v_λ. Thus there is no problem at all if v_λ is a cusp form. These methods allow the analytic continuation of the L-functions associated to an automorphic form for $GL(n, \mathbf{Z})$ which is an eigenfunction of all the Hecke operators. Many results in the following sections will only be described for $GL(3, \mathbf{Z})$.

2. Hecke operators for GL(n, Z). Suppose that f: $\mathcal{S}\mathcal{P}_n/GL(n, \mathbf{Z}) \to \mathbf{C}$. Then for any positive integer m, the m-th *Hecke operator* T_m is defined by

$$(2.1) \qquad T_m f(Y) = \sum_{A \in V_m} f((Y[A])^0), \text{ if } Y \in \mathcal{P}_n, \; Y^0 = |Y|^{-1/n} Y.$$

Here V_m denotes any complete system of representatives for $O_m/GL(n, \mathbf{Z})$, where $O_m = \{A \in \mathbf{Z}^{n\times n} \mid |A| = \pm m\}$. And we can take

$$V_m = \left\{ \begin{pmatrix} d_1 & d_{12} & \cdots & d_{1n} \\ 0 & d_2 & & \vdots \\ & & & \\ 0 & 0 & \cdots & d_n \end{pmatrix} \middle| \prod_{j=1}^{n} d_j = m,\ d_j > 0,\ d_{ij}(\operatorname{mod} d_i) \right\}.$$

This is easily proved by induction. Note that the center of $GL(n, \mathbf{R})$ will cancel out when one computes $(Y[A])^0$ for $Y \in \mathscr{P}_n$, $A \in GL(n, \mathbf{R})$.

Hecke operators for $SL(2, \mathbf{Z})$ were used by Hecke in **[6]** to study Euler products of L-functions corresponding to modular forms for $SL(2, \mathbf{Z})$ and similar groups. Such things also appear earlier in work of Hurwitz **[8]**. Maass studied the Hecke operators for the Siegel modular group $Sp(n, \mathbf{Z})$ in **[16]**. A good reference for the Hecke ring of a general group is Shimura's book [**18**, Ch. 3]. Here one finds an exposition of some work of Tamagawa (cf. **[21]**) connecting Hecke operators with combinatorial results about lattices as well as p-adic convolution operators and a p-adic version of some of Selberg's work in **[17]**. The Hecke operators for $Sp(n, \mathbf{Z})$ and $SL(n, \mathbf{Z})$ are also studied by Andrianov in **[1]** and **[2]**.

The Hecke operators (2.1) appear in many calculations involving the space $\mathscr{SP}_n$. For example set the function $f(Y) = 1$ for all Y in $\mathscr{SP}_n$. Then one sees from the formula for V_m that

$$\sum_{m\geq 1} T_m f(Y)\, m^{-s} = \sum_{m\geq 1} \Big(\sum_{A\in V_m} 1\Big) m^{-s} = \prod_{j=0}^{n-1} \zeta(s-j). \tag{2.2}$$

This function appears in Minkowski's formula for the volume of the fundamental domain for $\mathscr{SP}_n/GL(n, \mathbf{Z})$ (cf. [**22**, Ch. 4]). Solomon considers generalizations of such results in **[19]**. Such operators as the T_m are also very apparent in formulas connecting Eisenstein series like (1.15) defined as sums over $GL(n, \mathbf{Z})$ and higher dimensional Mellin transforms of the non-singular terms in a theta function. This will be seen in Proposition 1 of §3.

The basic facts about Hecke operators for $GL(n, \mathbf{Z})$ are contained in the following theorem.

THEOREM 1. 1. *The Hecke operator T_m maps automorphic forms $v: \mathscr{SP}_n \to \mathbf{C}$ for $GL(n, \mathbf{Z}) = \Gamma$ as in* (1.9) *to automorphic forms for $GL(n, \mathbf{Z})$ and preserves the eigenvalues of v under $SL(n, \mathbf{R})$-invariant differential operators on $\mathscr{SP}_n$.*

2) *T_m is hermitian with respect to the inner product*

$$(f, g) = \int_{W\in\mathscr{SP}_n/\Gamma} f(W)\,\overline{g(W)}\,dW.$$

3) *The ring of Hecke operators is commutative and thus has a set of simul-*

taneous eigenfunctions which span the space of all automorphic forms for $GL(n, \mathbf{Z})$.

4) *If* $(k, m) = 1$, *then* $T_k T_m = T_{km}$. *For* $GL(3, \mathbf{Z})$, *one has the following formula for the formal power series in the indeterminate* X: (*for prime* p)

$$\sum_{r \geq 0} T_{p^r} X^r = (I - T_p X + [(T_p)^2 - T_{p^2}]\, X^2 - p^3 X^3)^{-1}.$$

5) *Suppose that* $f \colon \mathcal{SP}_n \to \mathbf{C}$ *is an automorphic form for* $GL(n, \mathbf{Z})$ *such that* $T_m f = u_m f$, $u_m \in \mathbf{C} - 0$. *Then form the Dirichlet series* $L_f(S) = \sum_{m \geq 1} u_m m^{-s}$. *This series converges for* $\operatorname{Re} s > n/4$ *and can be analytically continued to a meromorphic function of* s *with functional equation*

$$\begin{aligned} \Lambda_{s,f}(Y) &= 2\pi^{-ns+n(n-1)/4} |Y|^s f(Y^0) L_f(2s) \prod_{j=1}^{n} \Gamma(s - a_j) \\ &= |Y|^{-n/2}\, \Lambda_{n/2-s, f^*}(Y^{-1}), \end{aligned}$$

where $f^*(W) = f(W^{-1})$ *and the* $a_j \in \mathbf{C}$ *are defined for* L^u *as in* (1.7) *by*

$$L^u(|X|^s f(X^0)) = (-1)^n \left\{ \prod_{j=1}^{n} (s - a_j) \right\} |X|^s f(X^0),$$

$X \in \mathcal{P}_m$. *For* $GL(3, \mathbf{Z})$, *it follows that* $L_f(s)$ *has the Euler product*

$$L_f(s) = \prod_{p \text{ prime}} (1 - u_p p^{-s} + ((u_p)^2 - u_{p^2}) p^{-2s} - p^{3-3s})^{-1}.$$

PROOF. 1) First note that the Hecke operators clearly commute with all the invariant differential operators on $\mathcal{SP}_n$. To see that $T_m f$ is invariant under $GL(n, \mathbf{Z})$, if the function f is invariant, one needs to know that

$$O_m = \bigcup_{A \in V_m} A\, GL(n, \mathbf{Z}) = \bigcup_{A \in V_m} GL(n, \mathbf{Z})\, A.$$

The inductive proof of the form of the set V_m makes this equality clear. Shimura gives another proof in **[18**, Ch. 3].

2) One need only imitate the proof of Petersson for $SL(2, \mathbf{Z})$ (cf. **[18]**) and Maass for $Sp(n, \mathbf{Z})$ in **[16]**. Setting $\Gamma = GL(n, \mathbf{Z})$, write

$$(T_m f, g) = \sum_{A \in O_m/\Gamma} \int_{\mathcal{SP}_n/\Gamma} f((W[A])^0)\overline{g(W)}\, dW.$$

If $A \in O_m$ and we set $g(W) = f((W[A])^0)$, then g is fixed by the congruence subgroup $\Gamma(m) = \{B \in \Gamma \mid B \equiv I (\text{mod } m)\}$. For $B \in \Gamma(m)$ and $A \in O_m$ imply that $A^{-1}BA \in \Gamma$. To see this note that $A^{-1} = (1/m)^t(\text{adj } A) \in (1/m)\mathbf{Z}^{n \times n}$. Thus $mA^{-1}BA \in \mathbf{Z}^{n \times n}$ and $mA^{-1}BA \equiv mA^{-1}A \equiv 0 \pmod m$.

Since the fundamental domain $\mathcal{SP}_n/\Gamma(m)$ consists of $[\Gamma : \Gamma(m)]$ copies of $\mathcal{SP}_n/\Gamma$, one sees that

$$\begin{aligned}(T_m f, g) &= \sum_{A \in V_m} [\Gamma : \Gamma(m)]^{-1} \int_{W \in \mathscr{S}\mathscr{P}_n/\Gamma(m)} f((W[A])^0)\overline{g(W)}\,dW \\ &= \sum_{A \in V_m} [\Gamma : A^{-1}\Gamma(m)A]^{-1} \int_{\mathscr{S}\mathscr{P}_n/A^{-1}\Gamma(m)A} f(x)\overline{g((X[A^{-1}])^0)}\,dX \\ &= (f, T_m g).\end{aligned}$$

The second equality is seen by making the substitution $X = W[A]$ and noting that $[\Gamma : \Gamma(m)] = [\Gamma : A^{-1}\Gamma(m)A]$.

3) This is proved in Shimura's book [**18**, p. 56] from the existence of the anti-automorphism of $GL(n)$ given by $X \to {}^tX$.

4) Here one can follow Shimura [**18**] or just multiply matrices, as follows. To see that $T_k T_m = T_{km}$, one need only note that the following equality holds:

$$\begin{pmatrix} d_1 & d_{12} & \cdots & d_{1n} \\ c & c_2 & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}\begin{pmatrix} c_1 & c_{12} & \cdots & c_{1n} \\ 0 & c_2 & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_n \end{pmatrix}$$

$$= \begin{pmatrix} d_1c_1 & d_1c_{12} + c_2 d_{12} & \cdots & d_1 c_{1n} + d_{12}c_{2n} + \cdots + d_{1n}c_n \\ 0 & d_2c_2 & \cdots & d_2c_{2n} \quad + \cdots + d_{2n}c_n \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_nc_n \end{pmatrix}.$$

For if d_{ij} runs through a complete set of representatives mod d_i and c_{ij} runs through a complete set of representatives mod c_i, then consider for $i < j$

$$d_i c_{ij} + d_{i,i+1}\, c_{i+1,j} + \cdots + d_{ij}c_j,$$

which is the i, j-th entry of the product. Inductively we can assume the terms $d_{ij'}$ with $j' < j$ and $c_{i'j}$ with $i' > i$ to be fixed. Thus what remains is $d_i c_{ij} + d_{ij}c_j + a$ fixed number. This gives a complete set of representatives modulo $d_i c_i$.

Next consider the proof of the formula which implies the Euler product for L-functions corresponding to eigenforms of Hecke operators for $GL(3, \mathbf{Z})$. The proof described below involves only matrix multiplication, but clearly becomes more complicated for $GL(n, \mathbf{Z})$, with $n > 3$. Thus Tamagawa's methods (cf. [**18**] and [**21**]) seem preferable. Those methods show that the p-part of the Hecke ring has n algebraically independent generators T_1^* coming from the double coset decomposition

$$GL(n, \mathbf{Z})A_i\, GL(n, \mathbf{Z}) = \bigcup_{B \in V_i} B\, GL(n, \mathbf{Z}), \text{ with } A_i = \begin{pmatrix} I_i & 0 \\ 0 & pI_{n-i} \end{pmatrix},$$

where I_k is the $K \times K$ identity matrix. Tamagawa proves that

$$\text{(a)} \qquad \sum_{r \geqq 0} T_{p^r} X^r = \left\{\sum_{j=0}^{n} (-1)^j p^{j(j-1)/2} T_j^*\right\}^{-1}.$$

However we need the formula involving T_{p^j} and not T_j^*. So let us sketch the proof for $GL(3, \mathbf{Z})$ that

$$\text{(b)} \qquad \sum_{r \geqq 0} T_{p^r} X^r = \{I - T_p X + [(T_p)^2 - T_{p^2}]\, X^2 - p^3 X^3\}^{-1}.$$

Note that b) agrees with a) for $n = 3$, provided that $(T_p)^2 - T_{p^2} = pT_2^*$. This formula does not appear to be totally obvious. The corresponding result for $Sp(3, \mathbf{Z})$ is due to Shimura and the generalization to $Sp(n, \mathbf{Z})$ appears to be open (cf. [2]).

Note first that $T_k T_m f(Y) = \sum_{A \in V_m} \sum_{B \in V_k} f(Y[BA]^0)$. It will also help to set up the following notation. Suppose that S is a subset of V_m and let $T(S)$ denote the operator $T(S) f(Y) = \sum_{A \in S} f(Y[A]^0)$.

Formula b) follows from the following two formulas, which are easily checked by multiplying the matrix representatives of the operators involved.

$$\text{(c)} \qquad T_p T_{p^r} = T_{p^{r+1}} + T(S_1^r) + T(S_2^r),$$

where

$$S_1^r = \left\{ \begin{pmatrix} p^e & p(a_1 \bmod p^e) & a_2 \bmod p^e \\ 0 & p^{f+1} & a_3 \bmod p^{f+1} \\ 0 & 0 & p^g \end{pmatrix} \middle| \; e \geqq 1;\, f, g \geqq 0;\, e + f + g = r \right\}$$

$$S_2^r = \left\{ \begin{pmatrix} p^e & a_1 \bmod p^e & p(a_2 \bmod p^e) \\ 0 & p^f & p(a_3 \bmod p^f) \\ 0 & 0 & p^{g+1} \end{pmatrix} \middle| \; e \geqq 1 \text{ or } f \geqq 1;\, g \geqq 0;\, e + f + g = r \right\}.$$

$$\text{(d)} \qquad \{(T_p)^2 - T_{p^2}\} T_{p^r} = p^3 T_{p^{r-1}} + T_{p^{r+1}} T_p - T_{p^{r+2}}, \text{ for } r \geqq 1.$$

To prove (c) look at the formulas

$$\begin{pmatrix} p^e & a_1 \bmod p^e & a_2 \bmod p^e \\ 0 & p^f & a_3 \bmod p^f \\ 0 & 0 & p^g \end{pmatrix} \begin{pmatrix} p & b_1 \bmod p & b_2 \bmod p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} p^{e+1} & c_1 \bmod p^{e+1} & c_2 \bmod p^{e+1} \\ 0 & p^f & c_3 \bmod p^f \\ 0 & 0 & p^g \end{pmatrix},$$

$$\begin{pmatrix} p^e & a_1 \bmod p^e & a_2 \bmod p^e \\ 0 & p^f & a_3 \bmod p^f \\ 0 & 0 & p^g \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & b_3 \bmod p \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} p^e & p(c_1 \bmod p^e) & c_2 \bmod p^e \\ 0 & p^{f+1} & c_3 \bmod p^{f+1} \\ 0 & 0 & p^g \end{pmatrix},$$

$$\begin{pmatrix} p^e & a_1 \bmod p^e & a_2 \bmod p^e \\ 0 & p^f & a_3 \bmod p^f \\ 0 & 0 & p^g \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} p^e & c_1 \bmod p^e & p(c_2 \bmod p^e) \\ 0 & p^f & p(c_3 \bmod p^f) \\ 0 & 0 & p^{g+1} \end{pmatrix}.$$

The first set of matrices gives $T_{p^{r+1}}$, except the $e + 1 = 0$ term. The second set of matrices gives the $e = 0$ term of $T_{p^{r+1}}$, but not the $f + 1 = 0$ term, and it also gives S_1^r. The third set gives the $e = f = 0$ term of $T_{p^{r+1}}$, as well as S_2^r.

To prove (d), use (c) with $r = 1$ to see that $(T_p)^2 - T_{p^2} = T(R_1) + T(R_2) + T(R_3)$, where

$$R_1 = \left\{\begin{pmatrix} p & p(b_1 \bmod p) & b_2 \bmod p \\ 0 & p & b_3 \bmod p \\ 0 & 0 & 1 \end{pmatrix}\right\},$$

$$R_2 = \left\{\begin{pmatrix} p & b_1 \bmod p & p(b_2 \bmod p) \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}\right\},$$

$$R_3 = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & p(b_3 \bmod p) \\ 0 & 0 & p \end{pmatrix}\right\}.$$

Then compute the matrix products to find that $T(R_j)T_{p^r} = T(Q_j)$, where

$$Q_1 = \left\{\begin{pmatrix} p^{e+1} & p(a_1 \bmod p^{e+1}) & a_2 \bmod p^{e+1} \\ 0 & p^{f+1} & a_3 \bmod p^{f+1} \\ 0 & 0 & p^g \end{pmatrix}\right\},$$

$$Q_2 = \left\{\begin{pmatrix} p^{e+1} & a_1 \bmod p^{e+1} & p(a_2 \bmod p^{e+1}) \\ 0 & p^f & p(a_3 \bmod p^f) \\ 0 & 0 & p^{g+1} \end{pmatrix}\right\},$$

$$Q_3 = \left\{\begin{pmatrix} p^e & p(a_1 \bmod p^e) & p(a_2 \bmod p^e) \\ 0 & p^{f+1} & p(a_3 \bmod p^{f+1}) \\ 0 & 0 & p^{g+1} \end{pmatrix}\right\}.$$

Now $T(Q_1)$ gives $T(S_1^{r+1})$ from (c). And $T(Q_2)$ gives the $e+1 \neq 0$ part of $T(S_2^{r+1})$ in (c). The $e = 0$ part of $T(Q_3)$ gives the remainder of $T(S_2^{r+1})$. The $e \geqq 1$ part of $T(Q_3)$ gives $p^3 T_{p^{r-1}}$, since

$$\begin{pmatrix} p^{e-1} & a_1 \bmod p^e & a_2 \bmod p^e \\ 0 & p^f & a_3 \bmod p^{f+1} \\ 0 & 0 & p^g \end{pmatrix}$$

$$= \begin{pmatrix} p^{e-1} & b_1 \bmod p^{e-1} + p^{e-1}(c_1 \bmod p) & b_2 \bmod p^{e-1} + p^{e-1}(c_2 \bmod p) \\ 0 & p^f & b_3 \bmod p^f + p^f(c_3 \bmod p) \\ 0 & 0 & p^g \end{pmatrix}$$

$$= \begin{pmatrix} p^{e-1} & b_1 \bmod p^{e-1} & b_2 \bmod p^{e-1} \\ 0 & p^f & b_3 \bmod p^f \\ 0 & 0 & p^g \end{pmatrix} \begin{pmatrix} 1 & c_1 \bmod p & c_2 \bmod p \\ 0 & 1 & c_3 \bmod p \\ 0 & 0 & 1 \end{pmatrix}.$$

One can also obtain an independent check of formula (b) by setting $v_0(Y) = 1$ for all Y in $\mathscr{S}\mathscr{P}_3$. Then we know that for $GL(3, \mathbf{Z})$

$$\sum_{m \geqq 1} T_m(v_0) m^{-s} = \prod_{j=0}^{2} \zeta(s-j) = \prod_{p \text{ prime}} (1-p^{-s})^{-1}(1-p^{1-s})^{-1}(1-p^{2-s})^{-1}$$
$$= \prod_{p \text{ prime}} \{1 - p^{-s}(1+p+p^2) + p^{-2s}(p+p^2+p^3) - p^{3-3s}\}.^{-1}$$

On the other hand formula (b) says that

$$\sum_{m \geqq 1} T_m(v_0) m^{-s} = \prod_p \{1 - T_p(v_0)p^{-s} + [(T_p)^2 v_0 - T_{p^2} v_0] p^{-2s} - p^{3-3s}\}^{-1}.$$

Now $T_m v_0 = \# V_m$ so that $T_p v_0 = p^2 + p + 1$ and $(T_p)^2 v_0 - T_{p^2} v_0 = p^3 + p^2 + p$. This completes the check.

5) In the case of $SL(2, \mathbf{Z})$, this is proved by relating $L_f(s)$ and the Mellin transform of f minus the constant term in its Fourier expansion. Here one cannot exactly do that, but almost. The argument involves the analytic continuation of higher dimensional Mellin transforms based on the methods of Maass and Selberg (cf. **[15]**). First one needs to know a specific Mellin transform evaluated by Maass in **[15**, §7] and giving an analogue of the gamma function in higher dimensions, which is attached to f: $\mathscr{S}\mathscr{P}_m \to \mathbf{C}$, an automorphic form for $GL(m, \mathbf{Z})$. The *gamma function* is

$$\int_{X \in \mathscr{P}_m} \exp[-Tr(XY^{-1})] f(X^0) |X|^{s-(m+1)/2} dX$$
$$= \pi^{m(m-1)/4} \left\{ \prod_{j=1}^{m} \Gamma(s - a_j) \right\} f(Y^0) |Y|^s, \tag{2.3}$$

where the a_j are as defined in 5).

It follows that, if one defines the *theta function* for $Y \in \mathscr{P}_n$, $X \in \mathscr{P}_m$ with

$1 \leqq m \leqq n$ by

$$\theta(Y, X) = \sum_{k=0}^{m} \theta_k(Y, X), \text{ with} \tag{2.4}$$
$$\theta_k(Y, X) = \sum_{\substack{A \in \mathbf{Z}^{n\times m} \\ rk\, A = k}} \exp[-\pi \mathrm{Tr}(Y[A]X)],$$

then one can use (2.3) to evaluate the following integral

$$\Lambda_{s,f}(Y) = \int_{\mathscr{P}_m/GL(m,\mathbf{Z})} f(X^0)\theta_m(Y, X)\,|X|^{s-(m+1)/2}\,dX. \tag{2.5}$$

The result is

$$\Lambda_{s,f}(Y) = 2\pi^{-ms+m(m-1)/4}\left\{\prod_{j=1}^{m} \Gamma(s - a_j)\right\} \sum_{\substack{A \in \mathbf{Z}^{n\times m}/GL(m,\mathbf{Z}) \\ rk\, A = m}} |Y[A]|^{-s} f((Y[A])^0). \tag{2.6}$$

Note the similarity with (1.15) except that then we mostly wanted to choose $m = n - 1$, whereas here one wants $m = n$.

Next suppose that $m = n$ and that f is an eigenform for all the Hecke opeators, i.e., $T_r f = u_r f$, for all $r \geqq 1$, with $u_r \in \mathbf{C} - 0$. Form the Dirichlet series $L_f(S) = \sum_{r\geqq 1} u_r r^{-s}$. It follows from (2.6) that

$$\Lambda_{s,f}(Y) = 2\pi^{-ns+n(n-1)/4}|Y|^{-s} f(Y^0)\, L_f(2s) \prod_{j=1}^{n} \Gamma(s - a_j). \tag{2.7}$$

Thus, if f is in $L^1(\mathscr{S}\mathscr{P}_n/GL(n, \mathbf{Z}))$, then $\Lambda_{s,f}(Y)$ and therefore $L_f(2s)$ converge for Re $s > n/2$, as was shown in §1. Riemann's method of analytic continuation of the Riemann zeta function using the theta function when $n = m = 1$ can be modified to yield the analytic continuation of $L_f(s)$ to all s in $\mathbf{C}$ as a meromorphic function with the functional equation stated in 5). This can be found in **[15]**, **[22]**, and **[24]**. The proof used a trick involving differential operators which was invented by Selberg. In the next section the possibility of doing without Selberg's trick will be investigated.

3. The analytic continuation of zeta functions attached to automorphic forms for GL(m, Z). Define the zeta function $Z_{s,\lambda}(Y)$ attached to the positive matrix Y in $\mathscr{P}_n$, the automorphic form v_λ for $GL(m, \mathbf{Z})$ from (1.9) and (1.14), and the complex variable s with Re $s > n/2$, for $1 \leqq m \leqq n$ by

$$Z_{s,\lambda}(Y) = \sum_{A \in \mathbf{Z}^{n\times m} rk\, m/GL(m,\mathbf{Z})} |Y[A]|^{-s}\, v_\lambda((Y[A])^0). \tag{3.1}$$

This is the Dirichlet series in formula (2.6). Such zeta functions have been analytically continued by Maass in [**15**, §**16**], using a method beginning with formula (2.5) and employing Selberg's trick. This trick uses differential operators to annihilate the integrals which arise from the singular terms of theta (the θ_k with $k < n$). In the case $m = n = 1$ the method gives a result of Riemann which evidently motivated Selberg. The idea is also discussed in [**22**] and [**24**]. The objects to be studied in the present section are precisely the terms in the analytic continuation of $Z_{s,\lambda}(Y)$ which Selberg's trick was designed to eliminate. It turns out that the constant terms in the Fourier expansions of the v_λ with respect to maximal parabolic subgroups $P(k, m - k)$ play a starring role in this drama.

Note that when $n = m$ and v_λ is an eigenfunction of all the Hecke operators for $GL(m, Z)$, then as at the end of the last section

$$Z_{s,\lambda}(Y) = v_\lambda(Y^0)|Y|^{-s}L_{v_\lambda}(2s),$$

where L_{v_λ} is the L-function defined in 5) of Theorem 1 in §2. Thus, in the more special case that $m = n$, $\lambda = 0$, $v_0(W) = 1$ for all W in $\mathcal{SP}_n$, one has

$$Z_{s,0}(Y) = |Y|^{-s}\prod_{j=1}^{n}\zeta(2s - j + 1),$$

(cf. (2.2)). The analytic continuation of this function in the form (3.1) has been of interest for the computation of the volume of the fundamental domain of $\mathcal{SP}_n/GL(n, \mathbf{Z})$, as well as in the theory of simple algebras, since it is the zeta function of the simple algebra of $n \times n$ matrices over the rationals.

In the case that $1 \leqq m \leqq n - 1$ and v_λ is an eigenfunction of all the Hecke operators for $GL(m, \mathbf{Z})$, the following proposition describes the relation between $Z_{s,\lambda}$ and the Eisenstein series $E_{s,\lambda}$ defined by (1.15). The formula involves the L-function L_{v_λ} defined in 5) of Theorem 1 in §2.

PROPOSITION 1. *Suppose that v_λ is an automorphic form for $GL(m, \mathbf{Z})$ which is an eigenfunction of all the Hecke operators, i.e., $T_r v_\lambda = u_\lambda(r)v_\lambda$, for some $u_\lambda(r) \in \mathbf{C} - 0$. Then*

$$Z_{s,\lambda}(Y) = L_{v_\lambda}(2s)E_{s,\lambda}(Y),$$

where

$$L_{v_\lambda}(s) = \sum_{r\geqq 1} u_\lambda(r)r^{-s}, \text{ for } \operatorname{Re} s > m/4.$$

PROOF. Use the matrix decomposition which says that summing over A in $\mathbf{Z}^{n\times m}$ rk $m/GL(m, \mathbf{Z})$ is the same as summing over $A = BC$, where $B \in \mathbf{Z}^{n\times m}$, $(B*) \in GL(n, \mathbf{Z})/P(m, n - m)$ and $C \in \mathbf{Z}^{m\times m}rk$ $m/GL(m, \mathbf{Z})$. Here $P(m, n - m)$ is defined in (1.10). The proof of this decomposition is an easy application of elementary divisor theory (cf. [**22**, Ch. 4]).

It follows that

$$\begin{aligned} Z_{s,\lambda}(Y) &= \sum_{B,C} |Y[BC]|^{-s} v_\lambda((Y[BC])^0) \\ &= \sum_{B} |Y[B]|^{-s} \sum_{r\geqq 1} r^{-2s} \sum_{\substack{|C|=r \\ C\in \mathbf{Z}^{m\times m}/GL(m,\mathbf{Z})}} v_\lambda(((Y[B]^0)[C]^0)) \\ &= L_{v_\lambda}(2s) E_{s,\lambda}(Y). \end{aligned}$$

The analytic continuation of $E_{s,\lambda}(Y)$ as a function of s starts with formula (2.5) of Maass, which writes

$$\Lambda_{s,f}(Y) = 2\pi^{-ms+m(m-1)1/4} \left\{ \prod_{j=1}^{m} \Gamma(s-a_j) \right\} Z_{s,\lambda}(Y) \tag{3.2}$$

as a Mellin transform of the non-singular part of a theta function. Then split the integral over $\mathscr{M}_m$ in (2.5) into two parts—that over $X\in\mathscr{M}_m$ with $|X| \geqq 1$ and that over $X\in\mathscr{M}_m$ with $|X| \leqq 1$. In the latter make the substitution $X \to X^{-1}$ to obtain

$$\begin{aligned} \Lambda_{s,\lambda}(Y) = &\int_{\substack{X\in\mathscr{M}_m \\ |X|\geqq 1}} \theta_m(Y,X)\, v_\lambda(X^{0^{-1}})\, |X|^{s-(m+1)/2}\, dx \\ &+ \int_{\substack{X\in\mathscr{M}_m \\ |X|\geqq 1}} \theta_m(Y, X^{-1})\, v_\lambda(X^0)\, |X|^{-s-(m+1)/2}\, dX. \end{aligned}$$

The transformation formula of the theta function, which is easily proved by the Poisson summation formula, says that

$$\sum_{k=0}^{m} \theta_k(Y, X^{-1}) = |Y|^{-m/2} |X|^{n/2} \sum_{k=0}^{m} \theta_k(Y^{-1}, X).$$

This implies, upon setting $v_{\lambda^*}(X) = v_\lambda(X^{-1})$,

$$\begin{aligned} \Lambda_{s,\lambda}(Y) = &\int_{\substack{X\in\mathscr{M}_m \\ |X|\geqq 1}} \theta_m(Y,X)\, v_\lambda(X^{0-1})\, |X|^{s-(m+1)/2}\, dX \\ &+ \int_{\substack{X\in\mathscr{M}_m \\ |X|\geqq 1}} \theta_m(Y^{-1}, X)\, |Y|^{-m/2}\, v_{\lambda^*}(X^{0^{-1}})\, |X|^{-s+(n-m-1)/2}\, dX \\ &+ \sum_{k=0}^{m-1} I_k(Y, s, \lambda), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} I_k(Y, s, \lambda) = &\int_{\substack{X\in\mathscr{M}_m \\ |X|\geqq 1}} \{ |Y|^{-m/2} |X|^{n/2-s}\, v_\lambda(X^0) \theta_k(Y^{-1}, X) \\ &- |X|^{s}\, v_{\lambda^*}(X^0) \theta_k(Y, X) \} |X|^{-(m+1)/2}\, dX. \end{aligned} \tag{3.4}$$

The terms I_k, for $1 \leqq k \leqq m-1$, are the trouble-makers. Koecher considered the case that v_0 is identically constant in [11] and obtained a formula for I_k by introducing a new variable. However, the formula which he obtains has only simple poles in s, while double poles occur in $\Lambda_{s,0}(Y)$, when $m = n$, since

$$\Lambda_{s,0}(Y) = 2|Y|^{-s}\pi^{-ms+m(m-1)/4}\left\{\prod_{j=0}^{m-1}\Gamma(s-j/2)\right\}\zeta(2s-j).$$

An explanation for the divergence of the $I_k(Y, s, 0)$ using integral formulas for $\mathscr{P}_n$ can be found in [23] and [24].

Recently Arakawa (cf. [3]) has obtained the analytic continuation of similar zeta functions attached to Siegel modular forms, but he does not seem to include the case that $m = n$, when the double poles arise. Arakawa uses Klingen's Eisenstein series for $\mathrm{Sp}(n, \mathbf{Z})$. This is related to the development that follows, since the ensuing analytic continuation of $Z_{s,0}$ when $n = 3$ and $m = 2$ involves Eisenstein series for $GL(3, \mathbf{Z})$ in the highest dimensional part of the spectrum. That is the analytic continuation of $Z_{s,0}$ is obtained from that of $Z_{s,\lambda}$, $\lambda = r(r-1)$ by analytic continuation to $r = 0$. For the purposes of harmonic analysis on $\mathscr{SP}_3/SL(3, \mathbf{Z})$, the case $n = 3$ and $m = 2$ suffices. Therefore we will leave the general result for future analysis. One expects that this kind of formula for the analytic continuation of Koecher's zeta function must come from that of the highest dimensional part of the spectrum $e_n(Y, s)$ by analytic continuation. In fact this has been proved by the author (cf. [22]). For the present argument, one must also have the explicit Fourier expansion of $e_{n-1}(Y, r)$ with respect to any maximal parabolic subgroup of $GL(n-1, \mathbf{Z})$. Thus the proper formulation of the induction argument should include the form of the constant term in the Fourier expansion of the Eisenstein series (results obtained in great generality by Langlands in [13]).

An adelic version of the analytic continuation of Eisenstein series which is related to that which follows can be found in the paper by Jacquet and Shalika in Inventiones Math. **38** (1976), 1–16. Another adelic method was found by Helen Strassberg in [20].

The term I_0 is no problem, since one has

$$(3.5)\quad I_0(Y, s, \lambda) = \begin{cases} 0, & \text{if } v_\lambda \text{ is orthogonal to the constants,} \\ \dfrac{|Y|^{-m/2}}{n/2 - s} - \dfrac{1}{s}, & \text{if } v_\lambda(W) = 1 \text{ for all } W \in \mathscr{SP}_m, \\ & \text{i.e., } \lambda = 0. \end{cases}$$

Recall that Epstein's zeta function with the complex variable in the critical strip must be orthogonal to the constants from Theorem 2 of [24].

To study I_k, for $0 < k < m$, one needs a decomposition of the rank k matrices $A \in \mathbf{Z}^{n\times m}$;

$$(3.6)\quad \{A \in \mathbf{Z}^{n\times m} \mid rk\,A = k\}$$
$$= \{B\,{}^tC \mid B \in \mathbf{Z}^{n\times k}\ rk\ k, (C*) \in GL(m, \mathbf{Z})/P(k, m-k)\}.$$

The proof of (3.6) is an exercise in elementary divisor theory.

It is also necessary to know the Jacobian of the following change of variables:

$$(3.7)\qquad W = \begin{pmatrix} u^{-1}T & 0 \\ 0 & uV \end{pmatrix}\begin{bmatrix} I & Q \\ 0 & 1 \end{bmatrix}, \text{ for } u > 0,\ T \in \mathcal{SP}_p,$$
$$V \in \mathcal{SP}_q,\ Q \in \mathbf{R}^{p\times q},\ p + q = n.$$

One finds using the normalization (1.5) and the Jacobian of the partial Iwasawa decomposition (cf. [**15**, pp. 149–150] and [**22**, Ch. 5, §1]) that

$$(3.8)\qquad dW = \frac{2pq}{n}\, u^{-pq-1}\, du\, dT\, dV.$$

It follows from (3.6)–(3.8) that

$$\frac{m}{2k(m-k)} I_k(Y, s, \lambda)$$
$$(3.9)\qquad = \int_{T,V,Q,u,t} t^{-s-1}\Big\{|Y|^{-m/2} t^{n/2} v_\lambda(W) \sum_B \exp[-\pi \mathrm{Tr}(Y^{-1}[B] t^{1/m} u^{-1} T)]$$
$$- v_{\lambda^*}(W) \sum_B \exp[-\pi \mathrm{Tr}(Y[B] t^{-1/m} u^{-1} T)]\Big\}$$
$$\cdot u^{-k(m-k)-1} du\, dt\, dT\, dV\, dQ.$$

Here the integration is over the set of $T \in \mathcal{SP}_k/GL(k, \mathbf{Z})$, $V \in \mathcal{SP}_{m-k}/GL(m-k, \mathbf{Z})$, $Q \in (\mathbf{R}/\mathbf{Z})^{k\times(m-k)}$, $u > 0$, $t \geqq 1$ and W is formed as in (3.7).

Suppose next that $v_\lambda(W)$ has the Fourier expansion

$$(3.10)\qquad v_\lambda(W) = \sum_{N \in \mathbf{Z}^{k\times(m-k)}} A_{N,\lambda}(u^{-1}T, uV) \exp[2\pi i \mathrm{Tr}({}^tNQ)],$$

if W is as in (3.7). Then the integral over Q in (3.9) kills all the terms in (3.10) except the $N = 0$ term—the so-called constant term of the Fourier expansion (even though it is no constant in general). Therefore

$$I_k(Y, s, \lambda) = \frac{2k(m-k)}{m}$$
$$(3.11)\qquad \int_{T,V,t,u} \Big\{A_{0,\lambda}(u^{-1}T, uV) t^{n/2-s} |Y|^{-m/2} \sum_B \exp[-\pi \mathrm{Tr}(Y^{-1}[B] t^{1/m} u^{-1} T)]$$
$$- A_{0,\lambda^*}(u^{-1}T, uV) t^{-s} \sum_B \exp[-\pi \mathrm{Tr}(Y[B] t^{-1/m} u^{-1} T)]\Big\}$$
$$\cdot u^{-k(m-k)-1} t^{-1} du\, dt\, dT\, dV,$$

where the integration is over the domain $T \in \mathcal{SP}_k/GL(k, \mathbf{Z})$, $V \in \mathcal{SP}_{m-k}/GL(m-k, \mathbf{Z})$, $t \geqq 1$, $u > 0$.

One cannot proceed further without a formula for $A_{0,\lambda}(u^{-1}T, uV)$. Such formulas exist for the general case (cf. [13] and [25]). However, let us restrict ourselves, for simplicity to the case of interest for harmonic analysis on $\mathcal{P}_3/GL(3, \mathbf{Z})$. Thus we will assume that $v_\lambda(X) = e_2(X, r)$. for $X \in \mathcal{SP}_2$, so that $\lambda = r(r-1)$. It is easy to show (cf. [**22**, Ch. 4 §5]) that in this case one has

$$A_{0,\lambda}(u^{-1}, u) = u^r + c(r)u^{1-r}, \tag{3.12}$$

with $c(r) = \Lambda(1-r)/\Lambda(r)$, $\Lambda(r) = 2\pi^{-r}\Gamma(r)\zeta(2r)$. One can handle the case $v_\lambda =$ constant (so that $\lambda = 0$), by taking residues at $r = 0$, as will be seen. And it is clear from (3.11) that $I_k = 0$ for all cusp forms v_λ. Thus $Z_{s,\lambda}$ is seen to be an entire function of s if v_λ is a cusp form. That takes care of all the possibilities for $m = 2$ and v_λ in a total orthonormal set of automorphic forms for $SL(2, \mathbf{Z})$ as in (1.14). One would expect to do similar things for general n, m using Langlands' formulas for the constant term in the Fourier expansion of Eisenstein series [**13**].

$$e_2(X, r) = e_2(X^{-1}, r), \text{ for all } X \text{ in } \mathcal{SP}_2. \tag{3.13}$$

To see this, one simply needs to write down the Dirichlet series defining $e_2(X, r)$. Thus $\lambda = \lambda^*$ in this case.

PROPOSITION 2. *In the special case that $n = 3$, $m = 2$, $v_\lambda(X) = e_2(X, r)$, with $\lambda = r(r-1)$, the analytic continuation of $Z_{s,\lambda}(Y)$ can be obtained from formula* (3.3) *with $I_0(Y, s, \lambda) = 0$ and*

$$I_1(Y, s, \lambda) = \frac{|Y|^{-1}\Lambda_3(Y^{-1}, 1-r)}{s-1-r/2} - \frac{\Lambda_3(Y, 1-r)}{s+(r-1)/2} + \frac{c(r)|Y|^{-1}\Lambda_3(Y^{-1}, r)}{s+(r-3)/2} - \frac{c(r)\Lambda_3(Y, r)}{s-r/2}, \tag{3.14}$$

where $\Lambda_3(Y, r)$ involves Epstein's zeta function and is defined by

$$\Lambda_3(Y, r) = \pi^{-r}\Gamma(r) \sum_{a \in \mathbf{Z}^3 - 0} Y[a]^{-r}, \tag{3.15}$$

if $Y \in \mathcal{P}_3$ and Re *$r > 3/2$. One may assume $\Lambda_3(Y, r)$ to be analytically continued to all complex numbers r as a meromorphic function with poles at $r = 0$ and $3/2$. This is the case that $m = 1$ in* (3.3)–(3.5). *Then $Z_{s,\lambda}(Y)$ satisfies the functional equation*

$$\Lambda_{s,\lambda}(Y) = |Y|^{-1}\Lambda_{3/2-s,\lambda^*}(Y^{-1}).$$

Taking residues at $r = 0$ will give the analytic continuation of Koecher's zeta function $Z_{s,0}(Y)$—the case $v_0 \equiv 1$.

PROOF. From (3.11) and (3.12) one has

$$I_1(Y, s, \lambda) = \int\limits_{\substack{u>0\\ t\geq 1}} (u^r + c(r)u^{1-r})t^{-s}\Big\{|Y|^{-1}t^{3/2} \sum_{b\in\mathbf{Z}^3-0} \exp(-\pi Y^{-1}[b]t^{1/2}u^{-1})$$

$$- \sum_{b\in\mathbf{Z}^3-0} \exp(-\pi Y[b]t^{-1/2}u^{-1})\, u^{-2}t^{-1}\Big\} du\, dt.$$

Break the integral over u into that over $(0, 1)$ and that over $(1, \infty)$. Replace u by u^{-1} in the first integral and use the transformation formula of the theta function in the second integral to obtain the analytic continuation of the integral to all values of r as well as formula (3.14). The functional equation follows most easily from (3.4) itself.

Next note that the formula for the analytic continuation of $\Lambda(r)$ and $\Lambda_3(Y, r)$, which is just the case $m = 1$ of (3.3)–(3.5) and is discussed in more detail in **[23**, p. 6], shows that

$$\lim_{r\to 0} \{r(r-1)\,\Lambda(r)\,I_1(Y, s, \lambda)\}$$
$$= \frac{|Y|^{-1}\Lambda_3(Y^{-1}, 1)}{s-1} - \frac{\Lambda_3(Y, 1)}{s-1/2} + \frac{\Lambda(1)|Y|^{-1}}{s-3/2} - \frac{\Lambda(1)}{s}.$$

This combined with (3.3) and (3.5) yields the analytic continuation of Koecher's zeta function $Z_{s,0}(Y)$ in the case $n = 3$ and $m = 2$. The final formula agrees with formula (3.16) in Koecher's paper **[11]**. One can also check that the poles and residues for $Z_{s,0}(Y)$, $Y \in \mathscr{P}_3$ are correct, since it can be shown that in the special case that $n = 3$ and $m = 2$, Koecher's zeta function is essentially Epstein's; more precisely that

$$\begin{aligned} Z_{s,0}(Y) &= \sum_{A\in\mathbf{Z}^{3\times 2}rk2/GL(2,\mathbf{Z})} |Y[A]|^{-s} \\ &= \frac{1}{2}|Y|^{-s}\zeta(2s-1) \sum_{b\in\mathbf{Z}^3-0} Y^{-1}[Ub]^{-s}, \end{aligned} \tag{3.16}$$

where U is the matrix

$$U = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

In order to generalize Proposition 2 to arbitrary m and n, one needs the Langlands formulas for the constant term in the Fourier expansion of Eisenstein series (cf. **[13]**). Note that one has to be very careful with arguments of the type just given. It appears likely that one must start with $v_\lambda(W) = e_{n-1}(W, r)$, $r \in \mathbf{C}^{n-1}$ to get the analytic continuation of $Z_{s,0}(Y)$, $Y \in \mathscr{P}_n$. Lower dimensional parts of the spectrum Σ_{n-1} will not work, as one can check by trying Epstein's zeta function instead.

We should note as a final remark that harmonic analysis on $\mathcal{SP}_n/SL(n, \mathbf{Z})$ will differ from that on $\mathcal{SP}_n/GL(n, \mathbf{Z})$. For example, when $n = 2$, Hejhal's tables [7] show that in fact there are odd cusp forms, which are thus necessary for harmonic analysis on $\mathcal{SP}_2/SL(2, \mathbf{Z})$, but not for harmonic analysis on $\mathcal{SP}_2/GL(2, \mathbf{Z})$. One might expect that harmonic analysis on $\mathcal{SP}_n/SL(n, \mathbf{Z})$ is essentially a "product" of that on $\mathcal{SP}_n/GL(n, \mathbf{Z})$ and that on $GL(n, \mathbf{Z})/SL(n, \mathbf{Z})$. A similar phenomenon is apparent for harmonic analysis for fundamental domains modulo congruence subgroups (cf. [14]).

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