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CYCLIC EXT

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1. Introduction. A classical theorem of Baer's [1] states that an abelian p-group G is determined by its endomorphism ring E. More is true; one can recover G as an E-module from the ring E. Let $G = B \oplus D$ with B reduced and D divisible. Richman and Walker [4] showed how to recover G as an E-module if B is unbounded or if $D = 0$. The case where B is bounded and $D \neq 0$ was handled by Kuebler and Reid [2], who first recover D and G/D as E-modules, then use an ingenious argument to recapture the E-module G. The exact sequence $0 \to D \to G \to G/D \to 0$ represents an element of $\mathrm{Ext}_E(G/D, D)$. Kuebler and Reid show that $\mathrm{Ext}_E(G/D, D)$ is cyclic as a module over the center of E and that if $0 \to D \to X \to G/D \to 0$ is any generator of this cyclic module, then X and G are isomorphic E-modules. Thus G is recovered by taking the middle term of any exact sequence that generates $\mathrm{Ext}_E(G/D, D)$.

Two aspects of this development are intriguing. First, since E determines G as an E-module, you should be able to construct the E-module G directly from E without resorting to homological machinery or going so far afield as Kuebler and Reid do. The theorem is basic and deserves an elementary, readily accessible proof. We provide this in §2. Second, Kuebler and Reid's proof of the startling phenomenon that $\mathrm{Ext}_E(G/D, D)$ is cyclic over the center of E is quite complicated, relying on the isomorphism of $\mathrm{Ext}_E(G/D, D)$ with a certain cohomology group, and on a characterization of the endomorphism ring of $\mathrm{Hom}_Z(G/D, D)$ as an E-bimodule. In §3 we show directly how to write a given element of $\mathrm{Ext}_E(G/D, D)$ as a multiple of the generator. Finally, in §4, we provide a concise development of Kuebler and Reid's approach, in a somewhat more general situation, and relate it to ours. In these endeavors, the question of when a direct sum of torsion-free groups of rank one is flat over its endomorphism ring arises. §5 settles that question.

2. Constructing G as an E-module from E. Let $G = B \oplus D$ be an abelian p-group, where B is reduced and D is divisible. Let E be the endomorphism ring of G. We want to reclaim the E-module G from the ring E. If B is unbounded, or if G is bounded, this is done in [3] in an elementary manner. So we assume that B is bounded and that $D \neq 0$. Let m be the smallest

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nonnegative integer such that $p^m B = 0$. Then E has a maximal idempotent π of additive order p^m (e.g., a projection of G onto B). So $G = \pi G \oplus (1 - \pi)G$, and from the requirement on π, it follows that πG is isomorphic to B and that $(1 - \pi)G = D$. Since B is bounded, it is a direct sum of cyclic groups, and hence has a summand of order p^m. Thus π contains a primitive idempotent e of order p^m. Since D is a divisible p-group, it is a direct sum of copies of $Z(p^\infty)$, and so $1 - \pi$ contains a primitive idempotent ρ of infinite additive order. Finally, let α be an element of E of order p^m such that $\alpha = \rho\alpha e$ (so α maps the cyclic summand given by e onto the p^m-socle of the $Z(p^\infty)$ given by ρ). Now it is easy to see that Ee is isomorphic to $G[p^m]$ and $D = \text{inj lim } E\rho/p^n E\rho$ as E-modules. The element α induces an E-map ϕ from $E\rho/p^m E\rho$ to Ee by defining $\phi(x\rho + p^m E\rho) = x\rho\alpha$. It is readily seen that ϕ simply identifies $E\rho/p^m E\rho$, which is isomorphic to $D[p^m]$, with the submodule of Ee that corresponds to $D[p^m]$. Hence we can construct G, as an E-module, by taking $Ee \oplus \text{inj lim } E\rho/p^n E\rho$ modulo the submodule $\{\phi(y) \oplus y : y \in E\rho/p^m E\rho\}$.

3. Cyclic Ext. In Theorem 1 below we give somewhat more general conditions than Kuebler and Reid for Ext to be cyclic.

THEOREM 1. *Let G be an R-module, $e \in R$ an idempotent, and $D = \{x \in G: ex = 0\}$. Suppose further that D is an R-submodule of G, the restriction map $E_R(D) \to \mathrm{Hom}_R(D \cap ReG, D)$ is onto, and ReG is flat. Then* $\mathrm{Ext}_R(G/D, D)$ *is a rank-one free $E_R(D \cap ReG)$-module, with generator $D \subset G \to G/D$.*

PROOF. Let $D \subset K \to G/D$ be an element of $\mathrm{Ext}_R(G/D, D)$ with π: $K \to G/D$. Note that D and ReG are fully invariant R-submodules of G. We shall construct a map ϕ: $G \to K$ such that $\pi\phi = \sigma$, the natural map from G to G/D. Define ϕ on ReG by $\phi(\sum f_i e x_i) = \sum f_i e \pi^{-1}\sigma x_i$. We must show that this is well defined. If $\sum f_i e e x_i = \sum f_i e x_i = 0$, then, from the flatness of ReG, there exist $y_j \in ReG$ such that $ex_i = \sum \lambda_{ij} y_j$ and $\sum f_i e \lambda_{ij} = 0$ [3; page]. Noting that $\sigma e = \sigma$, we get

$$\begin{aligned} \sum f_i e\pi^{-1}\sigma x_i &= \sum f_i e\pi^{-1}\sigma\lambda_{ij} y_j \\ = \sum f_i e\pi^{-1}\lambda_{ij}\sigma y_j &= \sum f_i e\lambda_{ij}\pi^{-1}\sigma y_j = 0. \end{aligned}$$

Now ϕ takes $D \cap ReG$ to D, which restriction can be extended to D, defining ϕ on all of G. Thus $D \subset G \to G/D$ is a generator of the $E_R(D)$-module $\mathrm{Ext}_R(G/D, D)$. If we show that the annihilator of this generator consists of those maps in $E(D)$ that are zero on $D \cap ReG$, we will be done. But $D \subset K \to G/D$ splits if and only if there is a map $G \to D$ that agrees with ϕ on D. Any such map must be zero on eG, hence on ReG. Thus ϕ is zero on $D \cap ReG$. Conversely if ϕ is zero on $D \cap ReG$, then the restriction of ϕ to D extends to a map $G \to D$ that takes eG to zero.

Suppose $G = B \oplus D$ is an abelian p-group with D divisible and B reduced. Let R be the endomorphism ring of G, and let e be the projection of G onto B with kernel D. Note that ReG is projective if B is bounded [**3**; Thm. 4] and $ReG = G$ is flat if B is unbounded [**3**; Thm. 2]. Then we have the setup of Theorem 1, and $E_R(D \cap ReG) = \operatorname{Hom}_R(D \cap ReG, D)$ is the ring of p-adic integers if B is unbounded and is the ring of the integers modulo p^k if p^k is the bound for B. Hence we have Kuebler and Reid's result on the cyclicity of $\operatorname{Ext}^1_R(G/D, D)$ [**2**; page 592].

4. Derivations. If A and B are R-modules, let $\operatorname{Ext}_{R;Z}(A, B)$ be the subgroup of $\operatorname{Ext}_R(A, B)$ consisting of those extensions which split as abelian groups. Let d: $R \to \operatorname{Hom}_Z(A, B)$ be a derivation. Then we can impose an R-module structure on $A \oplus B$ by setting $r(a, b) = (ra, rb + d(r)a)$. This gives a homomorphism from the group of derivations $\operatorname{Der}(R, \operatorname{Hom}_Z(A, B))$ onto $\operatorname{Ext}_{R;Z}(A, B)$ whose kernel is the group of inner derivations. Now $\operatorname{Hom}_Z(A, B)$ is an R-bimodule. Let Γ be the ring of biendomorphisms of $\operatorname{Hom}_Z(A, B)$. Then for $\gamma \in \Gamma$ and $d \in \operatorname{Der}(R, \operatorname{Hom}_Z(A, B))$, setting $(\gamma d)(r) = \gamma(d(r))$ makes $\operatorname{Der}(R, \operatorname{Hom}_Z(A, B))$ into a Γ-module, with the inner derivations forming a Γ-submodule. Thus $\operatorname{Ext}_{R;Z}(A, B)$ is a Γ-module. The thrust of Kuebler and Reid's paper [**2**] is that $\operatorname{Ext}_{R;Z}(G/B, B)$ is a cyclic Γ-module when $G = A \oplus B$ is a p-group, B is divisible, A is reduced, and R is the endomorphism ring of G. In this case $\operatorname{Ext}_{R;Z}(G/B, B) = \operatorname{Ext}_R(G/B, B)$. The following theorem generalizes this.

THEOREM 2. *Let A and B be left R-modules, and $\pi \in R$ such that $\pi A = 0$ and $\pi b = b$ for all b in B. If d is a derivation from R to* $\operatorname{Hom}_Z(A, B)$ *such that π is in the center of the ring of constants of d, then $d(R)$ is an R-bisubmodule of* $\operatorname{Hom}_Z(A, B)$ *and, if e is a derivation from R to* $\operatorname{Hom}_Z(A, B)$, *then there is an R-bimodule map ϕ from dR to* $\operatorname{Hom}_Z(A, B)$ *such that $\phi d - e$ is an inner derivation.*

PROOF. That $d(R)$ is an R-bisubmodule of $\operatorname{Hom}_Z(A, B)$ follows from the equation $xd(y) = d(x\pi y)$, and $d(y)x = d(y(1 - \pi)x)$. The map ϕ is defined by $\phi d(x) = e(x) - xe(\pi) + e(\pi)x$.

If d maps R onto all of $\operatorname{Hom}_Z(A, B)$, then Theorem 2 asserts that $\operatorname{Ext}_{R;Z}(G/B, B) = \operatorname{Ext}_R(G/B, B)$ is a cyclic Γ-module because ϕ is then in Γ.

Let A and B be abelian groups with $\operatorname{Hom}(B, A) = 0$ and let $R = E(A \oplus B)$ be the endomorphism ring of $A \oplus B$. Then B is an R-submodule of $A \oplus B$, and A is an R-module via its identification with the R-module $(A \oplus B)/B$. Then we have the exact sequence $\varepsilon: 0 \to B \to A \oplus B \to A \to 0$ of R-modules. Let π be the projection of $A \oplus B$ onto B with kernel A. Then the map d: $R \to \operatorname{Hom}_Z(A, B)$ given by $d(r)(a) = \pi r(a)$ is a derivation, and π is in the center of its ring of constants. Therefore, by Theorem 2,

$\mathrm{Ext}_R(A, B)$ is cyclic, and viewing $\mathrm{Ext}_R(A, B)$ as short exact sequences, ε is a generator since it is in the extension corresponding to the derivation d. In Kuebler and Reid's case [**2**; Prop. 2.1, p. 589], A is reduced and B is divisible, so $\mathrm{Hom}_Z(A, B) = 0$, whence $\mathrm{Ext}_R(A, B)$ is cyclic.

Both Theorems 1 and 2 yield Kuebler and Reid's result on the cyclicity of Ext. These two theorems are not directly comparable since they involve different rings in general. The following example shows that Theorem 2 yields a cyclic Ext when the flatness hypothesis of Theorem 1 is not satisfied.

EXAMPLE 3. Let A and B be rank-one torsion-free groups with $\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A)$, and $G = A \oplus B \oplus Q$. Let $L = \mathrm{Hom}_Z(A \oplus B, Q) \subset E(G)$. Let λ be an imbedding of A in Q and μ an imbedding of B in Q. Extend λ and μ to G by defining $\lambda(B \oplus Q) = 0$ and $\mu(A \oplus Q) = 0$. Then λ and μ form a basis for L over Q. Now Q is an $E(G)$-submodule of G, and identification of $(A \oplus B \oplus Q)/Q$ with $A \oplus B$ makes $A \oplus B$ an $E(G)$-module and thus $\mathrm{Hom}_Z(A \oplus B, Q)$ an $E(G)$-bimodule. Let Γ be the endomorphism ring of L as an $E(G)$-bimodule. Then $Q \times Q \subset \Gamma$ under the correspondence taking a pair $(p, q) \in Q \times Q$ to the map taking λ to $p\lambda$ and μ to $q\mu$. Moreover if $\gamma \in \Gamma$, then $\gamma(\lambda) = \gamma(\lambda\pi_A) = \gamma(\lambda)\pi_A$ so $\gamma(\lambda) = p\lambda$ for some p in Q. Similarly $\gamma(\mu) = q\mu$ for some q in Q, so $\Gamma = Q \times Q$. From Theorem 2 we get that $\mathrm{Ext}_R(A \oplus B, Q)$ is a free Γ-module. Theorem 1 does not apply because G is not flat over $E(G)$, as is easily verified, or follows from Theorem 4 below. In fact the conclusion of Theorem 1 does not hold as $\mathrm{Ext}_R(A \oplus B, Q)$ is rank 2, so is not a cyclic Q-module.

5. Direct sums of rank one torsion-free groups. Example 3 makes pertinent the question as to when a direct sum of torsion-free groups of rank one is flat over its endomorphism ring. The following theorem has also been proven by Dave Arnold in an unpublished paper.

THEOREM 4. *Let $\{M_i\}_{i\in I}$ be a family of rank-one torsion free groups. Then $G = \sum M_i$ is a flat module over its endomorphism ring if and only if whenever* $\mathrm{Hom}(M_i, M_k)$ *and* $\mathrm{Hom}(M_j, M_k)$ *are both nonzero, then there is m in I such that* $\mathrm{Hom}(M_m, M_i)$ *and* $\mathrm{Hom}(M_m, M_k)$ *are both nonzero.*

PROOF. We identify $\mathrm{Hom}(M_x, M_y)$ with the set of elements in $E(G)$ that takes M_x into M_y and kill the complement of M_x. Suppose G is a flat $E(G)$-module and $\lambda \in \mathrm{Hom}(M_i, M_k)$ and $\mu \in \mathrm{Hom}(M_j, M_k)$ are nonzero. Then there are nonzero elements $a \in M_i$ and $b \in M_j$ such that $\lambda a = \mu b$. Hence $a = \sum r_q a_q$, $b = \sum s_q b_q$, and $\lambda r_q = \mu s_q$. We may assume that r_q maps into M_i and s_q maps into M_j. Choose q so that $s_q b_q \neq 0$. Then $s_q M_m \neq 0$ for some m, so $\mathrm{Hom}(M_m, M_j) \neq 0$. Hence also $r_q M_m \neq 0$, so $\mathrm{Hom}(M_m, M_i) \neq 0$.

Conversely, if $a \in M_x$, then the cyclic $E(G)$-submodule generated by a is projective, as the annihilator of a is the annihilator of M_x, which is a summand of $E(G)$. If the condition of the theorem holds, then G is a direct sum of direct limits of such cyclic submodules, hence is flat.

The simplest completely decomposable G which is not flat as an $E(G)$-module is $G = Q \oplus M \oplus N$ with M and N reduced and $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M) = 0$. Here $\mathrm{Ext}(G/Q, Q)$ is rank-2 over our ring because $G/Q = M \oplus N$ as an $E(G)$-module. However it is rank-one free over the ring Γ of $E(G)$-bimodule endomorphisms of $\mathrm{Hom}(G/(Q, Q)$. Such rings Γ are identified in the next theorem, but whether the relevant Ext is Γ-free or not we do not know.

THEOREM 5. *Let $G = \sum_I A_i$ be a finite direct sum of rank-one torsion-free groups, and $D = \sum_J A_i$ a fully invariant subgroup of G. Let $\sim$ denote the equivalence relation on $L = \{(i, j): i \in I \backslash J \text{ and } j \in J\}$ generated by declaring (i, j) and (u, v) equivalent if* $\mathrm{Hom}(A_u, A_i)$ *and* $\mathrm{Hom}(A_j, A_v)$ *are both nonzero. If τ is an equivalence class of L, set*

$$R_\tau = \bigcap \{E(A_j): (i, j) \in \tau\}.$$

Then the ring Γ of all $E(G)$-bimodule endomorphisms of $\mathrm{Hom}(G/D, D)$ *is isomorphic to* ΠR_τ.

PROOF. If $(i, j) \in L$, then $\mathrm{Hom}(A_i, A_j)$ is a subgroup of $\mathrm{Hom}(G/D, D)$ which is invariant under Γ. The endomorphism ring of a nontrivial $\mathrm{Hom}(A_i, A_j)$ is $E(A_j)$. Suppose $(i, j) \in L$ and $\mathrm{Hom}(A_u, A_i)$, $\mathrm{Hom}(A_i, A_j)$, and $\mathrm{Hom}(A_j, A_v)$ are all nonzero. Then any bimodule endomorphism of $\mathrm{Hom}(G/D, D)$ induces the same map on $\mathrm{Hom}(A_u, A_v)$ as on $\mathrm{Hom}(A_i, A_j)$, hence yields an element of ΠR_τ. This clearly gives a ring homomorphism from Γ to ΠR_τ. The kernel of this homomorphism is zero because the $\mathrm{Hom}(A_i, H_j)$ generate $\mathrm{Hom}(G/D, D)$. On the other hand, given an element of ΠR_τ, we get endomorphisms of $\mathrm{Hom}(A_i, A_j)$ for each $(i, j) \in L$. It is readily seen that these fit together to give an $E(G)$-bimodule endomorphism of $\mathrm{Hom}(G/D, D)$.

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