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CHOICE SETS AND MEASURABLE SETS

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Call two real numbers *equivalent* if their difference is rational. Call $S \subset R$ a *choice set* if S is a set of representatives of the equivalence classes of R. J. A. Andrews [1] observed that the set $\{\lambda S: S \in \mathscr{F}\}$ is dense in the unit interval [0, 1] where λ denotes Lebesgue outer measure and $\mathscr{F}$ denotes the family of all choice sets $\subset [0, 1]$. In this note we prove that in fact $\{\lambda S: S \in \mathscr{F}\} = (0, 1]$. More generally we prove the following theorem.

THEOREM 1. *There exists a set* $E \subset R$ *such that*

(i) $\lambda(E \cap A) = \lambda(A)$ *where* A *is any Lebesgue measurable set, and*

(ii) $E \cap (r + E) = \emptyset$ *where* r *is any nonzero rational number. Moreover, if* I *is any interval in* R, *and* S *is any extension of the set* $E \cap I$ *to a choice set* $S \subset I$, *then* $\lambda S = \lambda I$.

PROOF. Let Q be the field of rational numbers. Say that $x, y \in R \backslash Q$ are Q-equivalent if $y \in Qx + Q$. This divides $R\backslash Q$ into Q-equivalence classes. Let $W \subset (0, 1)$ be a set of representatives of the Q-equivalence classes. Now $R\backslash Q \subset \bigcup_{a,b \in Q}(aW + b)$ and $\lambda(aW + b) = a\lambda W$. It follows that $0 < \lambda W \leqq 1$.

We use the Vitali covering theorem to a.e. cover W with countably many pairwise disjoint closed intervals I_j with rational endpoints such that $\lambda I_j < 2^{-1}\lambda W$ for each j and $\sum_j \lambda(I_j) < (1 + 2^{-1})\,\lambda W$. For some index j, $\lambda(I_j) < (1 + 2^{-1})\lambda(I_j \cap W)$. Let K_1 be this I_j. Then

$$\lambda(W\backslash K_1) \geqq \lambda W - \lambda K_1 \geqq \lambda W - 2^{-1}\,\lambda W > 0.$$

We use the Vitali covering theorem to a.e. cover $W\backslash K_1$ with countably many pairwise disjoint closed intervals J_j with rational endpoints, and disjoint from K_1, such that $\lambda J_j < 2^{-1}(\lambda(W\backslash K_1))$ for each j and $\sum_j \lambda(J_j) < (1 + 2^{-2})\lambda(W\backslash K_1)$. For some index j, $\lambda(J_j) < (1 + 2^{-2})\lambda(J_j \cap W)$. Let K_2 be this interval J_j. Then

$$\lambda(W\backslash K_1\backslash K_2) \geqq \lambda(W\backslash K_1) - \lambda K_2 > 2^{-1}\lambda(W\backslash K_1) > 0.$$

We use the Vitali covering theorem to a.e. cover $W\backslash K_1\backslash K_2$ with countably many pairwise disjoin closed intervals L_j with rational endpoints, and disjoint from $K_1 \cup K_2$, such that $\lambda L_j < 2^{-1}\lambda(W\backslash K_1\backslash K_2)$ for each j and

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$\sum_j \lambda L_j < (1 + 2^{-3})\lambda(W \backslash K_1 \backslash K_2)$. Then for some index j, $\lambda L_j < (1 + 2^{-3})\lambda(L_j \cap W)$. Let K_3 be this interval L_j. Also

$$\lambda(W \backslash K_1 \backslash K_2 \backslash K_3) \geqq \lambda(W \backslash K_1 \backslash K_2) - \lambda K_3 > 2^{-1}\lambda(W \backslash K_1 \backslash K_2) > 0.$$

We continue by induction on n to produce a sequence of pairwise disjoint closed intervals (K_n) with rational endpoints such that $\lambda K_n < (1 + 2^{-n}) \lambda(K_n \cap W)$ for each n.

Now let (I_n) be a sequence of closed intervals with rational endpoints such that if I is any closed interval with rational endpoints, $I = I_n$ for infinitely many indices n. Then for each n there is a unique increasing surjective linear function f_n: $K_n \to I_n$ of the form $f_n(x) = a_n x + b_n$ (a_n, $b_n \in Q$, $a_n \neq 0$). Let $E = \bigcup_{n=1}^{\infty} f_n(K_n \cap W)$.

Suppose $f_n(x), f_m(y) \in E$ where $x \in K_n \cap W$, $y \in K_m \cap W$, and $f_n(x) \neq f_m(y)$; then clearly $x \neq y$. If $n = m$, then $f_n(x) - f_m(y) = a_n(x - y) \notin Q$ since $x, y \in W$. If $n \neq m$, then $f_n(x) - f_m(y) = a_n x - a_m y + b_n - b_m \notin Q$ since $x, y \in W$. In either case, $f_n(x) - f_m(y) \notin Q$. Thus E satisfies (ii).

Now let I be any closed interval with rational endpoints. Say $I = I_n$. Then

$$\lambda(E \cap I)/\lambda I \geqq \lambda f_n(K_n \cap W)/\lambda I_n = \lambda(K_n \cap W)/\lambda K_n > (1 + 2^{-n})^{-1}.$$

Since $I = I_n$ for infinitely many indices n, we have $\lambda(E \cap I) = \lambda I$. It follows that if J is any open interval, $\lambda(E \cap J) = \lambda J$. (Just express J as the union of an expanding sequence of closed intervals with rational endpoints.) So if U is any open set, $\lambda(E \cap U) = \lambda U$.

Finally, let $A \subset R$ be any Lebesgue measurable set with $\lambda A < \infty$. There is an open set $U \supset A$ such that $\lambda(U \backslash A) < 1$. Then $\lambda(E \cap U) = \lambda U$. Since A is measurable, we obtain

$$\lambda(E \cap U) = \lambda(E \cap A) + \lambda(E \cap (U \backslash A)) = \lambda U = \lambda A + \lambda(U \backslash A) < \infty.$$

It follows that $\lambda(E \cap (U \backslash A)) = \lambda(U \backslash A)$ and $\lambda(E \cap A) = \lambda A$. Thus E satisfies (i).

If I is any interval in R, we extend the set $E \cap I$ to a choice set $S \subset I$, and then $\lambda I = \lambda(E \cap I) = \lambda S$. This completes the proof.

E. Hewitt and K. Stromberg, J. Australia Math. Society **18** (1974), 236–238, presented a set that can be shown to satisfy (i) but not (ii). H. W. Pu also presented a set satisfying (i) in the same journal, **13** (1972), 267–270.

It is easy to see that no two elements in E are Q-equivalent. The field Q can be replaced by a larger subfield F of R provided only that $\lambda W > 0$. The problem can be generalized to R^n, where $E \cap (r + E) = \emptyset$ for any nonzero vector $r \in R^n$, all of whose coordinates are rational. The proof, however, is more awkward.

Reference

1. J. A. Andrews, *Problem* E 2710, American Mathematical Monthly **85** (1978), 276.

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