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GENERALIZED CONTINUOUS AND HYPERCONTINUOUS LATTICES

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A class of complete lattices which have recently received a considerable deal of attention is the class of continuous lattices introduced by D. Scott [13] (see also [3]). One of the interesting features of this class of lattices is the fact that these lattices admit a unique compact Hausdorff topology for which the meet operation is continuous (i.e., they admit the structure of a compact topological semilattice). This topology turns out to be an "intrinsic" topology, i.e., one that can be defined directly from the lattice structure. We refer to this topology as the CL-topology.

A major goal of this paper is to give a more detailed examination of this CL-topology. For any complete lattice this topology is always compact and T_1. We characterize those complete lattices for which it is Hausdorff; because these lattices have many characteristics reminiscent of continuous lattices, we call them generalized continuous lattices. They seem to be an interesting class of lattices in their own right; hence we develop some of their fundamental properties.

One of the oldest of the intrinsic topologies is Frink's interval topology. We address ourselves to the question of for what continuous lattices do the CL-topology and the interval topology coincide. This turns out to be precisely the class of lattices which we call "hypercontinuous". We turn our attention to these and point out some surprising connections between these lattices and generalized continuous lattices.

0. Preliminaries.

In this preliminary section we collect some well-known notations, definitions and results needed later on.

DEFINITION 0.1. If L is any lattice and if $A \subseteqq L$ is a subset of L, then A is called on *upper set* provided that for $a, b \in L$, $a \leqq b$ and $a \in A$ implies $b \in A$. If A is any subset of L, then we denote by $\uparrow A$ the smallest upper set of L which contains A, i.e., $\uparrow A = \{x$: there is an $a \in A$ with $a \leqq x\}$. *Lower sets* and $\uparrow A$ are defined dually. An upper set (lower set), which is at the same time a sublattice of L is called a *filter* (*ideal*). A subset $I \subseteqq L$ is called *down-directed* (*up-directed*), if for each pair of elements a, $b \in I$

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there is a $c \in I$ such that $a \wedge b \geqq c$ ($a \vee b \leqq c$). Note that I is down-directed (up-directed) if and only if $\uparrow I$ is a filter ($\downarrow I$ is an ideal) of L.

DEFINITION 0.2. Let L be a complete lattice and let $a, b \in L$ be two elements. Then we say that a is *way below* b (and write $a \ll b$), provided that every up-directed subset I of L with a supremum greater than or equal to b gets eventually above a, i.e., $b \leqq \sup I$ implies the existence of a $d \in I$ such that $a \leqq d$. An element which is way below itself is called *compact*. A complete lattice L is called a *continuous lattice*, if every element $a \in L$ is the supremum of all elements way below a. An *algebraic lattice* is a complete lattice in which every element is the supremum of compact elements. Note that every algebraic lattice is in fact continuous.

The way below relation on every continuous lattice has the following interesting interpolation property. If $a, b \in L$ are two elements of a continuous lattice L with $a \ll b$, then there is an element $c \in L$ such that $a \ll c$ and $c \ll b$.

DEFINITION 0.3. Among the several possibilities of defining a topology on a complete lattice L, we introduce in this preliminary section only the following one. Let $U \subseteqq L$ be a subset. Then U is called *Scott open* if (i) U is an upper set and (ii) if D is an up-directed subset of L and if $\sup D \in U$, then there is $d \in D$ with $d \in U$. It is easy to see that the Scott open subsets of L form a topology, which we shall call the *Scott topoplogy*.

In terms of the Scott topology we can give the following characterization of continuous lattices. A complete lattice L is a continuous lattice if and only if for each $a \in L$ we have $a = \sup \{\inf U: a \in U, U \text{ is Scott open}\}$. In fact this characterization was the original definition given by D. Scott.

EXAMPLE 0.4. An important example of continuous lattices is the following. Let X be a locally compact topological space and let $O(X)$ denote the (complete) lattice of open subsets of X. Then $O(X)$ is a continuous lattice and we have $U \ll V$ if and only if $\bar{U}$ is compact and $\bar{U} \subseteqq V$. Moreover, for Hausdorff spaces X these are the only examples of this type. If we consider spaces X which are no longer Hausdorff, the question whether or not $O(X)$ is a continuous lattice is more complicated. All we can say in full generality is the following. If X is any topological space, then $O(X)$ is a continuous lattice if and only if for every point $x \in X$ and every neighborhood U of x there is a neighborhood V of x which is relatively compact in U. For a proof of this simple fact and more details concerning $O(X)$ we refer to [7].

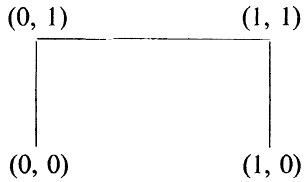
1. **Generalized continuous lattices.** We give here a generalization of continuous lattices which we stumbled upon in another context (to which we turn later on). Besides having application to later situations, these

objects appear worthy of study in their own right. Their theory bears rather striking resemblances to that of continuous lattices and gives a new perspective to earlier work.

DEFINITION 1.1. Let L be a complete lattice. If $F \subseteq L$ is a finite subset and if $x \in L$ is an element, we write $F \ll x$ if for every up-directed set D, $\sup D \geqq x$ implies $y \leqq d$ for some $d \in D$, $y \in F$. In this case we say F *guards* x *(from below)*. The idea here is that x cannot be penetrated without overrunning some member of F.

The following is our original example.

EXAMPLE 1.2. Consider the following subset of the unit square ordered by coordinatewise ordering, $(x_1, y_1) \leqq (x_2, y_2)$ if $x_1 \leqq x_2$ and $y_1 \leqq y_2$.



Thus L is an upside down U; L is a complete lattice with respect to the induced order although the meet operation is somewhat peculiar. Now let $F = \{(1, 1/2), (1/2, 1)\}$. Then $F \ll 1$ $(= (1, 1))$ and as a matter of fact such sets (but not singletons) can be picked arbitrarily close to 1. Note that sets of the form $(1/2 - \varepsilon, 1)$, $(1, 1 - \varepsilon)$ guard the point $(1/2, 1)$ and that the second points are necessary to prevent rear attacks.

We turn now to the definition of a generalized continuous lattice. The idea is that each point can be guarded from finitely many locations "arbitrarily near" to x. In continuous lattices this is always possible from even one position. One has simply to pick an element way below x, and these elements can be picked arbitrarily close to x, i.e., x is the supremum of all those elements.

DEFINITION 1.3. Let L be a complete lattice. Then L is a *generalized continuous lattice* (henceforth denoted GCL) if for all $x, y \in L$ such that $x \nleqq y$, there exists a finite set F such that $F \ll x$ and $\downarrow y \cap F = \emptyset$.

This definition is a smoother version of our first one which we include in the following proposition (whose proof is quite straightforward and hence omitted).

PROPOSITION 1.4. *Let L be a complete lattice. For each $x \in L$, let $\mathscr{F}_x = \{F \subseteq L : |F| < \infty$ and $F \ll x\}$. Then the following conditions are equivalent:*

(1) *L is a* GCL.

(2) *For each* $x \in L$ *and for each choice function* $\alpha \in \Pi_{F \in \mathscr{F}_x} F$, *we have* $x \leqq \sup\{\alpha(F): F \in \mathscr{F}_x\}$.

One of the key properties of continuous lattices which makes everything work nicely is the interpolation property for $\ll$. We need also an interpolation property for GCL's, but it is more elusive in this setting. Hence we need to introduce and develop additional machinery. Let X be a T_0 topological space. Then X has a partial ordering induced by the topology by defining $x \leqq y$ if and only if $y \in \mathrm{cl}\{x\}$. Conversely, let $(X, \leqq)$ be a poset. Then this ordering induces a topology on X by defining all sets of the form $\uparrow x$ to be a subbase for the closed sets. The ordering induced by this topology is precisely the original ordering. This topology is not new, but it seems to have no standard name. In [7] it is called the INF-topology while in [**11**] it is called the closure of points (COP) topology. Since the open sets are lower sets, we refer to this topology as the *lower topology*. The *upper topology* is defined dually.

Let X be a topological space, $A \subseteqq B \subseteqq X$. We say that A is precompact in B if every open cover of B contains a finite subcover of A. The following is a mild generalization of Alexander's Lemma.

PROPOSITION 1.5. *Let* $\mathscr{S}$ *be a subbasis for the topology on* X. *If* A *is precompact in* B *for the subbasis* $\mathscr{S}$, *then* A *is precompact in* B.

PROOF. Suppose A is not precompact in B. Then there exists an ultrafilter $\mathscr{F}$ with $A \in \mathscr{F}$ such that $\mathscr{F}$ does not cluster (equivalently, converge) to any point in B. Hence for each point $x \in B$, there exists a basic open set U_x sich that $x \in U_x \notin \mathscr{F}$. Since U_x is basic, there exist subbasic open sets $S_1, \ldots, S_n \in \mathscr{S}$ such that $U_x = S_1 \cap \cdots \cap S_n$. If each $S_i \in \mathscr{F}$, then $U_x \in \mathscr{F}$. Hence there is an $S_x \in \mathscr{S}$ such that $x \in S_x$, $S_x \notin \mathscr{F}$. Since A is precompact in B with respect to $\mathscr{S}$, there exist $x_1, \ldots, x_k \in B$ such that $A \subseteqq X_{x_1} \cup \cdots \cup S_{x_k}$, but $S_{x_i} \notin \mathscr{F}$ for all $1 \leqq i \leqq k$. Thus $X \backslash S_{x_i} \in \mathscr{F}$ for all i, and hence $\varnothing = A \cap (X \backslash S_{x_1}) \cap \ldots \cap (X \backslash S_{x_k}) \in \mathscr{F}$, a contradiction.

We now present a mild generalization of definition (1.1).

DEFINITION 1.6. Let L be a complete lattice, let $F, G \subseteqq L$. Then we write $F \ll G$ is $\sup D \geqq g$ for some directe set D and some element $g \in G$, then $y \leqq d$ for some $y \in F$, $d \in D$. In this case F is said to *guard* G.

PROPOSITION 1.7. *Let* L *be a complete lattice*, $F, G \subseteqq L$. *Then* $F \ll G$ *if and only if* $L \backslash \uparrow F$ *is precompact in* $L \backslash \uparrow G$ *for the lower topology on* L.

PROOF. Suppose $L \backslash \uparrow F$ is precompact in $L \backslash \uparrow G$. Let D be a directed set such that $\sup D \geqq g$ for some $g \in G$. Then $L \backslash \uparrow G \subseteqq \bigcup\{L \backslash \uparrow d \colon d \in D\}$, since $\uparrow G \supseteqq \uparrow g \supseteqq \bigcap\{\uparrow d \colon d \in D\}$. Since each $L \backslash \uparrow d$ is open in the lower topology, there exist $d_1, \ldots, d_n \in D$ such that $L\backslash\uparrow F \subseteqq L\backslash\uparrow d_1 \cup \cdots \cup L\backslash\uparrow d_n$. Let $d \in D$ such that $d_1, \ldots, d_n \leqq d$. Then $L\backslash\uparrow F \subseteqq L\backslash\uparrow d$, i.e., $\uparrow d \subseteqq \uparrow F$ and hence $d \geqq f$ for some $f \in F$. Conversely suppose $F \ll G$. Let $\{L\backslash\uparrow x\}_{x\in A}$ be a collection of subbasic open sets which covers $L\backslash\uparrow G$. Let D be the up-directed set of all suprema of finite subsets of A. If $B \subseteqq A$ is a finite subset, then $L\backslash\uparrow(\sup B) \supseteqq \bigcup_{b\in B} L\backslash\uparrow b$. Hence $\{L\backslash\uparrow d\colon d \in D\}$ is a cover of $L\backslash\uparrow G$. Thus $\sup D \in \uparrow G$. Since $F \ll G$, there exist $y \in F$ and a finite set $B \subseteqq A$ such that $y \leqq \sup B$. Now we can conclude that $L\backslash\uparrow F \subseteqq \uparrow y \subseteqq L\backslash\uparrow(\sup B) = \bigcup_{b\in B} L\backslash\uparrow b$. Hence finitely many of the collection $\{L\backslash\uparrow x\colon x \in A\}$ cover $L\backslash F$. Thus by Proposition (1.5) $L\backslash\uparrow F$ is precompact in $L\backslash\uparrow G$.

THEOREM 1.8. *Let L be a generalized continuous lattice.*

(i) *If $F \ll A$ and $G \ll A$, then $F \vee G \ll A$ (where $F \vee G = \{x \vee y\colon x \in F, y \in G\}$).*

(ii) *If A is closed in the lower topology and if $B \ll A$, then there exists a finite set F such that $A \subseteqq \uparrow F \subseteqq \uparrow B$ and furthermore $B \ll F \ll A$. In particular, if $x \in L$, and if G is a finite set such that $G \ll x$, then there exists a finite set F such that $G \ll F \ll x$ (the interpolation property).*

PROOF. Let $F, G \ll A$ and let D be a directed set such that $a \leqq \sup D$ for some $a \in A$. Then there exist $x \in F$ and $y \in G$ and $d_1, d_2 \in D$ such that $x \leqq d_1$ and $y \leqq d_2$. Pick a $d \in D$ such that d_1 and d_2 are less than or equal to d. Then we have $x \vee y \leqq d$. This proves (i).

(ii) Suppose that A is closed in the lower topology, and that $B \ll A$. Let $\mathscr{F} = \{\uparrow F\colon F$ is finite and there is a finite set G such that $F \ll G \ll A\}$. We first show $A = \bigcap \mathscr{F}$. Let $y \in L\backslash A$. Since A is closed in the lower topology, A is the intersection of sets of the form $\uparrow G$ where G is finite. Hence there exists some set G which is finite such that $y \notin G$ and $A \subseteqq \uparrow G$. Since L is a GCL, for each $g \in G$ there is a finite set F_g such that $F_g \ll g$ and $y \notin \uparrow F_g$. Let $F = \bigcup\{F_g\colon g \in G\}$. Then $F \ll G$ and $y \notin \uparrow F$. Repeat this process with F to obtain an $F_1 \ll F$ such that $y \notin F_1$. Then $F_1 \ll F \ll A$, and hence $F_1 \in \mathscr{F}$. Thus $y \notin \bigcap \mathscr{F}$. Since y was arbitrary, we have $A = \bigcap \mathscr{F}$. Since each member of $\mathscr{F}$ is closed in the lower topology and $A = \bigcap \mathscr{F}$, $\{L\backslash\uparrow F\colon \uparrow F \in \mathscr{F}\}$ is an open cover of $L\backslash A$. By proposition (1.7) $L\backslash\uparrow B$ is precompact in $L\backslash A$. Hence we can find finite sets $F_1, \ldots, F_n$ such that each $\uparrow F_i$ is a member of $\mathscr{F}$ and such that $L\backslash\uparrow B \subseteqq L\backslash\uparrow F_1 \cup \cdots \cup L\backslash\uparrow F_n$. For each i, there exists G_i finite such that $F_i \ll G_i \ll A$. Let $F = F_1 \vee \cdots \vee F_n$ and $G = G_1 \vee \cdots \vee G_n$. By part (i) we have $G \ll A$ and $F \ll G$. Since $\uparrow F = \uparrow F_1 \cap \cdots \cap \uparrow F_n$, we can conclude that $L\backslash\uparrow B \subseteqq L\backslash\uparrow F$, i.e., $\uparrow F \subseteqq \uparrow B$. For the final part let $A = \uparrow x$.

COROLLARY 1.9. *Let L be a complete lattice. The following statements are equivalent:*

(1) *L is a* GCL.

(2) *The lattice $O_1(L)$ of sets open in the lower topology is a continuous lattice* (*with respect to the operation of intersection*).

PROOF. (1) $\Rightarrow$ (2). Let U be open in the lower topology, and let $x \in U$. Then $L \backslash U$ is closed and $x \notin L \backslash U$. Hence as in the proof of theorem (1.8), we can find a finite set F such that $L \backslash U \subseteqq \uparrow F$ and $F \ll L \backslash U$ (equivalently $\uparrow F \ll L \backslash U$) and $x \notin \uparrow F$. By proposition (1.7) we have $x \in V = L \backslash \uparrow F \subseteqq U$ and V is precompact in U. Since x was arbitrary, $U = \bigcup\{V\colon V$ is precompact in $U\}$. This is precisely the condition needed for $O_1(L)$ to be a continuous lattice.

(2) $\Rightarrow$ (1). Let $x, y \in L$, $x \nleqq y$. Then $y \in L \backslash \uparrow x := U$, which is open in the lower topology. By hypothesis there exists an open set V such that $y \in V$ and V is precompact in U. Since $O_1(L)$ is a continuous lattice and hence has the interpolation property, there is an open set W such that V is precompact in W and W is precompact in U. Since $K = L \backslash W$ is closed, as in the proof of the preceding theorem K is the intersection of a descending family of upper sets of finite sets. Since V is precompact in W, there exists a finite set F such that $K \subseteqq \uparrow F$ and $V \subseteqq L \backslash \uparrow F$. Thus $y \notin \uparrow F$ and by Proposition (1.7) we have $K \ll x$ and hence $F \ll x$ since $K \subseteqq \uparrow F$. Thus L is a GCL.

2. **Topologies on generalized continuous lattices.**

DEFINITION 2.1. Let $(X, \leqq)$ be a partially ordered set. A topology $\mathcal{O}$ on X is said to be *order consistent* if (i) for all $x \in X$, $\mathrm{cl}(\{x\}) = \downarrow x$, and (ii) if D is an ascending subset of X and $z = \sup D$, then considering D as a net, D converges to z in $\mathcal{O}$.

If X is a T_0 space, then a partial order may be defined on X by $y \leqq x$ if $y \in \mathrm{cl}(\{x\})$. If a topology on a partially ordered set is order consistent, then this induced order is precisely the original order.

PROPOSITION 2.2. *On a partially ordered set $(X, \leqq)$ the upper topology is the coarsest of the order consistent topologies and the Scott topology is the finest.*

PROOF. Since for every closed set A in the upper or Scott topology $\downarrow A = A$ and since $\downarrow x$ is closed for all x, we have $\mathrm{cl}(\{x\}) = \downarrow x$ for both of these topologies.

Let D be an ascending subset of X. If D does not converge to $z = \sup D$ in the upper topology, then there exists a subbasic open set $L \backslash \downarrow x$ such that $z \in L \backslash \downarrow x$ but some cofinal subset of D misses $L \backslash \downarrow x$. But since D is ascending, we have that there exists $d \in D$ such that $d' \leqq x$ for $d \leqq d'$. Hence we have $x \geqq \sup D$, i.e., $x \geqq z$. This contradicts $z \in L \backslash \downarrow x$. Therefore the upper topology is order consistent.

If D does not converge to z in the Scott topology, then there is an open set U such that $z \in U$, but some cofinal subset of D misses U. Since $U = \uparrow U$, D itself misses U. Since U is open, $z = \sup D \notin U$, a contradiction. Thus the Scott topology is order consistent.

Now let $\mathcal{O}$ be any order consistent topology. Since for all $x \in X$, $\downarrow x = \mathrm{cl}(\{x\})$, which is closed, the topology $\mathcal{O}$ is finer than the upper topology. Let C be a closed set in X for the $\mathcal{O}$-topology. Then $\downarrow C = C$ (from condition (i) for order consistency). Let D be an up-directed set, $D \subseteqq C$. Then D converges to $\sup D$ since $\mathcal{O}$ is order consistent and hence $\sup D \in C$ since C is closed. Thus C is closed in the Scott topology.

We return to these "one-sided" topologies shortly, but first we have need to introduce some "two-sided" ones.

DEFINITION 2.3. Let L be a complete lattice. We define the CL (or Lawson) topology on L to be the topology for which all Scott open and all lower open sets form a subbase. We define the LimInf topology (LI) by declaring a set A to be closed if every ultrafilter which has A as a member has the liminf of the ultrafilter in A (if $\mathcal{F}$ is an ultrafilter, $\liminf \mathcal{F} = \sup \{\inf F\colon F \in \mathcal{F}\}$). Equivalently A is closed if for every universal net in A, the liminf of the net is again in A.

PROPOSITION 2.4. *Let L be a complete lattice. Then the* CL *and the* LI *topologies are compact and T_1. The identity mapping is continuous from* (L, LI) *to* (L, CL).

PROOF. We first show the latter statement. Let K be a closed set in the lower topology. Let $\mathcal{F}$ be an ultrafilter such that $K \in \mathcal{F}$. Let M be a finite set such that $K \subseteqq \uparrow M$. It suffices to show $\liminf \mathcal{F} \in \uparrow M$ since by definition of the lower topology K is the intersection of such sets. Since $\mathcal{F}$ is an ultrafilter and since $\uparrow M = \bigcup\{\uparrow x\colon x \in M\}$ and since this latter collection is finite, we can find $x \in M$ such that $\uparrow x \in \mathcal{F}$. Thus $x \leqq \liminf \mathcal{F}$ and hence $\liminf \mathcal{F} \in \uparrow M$.

Now let A be a Scott closed set and let again $\mathcal{F}$ be an ultrafilter such that $A \in \mathcal{F}$. For any $F \in \mathcal{F}$, $F \cap A \neq \emptyset$ and hence $\inf F \leqq a$ for some $a \in A$. Since $A = \downarrow A$, $\inf F \in A$. Since A contains sups of up-directed sets, $\liminf \mathcal{F} \in A$. Thus both lower closed sets and Scott closed sets are closed in the LI topology and therefore the identity function is continuous.

Furthermore, let L be equipped with the LI topology. Then in this topology ultrafilters still converge to their $\liminf \mathcal{F}$ (a general characteristic of defining a topology in terms of filters or nets), although additional limit points may also exist in the topology. In particular since every ultrafilter has a point of convergence, (L, LI) is compact. By continuity (L, CL) is compact, too.

Finally, (L, CL) is T_1 since $\{x\} = \downarrow x \cap \uparrow x$ and $\uparrow x$ is closed in the lower topology and $\uparrow x$ is closed in the Scott topology. Hence by continuity of the identity mapping (L, LI) is also T_1.

We come at this point to a major theorem.

THEOREM 2.5. *Let L be a complete lattice. Then the following statements are equivalent:*
(1) *L is a* GCL.
(2) (L, CL) *is Hausdorff.*
(3) (L, LI) *is Hausdorff.*
Furthermore if any of these equivalent conditions are satisfied, then the LI *and* CL *topology agree and the partial order $\leqq$ has closed graph for this topology. Furthermore, this topology has a subbase of open sets all sets of the form $\{s: F \ll s\}$ where F is some finite set and $L \backslash \uparrow x$ where $x \in L$.*

PROOF. (1) $\Rightarrow$ (2). Let L be a GCL. Let $x, y \in L$ and suppose that $x \nleqq y$. Then there is a finite set F such that $F \ll x$ and $y \notin \uparrow F$. Let $U = \{s: F \ll s\}$ and let $V = L \backslash \uparrow F$. Then $U \cap V = \emptyset$ and V is open in the lower, and hence CL, topology. To finish the proof we show that U is Scott open. Let D be an up-directed set such that $p = \sup D \in U$. By the interpolation property (1.8 (ii)), we can find a finite set G such that $F \ll G \prec p$. Thus there exists $d \in D$ such that $b \leqq d$ for some $b \in G$. But $F \ll G$ implies $F \ll b$ and thus $F \ll d$. Hence $\sup D \in U$ implies $d \in U$ for some $d \in D$. Thus U is Scott open.

Note that by the preceding paragraph sets of the form $\{s: F \ll s\}$ and $L \backslash \uparrow x$, $x \in L$ generate a Hausdorff topology if L is a GCL, and that these sets are open in the CL topology. Since by proposition (2.4) the CL topology is compact, these sets must generate precisely the CL topology. Hence the last statement of the theorem holds.

(2) $\Rightarrow$ (3). This is immediate since by (2.4) the identity function from $(L. \mathrm{LI})$ to (L, CL) is continuous.

(3) $\Rightarrow$ (1). Note first that if $\mathscr{F}$ is an ultrafilter on L, then $\mathscr{F}$ converges to $\operatorname{liminf} \mathscr{F}$ and to that point alone in the LI topology (since the convergence by definition implies convergence in the topology and since by Hausdorffness there is at most one point of convergence).

We show now that the relation $\leqq$ is closed in $L \times L$ (with each factor equipped with the LI topology). Let $\mathscr{G}$ be an ultrafilter contained in $\{(x, y): x \leqq y\}$, i.e., the set $\leqq$ is a member of $\mathscr{G}$. Let $\mathscr{G}_1 = \{\pi_1(G): G \in \mathscr{G}\}$ and $\mathscr{G}_2 = \{\pi_2(G): G \in \mathscr{G}\}$. For each $G \in \mathscr{G}$ such that $G \subseteqq \{(x, y): x \leqq y\}$, we have for the first and second projections, $\pi_1(G)$ and $\pi_2(G)$, $\inf(\pi_1(G)) \leqq \inf(\pi_2(G))$. Thus

$$\operatorname{liminf}\{\pi_1(G): G \in \mathscr{G}\} \leqq \operatorname{liminf}\{\pi_2(G): G \in \mathscr{G}\}.$$

Since $\mathscr{G}$ is an ultrafilter, $\pi_1(\mathscr{G})$ and $\pi_2(\mathscr{G})$ are also ultrafilters. Hence if

$a = \operatorname{liminf}\{\pi_1(G)\colon G \in \mathscr{G}\}$ and $b = \operatorname{liminf}\{\pi_2(G)\colon G \in \mathscr{G}\}$, then $\pi_1(\mathscr{G})$ and $\pi_2(\mathscr{G})$ converge to a and b respectively, and therefore $\mathscr{G}$ converges to (a, b). Hence the relation $\leqq$ is closed.

We now have that under the hypothesis (3) L is a compact Hausdorff space with a closed partial order with respect to the LI topology. Hence we may invoke the known properties of such structures. Let $x, y \in L$ with $x \nleqq y$. We wish to find a finite set F such that $F \ll x$ and $y \notin \uparrow F$. It suffices to find a finite F such that $y \notin \uparrow F$ and there exists an open set U such that $U = \uparrow U$ and $x \in U \subseteqq \uparrow F$ (for if D is an up-directed set with $\sup D \geqq x$, then since D converges to its supremum, $d \in U \subseteqq \uparrow F$ for some $d \in D$). Thus suppose for every open set $U \subseteqq \uparrow U$ with $x \in U$ and for every finite F such that $y \notin \uparrow F$, $U \nsubseteq \uparrow F$, i.e., $U \backslash \uparrow F \neq \varnothing$. Then these sets form a filter base; extend this base to an ultrafilter $\mathscr{F}$. Since $\mathscr{F}$ contains all open neighborhoods $U = \uparrow U$ of x, the limit of $\mathscr{F}$ is a member of $\uparrow x$. Since $\{\inf L\colon L \in \mathscr{F}\}$ converges to the liminf of $\mathscr{F}$ which must be the limit of $\mathscr{F}$, there exists $L \in \mathscr{F}$ such that $\inf L \nleqq y$. We thus have $\uparrow z \in \mathscr{F}$ where $z = \inf L$ and $L \backslash \uparrow z \in \mathscr{F}$ from the original definition. However this is impossible; so the argument is complete.

We have shown in this argument that $\leqq$ is closed. Since (L, LI) is compact, if (L, CL) is Hausdorff, then the two topologies agree by (2.4).

The next corollary is due to K.H. Hofmann.

COROLLARY 2.6. *Let L be a meet continuous complete lattice. Then if L is a* GCL, *it is a continuous lattice.*

PROOF. By [5] a meet continuous complete lattice for which the CL topology is Hausdorff is a continuous lattice. Hence the corollary follows from theorem (2.5).

We return now to a more detailed consideration of the lower topology.

PROPOSITION 2.7. *Let L be a complete lattice.*

(i) *If $\mathscr{F}$ is an ultrafilter on L, the set of cluster ($=$convergence) points is* $\uparrow(\operatorname{liminf} \mathscr{F})$ *in the lower topology.*

(ii) *A subset M of L is closed in the lower topology if and only if $\uparrow M = M$ and if for every ultrafilter $\mathscr{F}$ with $M \in \mathscr{F}$,* $\operatorname{liminf} \mathscr{F} \in M$.

(iii) *A subset M of L is closed in the lower topology if and only if $\uparrow M = M$ and if M is closed in the* CL *topology.*

PROOF. (i) The cluster points of an ultrafilter consists of all points in the intersection of the closure of all sets in the ultrafilter. If A is the set of cluster points, we have

$$A = \bigcap\{\bar{F}\colon F \in \mathscr{F}\} \subseteqq \bigcap\{\uparrow(\inf F)\colon F \in \mathscr{F}\} = \uparrow(\operatorname{liminf} \mathscr{F}).$$

Conversely, let $y \geqq \operatorname{liminf} \mathscr{F}$. If $y \notin \bar{F}$, then by definition of the lower topology, there is a finite set K such that $\bar{F} \subseteqq \uparrow K$ and $y \notin \uparrow K$. Since $F \subseteqq \bar{F} \subseteqq \uparrow K$, we have $\uparrow K \in \mathscr{F}$. Since $\mathscr{F}$ is an ultrafilter, $\uparrow x \in \mathscr{F}$ for some $x \in K$, because K is finite. Hence $x \leqq \liminf \mathscr{F}$, a contradiction to $x \nleqq y$. Thus $y \in \bar{F}$ for all $F \in \mathscr{F}$ and therefore y is a cluster point.

(ii) Suppose M is closed in the lower topology. It follows immediately that $M = \uparrow M$. Since M is closed, every ultrafilter containing M converges to points which are contained in M. Hence by part (i), we have $\liminf \mathscr{F} \in M$. Conversely suppose $\uparrow M = M$ and $M \in \mathscr{F}$ implies $\liminf \mathscr{F} \in M$ for every ultrafilter $\mathscr{F}$. Let $y \in \bar{M}$. Then we can find an ultrafilter $\mathscr{F}$ with $M \in \mathscr{F}$ converging to y. By part (i), $y \geqq \liminf \mathscr{F}$. Since $\liminf \mathscr{F} \in M$ and $M = \uparrow M$, we can conclude that $y \in M$. Thus M is closed.

(iii) If M is closed in the lower topology, then it is closed in the CL topology by definition. Conversely suppose $M = \uparrow M$ is closed in the CL topology. Then by proposition (2.4) it is closed in the LI topology and hence by part (ii) in the lower topology, too.

COROLLARY 2.8. *Let L be a complete lattice and let $\mathcal{O}$ be an order consistent topology on L. The set of cluster points for an ultrafilter $\mathscr{F}$ is contained in* $\uparrow(\liminf \mathscr{F})$.

PROOF. This corollary follows easily from (2.7. (i)) and (2.2.)

PROPOSITION 2.9. *Let L be a complete lattice. A subset U of L is open in the Scott topology if and only if $U = \uparrow U$ and U is open in the* CL *topology. Furthermore the following conditions are equivalent*:

(1) *L is a* GCL.

(2) *For every ultrafilter $\mathscr{F}$ the set of cluster points of $\mathscr{F}$ for the Scott topology is* $\downarrow(\liminf \mathscr{F})$.

PROOF. By definition of the CL topology, if U is Scott open, then U is CL open. Conversely suppose $U = \uparrow U$ is CL open. Let D be an up-directed set (with $x = \sup D$) in $L \backslash U$. It is easily verified that as a net D converges to every point of $\downarrow x$ in the lower topology, to every point of $\downarrow x$ in the Scott topology and hence precisely to x in the CL topology. Since $L \backslash U$ is CL closed, $x \in L \backslash U$. Hence $L \backslash U$ is Scott closed, i.e., U is Scott open.

(1) $\Rightarrow$ (2). Let $\mathscr{F}$ be an ultrafilter, $y = \liminf \mathscr{F}$. If $x \nleqq y$, then we may find a finite set K such that $K \ll x$ and $y \notin \uparrow K$. Set $W = \{w : K \ll w\}$. As in the proof of theorem (2.5), W is Scott open. Hence x is not a cluster point of $\mathscr{F}$ for the Scott topology. Let $F \in \mathscr{F}$ and let $z \leqq y$ with U a Scott open set such that $z \in U$. Since U is Scott open, $y \in U$ and hence $\inf F' \in U$ for some $F' \in \mathscr{F}$. Since $F' \subseteqq \uparrow(\inf F')$, $F' \subseteqq U$. Also $F \cap F' \neq \varnothing$ implies $F \cap U \neq \varnothing$. Since U was an arbitrary Scott open neighborhood of z, the Scott closure $\bar{F}$ of F contains z. Since F was arbitrary, z is a cluster point of $\mathscr{F}$ (this inclusion holds even if L is not a GCL).

(2) $\Rightarrow$ (1). By (2.7. (i)), the hypothesis and the definition of the CL topology, the only possible cluster point for an ultrafilter $\mathscr{F}$ is liminf $\mathscr{F}$. Hence the CL topology is Hausdorff. The conclusion follows now from theorem (2.5).

PROPOSITION 2.10. *Let L be a complete lattice equipped with a compact topology for which the closed upper sets are precisely the sets closed in the lower topology and for which the partial order is closed. Then L is a* GCL.

PROOF. It is well known that the closed upper sets of a compact space with closed order form a continuous lattice (see **[4]**). The proposition follows now from proposition (1.9).

3. **Morphisms between generalized continuous lattices.** In this section we want to describe under which conditions a mapping between generalized continuous lattices is continuous for the various topologies. In this context it seems to be reasonable to restrict ourselves to monotone mappings. Let us start with a well-known result (see **[13]**):

PROPOSITION 3.1. *Let $\zeta: L \to L'$ be a monotone mapping. Then ζ is continuous for the Scott topology on L and L' resp. if and only if ζ preserves suprema of up-directed families.*

PROPOSITION 3.2. *Let $\zeta: L \to L'$ be a mapping preserving up-directed suprema. Then ζ is continuous for the lower topologies on L and L' resp. if and only if ζ preserves liminf's of ultrafilters. Moreover, any monotone mapping preserving liminf's of ultrafilters is continuous.*

PROOF. Let us first assume that ζ is continuous. Pick an ultrafilter $\mathscr{F}$ on L. Then $\zeta(\operatorname{liminf} \mathscr{F})$ is a clusterpoint of the ultrafilter $\zeta(\mathscr{F})$ by the continuity of ζ. This implies $\operatorname{liminf} \zeta(\mathscr{F}) \leqq \zeta(\operatorname{liminf} \mathscr{F})$ by corollary (2.8). Conversely, $\zeta(\operatorname{liminf} \mathscr{F}) = \zeta(\sup\{\inf F\colon F \in \mathscr{F}\})$ and the latter supremum is up-directed. Hence

$$\begin{aligned}\zeta(\operatorname{liminf} \mathscr{F}) = \sup \zeta\{(\inf F)\colon F \in \mathscr{F}\} &\leqq \sup\{\inf \zeta(F)\colon F \in \mathscr{F}\}\\ &= \operatorname{liminf} \zeta(\mathscr{F})\end{aligned}$$

and this inequality holds because ζ is monotone.

Now let ζ by any monotone mapping preserving liminf's of ultrafilters. Then ζ is continuous. Let $A' \subseteq L'$ be closed in the lower topology and let $A = \zeta^{-1}(A')$. Then A is an upper set because ζ is monotone and A is closed under liminf's of ultrafilters containing A because A' is. Hence A is closed in the lower topology by (2.7).

PROPOSITION 3.3. *Let $\zeta: L \to L'$ be a monotone mapping between two complete lattices L and L'. If ζ preserves up-directed suprema and liminf's* *of ultrafilters, then* ζ *is continuous for the* CL *topologies on* L *and* L' *resp. Moreover, if* L *is a* GCL, *then the converse also holds.*

PROOF. By (3.1), (3.2) and the definition of the CL topology we need only give a proof of the second statement. Let us assume that L is a GCL and that $\zeta: L \to L'$ is CL continuous. If $U \subseteq L'$ is Scott open, then $\zeta^{-1}(U)$ is an open upper set and hence open in the Scott topology by (2.9). Hence ζ is Scott continuous and therefore preserves suprema of up-directed sets by (3.1). Next, let $a' \in L'$ and let $A = \zeta^{-1}(\uparrow a')$. Then A is a CL closed upper set of L and therefore closed in the lower topology on L by (2.7). Now we can conclude that ζ is continuous in the lower topology and hence preserves liminf's of ultrafilters by (3.2).

In the remainder of this section we will show that generalized continuous lattices are preserved under a weak kind of quotients, subojects, and products.

PROPOSITION 3.4. *Let* L *be a* GCL, L' *a complete lattice, and* $\zeta: L \to L'$ *be a surjective mapping which preserves upwards directed suprema and which is continuous for the lower topology* (*i.e.,* ζ *is monotone and continuous for the* CL *topologies*). *Then* L' *is a* GCL.

PROOF. Endow L with the liminf topology. Then (L, LI) is compact and Hausdorff by (2.5). Moreover, $\ker \zeta = \{(a, b): \zeta(a) = \zeta(b)\}$ is closed in $L \times L$. Indeed, let $\mathscr{G}$ be an ultrafilter on $L \times L$ containing $\ker \zeta$. We have to show that

$$\lim \mathscr{G} = (\lim \pi_1\mathscr{G}, \lim \pi_2\mathscr{G}) = (\operatorname{liminf} \pi_1\mathscr{G}, \operatorname{liminf} \pi_2\mathscr{G})$$

is contained in $\ker \zeta$, where $\pi_1: L \times L \to L$ and $\pi_2: L \times L \to L$ denote the first and second projection resp. First, note that $M \subseteq \ker \zeta$ implies $\zeta \circ \pi_1(M) = \zeta \circ \pi_2(M)$ and that ζ preserves liminf's of ultrafilters by the assumptions (3.4) and proposition (3.2). Hence

$$\zeta(\operatorname{liminf} \pi_1\mathscr{G}) = \operatorname{liminf} \zeta \circ \pi_1\mathscr{G} = \operatorname{liminf} \zeta \circ \pi_2\mathscr{G} = \zeta(\operatorname{liminf} \pi_2\mathscr{G}).$$

Now we can conclude that the quotient topology of ζ on L' is Hausdorff. A similar argument shows that the order $\leqq$ is closed in the quotient topology. If $A' \subseteq L'$ is closed in the lower topology, then $\zeta^{-1}(A')$ is closed in the lower topology of L by the assumptions of (3.4). Hence $\zeta^{-1}(A')$ is closed in the CL topology of L and therefore A' is closed in the quotient topology. Conversely, let $A' = \uparrow A' \subseteq L'$ be an upper set which is closed in the quotient topology. Then A' is closed under liminf's of ultrafilters. Indeed, let $\mathscr{F}'$ be an ultrafilter on L' containing A'. Pick an ultrafilter $\mathscr{F}$ containing $\zeta^{-1}(\mathscr{F}')$. Then $\zeta^{-1}(A') \in \mathscr{F}$ and $\zeta(\mathscr{F}) = \mathscr{F}'$. As $\zeta^{-1}(A')$ is closed in the CL topology of L, $\operatorname{liminf} \mathscr{F} \in \zeta^{-1}(A')$. Hence

$$\zeta(\liminf \mathscr{F}) = \liminf \zeta(\mathscr{F}) = \liminf \mathscr{F}' \in A',$$

because ζ preserves liminf's of ultrafilters by (3.3). Now (2.7) yields that A' is closed in the lower topology. Therefore the lower closed sets of L' are exactly the upper sets which are closed in the quotient topology. This completes the proof by (2.10).

DEFINITION 3.5. A CL-morphism between complete lattices L and L' is a mapping preserving arbitrary infima and up-directed suprema.

Clearly, every CL-morphism preserves liminf's of ultrafilters. This yields the following proposition.

PROPOSITION 3.6. *Every* CL-*mrophism is continuous for the lower topologies, the Scott topologies, the* CL *topologies, and the* LI *topologies on* L *and* L' *resp.*

PROPOSITION 3.7. *Let* $\zeta: L \to L'$ *be a* CL-*morphism between complete lattices* L *and* L'.

(i) *If* ζ *is surjective and if* L *is a* GCL, *then so is* L'.
(ii) *If* ζ *is injective and if* L' *is a* GCL, *then so is* L.

PROOF. (i) follows from (3.4) and (3.6) and (ii) follows from the observation that $\zeta(L)$ is a subsemilattice of L, which is closed in the LI topology.

Now let L be a complete lattice and let $P(L)$ its ideal lattice. From [**8**] we know that L is a continuous lattice if and only if the mapping $I \mapsto \sup I$: $P(L) \to L$ is a CL-morphism and that L is meet continuous if and only if $I \mapsto \sup I$: $P(L) \to L$ preserves finite infima. We add one more condition.

THEOREM 3.8. *Let* L *be a complete lattice. Then* L *is a* GCL *if and only if the mapping* $I \mapsto \sup I$: $P(L) \to L$ *is continuous for the lower topology* (*or equivalently for the* CL *topology*) *on* $P(L)$ *and* L *resp.*

PROOF. As the mapping $I \mapsto \sup I$ always preserves up-directed suprema, one direction is clear by (3.4). Conversely, let L be a GCL and let $a \in L$ be a point. We have to prove that $\{I \in P(L)\colon a \nleqq \sup I\}$ is open in the lower topology of $P(L)$. Let $J \in \{I \in P(L) : a \nleqq \sup I\}$. Then $\sup J \ngeqq a$. Hence we can find a finite set $F \subseteqq L \backslash \downarrow \sup J$ which guards a from below. Now let $O = \{I \in P(L)\colon \uparrow f \nsubseteqq I$ for all $f \in F\}$. Then $J \in O$ and O is open in the lower topology of $P(L)$. Moreover, $I \in O$ implies $\sup I \ngeqq a$, because otherwise we would have $\uparrow f \cap I \neq \varnothing$ for some $f \in F$, and thus $f \in I$, i.e., $\downarrow f \subseteqq I$. Hence $J \in O \subset \{I \in P(L)\colon a \nleqq \sup I\}$.

COROLLARY 3.9. *A complete lattice* L *is a* GCL *if and only if it is the quotient of a continuous lattice* L' *under a mapping which is continuous for* *the lower topologies on L and L' resp. and which preserves up-directed suprema.*

We conclude this section with two results, which may perhaps illustrate that many properties holding for continuous lattices are also true for generalized continuous lattices.

PROPOSITION 3.10. (See [**8**]). *If L and L' are* GCL *and if $\gamma: L \to L'$ is a mapping preserving arbitrary infima, then γ preserves up-directed suprema if and only if $F \ll G$ in L' implies $\delta F \ll \delta G$ in L, where $\delta: L' \to L$ is the right adjoint of γ (i.e., $\delta(a) = \inf \gamma^{-1}(\uparrow a)$).*

PROOF. Assume that γ preserves up-directed suprema and that $\sup D \geqq \delta(g)$ for some $g \in G$ and an up-directed family $D \subseteqq L$. Then $\gamma(\sup D) = \sup \gamma(D) \geqq g$, hence $\gamma(d) \geqq f$ for some $d \in D$ and some $f \in F$. This implies $d \geqq \delta(f)$ for some $d \in D$ and some $f \in F$, i.e., $\delta F \ll \delta G$.

Conversely, assume that $F \ll G$ implies $\delta(F) \ll \delta(G)$. Let $D \subseteqq L$ be an up-directed set. We have to show that $\gamma(\sup D) = \sup \gamma(D)$. Let $F \subseteqq L'$ be a finite subset which guards $\gamma(\sup D)$ from below. By hypothesis we have $\delta F \ll \delta\gamma(\sup D) \leqq \sup D$, and hence we can find an $f \in F$ and a $d \in D$ such that $\delta f \leqq d$, i.e., $f \leqq \gamma d$. Therefore we can conclude that $\sup \gamma(D) \in \bigcap\{\uparrow F: F \ll \gamma \sup D\} = \uparrow\gamma \sup D$; the last equality follows from (1.4). But this yields $\gamma \sup D \leqq \sup \gamma(D)$. The other inequality holds for every monotone mapping.

Our second example is the lemma on primes.

LEMMA 3.11. (The Lemma on Primes. See [**4**]). *Let L be a complete lattice and let L' be a* GCL. *Furthermore, let $\zeta: L' \to L$ be a mapping preserving up-directed suprema and let $p \in L$ be an element of L satisfying the following primality condition.*

(P) *For every finite subset $A \subseteqq L'$, $\inf \zeta(A) \leqq p$ implies the existence of an $a \in A$ such that $\zeta(a) \leqq p$. Then for every* CL *compact subset $K \subseteqq L'$, $\inf \zeta(K) \leqq p$ implies the existence of a $k \in K$ such that $\zeta(k) \leqq p$.*

PROOF. Let $K \subseteqq L'$ be a CL compact subset of L such that $\inf \zeta(K) \leqq p$ and assume that $K \cap \zeta^{-1}(\downarrow p) = \varnothing$. Then K is contained in the Scott open set $L \backslash \zeta^{-1}(\downarrow p)$. Hence for every $k \in K$ we can find a finite subset $F_k \subseteqq L \backslash \zeta^{-1}(\downarrow p)$ with $F_k \ll k$ (if every finite subset $F \ll k$ would intersect $\zeta^{-1}(\downarrow p)$, then $\bigcap\{\uparrow F \cap \zeta^{-1}(\downarrow p): F \ll k\}$ would be non-empty, because L' is CL compact. But

$$\begin{aligned}\bigcap\{\uparrow F \cap \zeta^{-1}(\downarrow p): F \ll k\} &= \zeta^{-1}(\downarrow p) \cap \bigcap\{\uparrow F: F \ll k\}\\ &= \zeta^{-1}(\downarrow p) \cap \uparrow k = \varnothing.\end{aligned}$$

Now we have $K \subseteqq \bigcup_{k \in K}\{x: F_k \ll x\}$ and hence

$$K \subseteq \bigcup_{i=1}^{n} \{x : F_{k_i} \ll x\}$$

for certain $k_1, \ldots, k_n \in K$ by the compactness of K (Recall from the proof of (2.5) that the set $\{x: F_k \ll x\}$ is open in the Scott topology). Now, clearly,

$$p \geqq \inf \zeta(K) \geqq \inf \zeta(\bigcup_{i=1}^{n} F_{k_i})$$

and $\bigcup_{i=1}^{n} F_{k_i}$ is finite. Hence the primality condition (p) yields an element $f \in \bigcup_{i=1}^{n} F_{k_i}$ with $\zeta(f) \leqq p$, a contradiction to $\bigcup_{i=1}^{n} F_{k_i} \subseteq L \backslash \zeta^{-1}(\downarrow p)$. Therefore there is an element $k \in K$ with $\zeta(k) \leqq p$.

4. **Complete lattices in the interval topology.** In this section we discuss the symmetric interval topology and its relations to generalized continuous lattices.

DEFINITION 4.1. Let L be a complete lattice. We define the interval topology (= IV topology) on L to be the topology which has as a subbase for the closed sets the collection of all closed intervals $[a, b] = \{x: a \leqq x \leqq b, a, b \in L\}$.

It is immediate that the IV topology is the supremum of the lower and the upper topology on L. This observation proves a large part of the following proposition.

PROPOSITION 4.2. *Let L be a complete lattice and let $\mathscr{F}$ be an ultrafilter on L. Then the set of all cluster points of $\mathscr{F}$ in the* IV *topology is exactly the set* [liminf $\mathscr{F}$, limsup $\mathscr{F}$], *and this set is not empty. Hence L is a compact T_1 space in the* IV *topology and the* IV *topology is coarser than the* LI *topology and the* CL *topology on L (or L^{op}).*

PROOF. Clearly, for every filter $\mathscr{F}$ we have $\sup\{\inf M: M \in \mathscr{F}\} \leqq \inf\{\sup M: M \in \mathscr{F}\}$ and hence [liminf $\mathscr{F}$, limsup $\mathscr{F}$] $\neq \emptyset$. By (2.7) and the above remark, [liminf $\mathscr{F}$, limsup $\mathscr{F}$] is exactly the set of all cluster points of an ultrafilter $\mathscr{F}$.

PROPOSITION 4.3. *Let L be a complete lattice. Then the following statements are equivalent.*

(i) *The* IV *topology on L is Hausdorff.*

(ii) *For every ultrafilter $\mathscr{F}$ on L we have* liminf $\mathscr{F}$ = limsup $\mathscr{F}$.

(iii) *L is a generalized bicontinuous lattice (i.e., L and L^{op} are both* GCL *and the* CL *topologies on L and L^{op} agree). In each of these cases, the* IV *topology and the* CL *topology agree.*

PROOF. (i) $\Leftrightarrow$ (ii) holds by (4.2).

(i) $\Rightarrow$ (iii). By (4.2), the IV topology is coarser than both the CL topology on L and the CL topology on L^{op}. If the IV topology is Hausdorff, then all the three topologies agree. Hence (iii) follows by (2.5).

(iii) $\Rightarrow$ (ii) is an easy consequence on (2.5).

COROLLARY 4.4. *Let L be a meet and join continuous lattice. Then the following conditions are equivalent.*

(i) *L is bicontinuous.*

(ii) *The interval topology on L is T_2.*

(iii) *The* CL *topology on L and the interval topology on L agree.*

Here again, a *bicontinuous lattice L* is a complete lattice with the property that L and L^{op} are continuous lattices and the CL topologies on L and L^{op} resp. agree. We will give another description of bicontinuous lattices later on. For the moment let us return to lattices which are Hausdorff in their interval topology. This type of lattice is preserved under a certain kind of quotients.

PROPOSITION 4.5. *Let L and L' be two complete lattices and let $\zeta: L' \to L$ be a surjective mapping preserving up-directed supprema and down-directed infima. Assume moreover that the* IV *topology on L' is Hausdorff. Then the* IV *topology on L is also Hausdorff.*

PROOF. Let $\mathscr{F}$ be an ultrafilter on L and let $\mathscr{F}'$ be an ultrafilter on L' which is mapped onto $\mathscr{F}$ under ζ. As $\limsup \mathscr{F}' = \liminf \mathscr{F}'$ in L' and as ζ preserves up-directed suprema and down-directed infima, we can compute

$$\begin{aligned}
\limsup \mathscr{F} &= \limsup \zeta(\mathscr{F}') \\
&= \inf\{\sup \zeta(M) : M \in \mathscr{F}'\} \leqq \inf\{\zeta(\sup M) : M \in \mathscr{F}'\} \\
&= \zeta(\inf\{\sup M : M \in \mathscr{F}'\}) = \zeta(\limsup \mathscr{F}') = \zeta(\liminf \mathscr{F}') \\
&= \zeta(\sup\{\inf M : M \in \mathscr{F}'\}) = \sup\{\zeta(\inf M) : M \in \mathscr{F}'\} \\
&\leqq \sup \inf\{\zeta(M) : M \in \mathscr{F}'\} = \liminf \zeta(\mathscr{F}') = \liminf \mathscr{F}.
\end{aligned}$$

Hence $\liminf \mathscr{F} = \limsup \mathscr{F}$.

COROLLARY 4.6. *Let L' be a completely distributive lattice and let L be a complete lattice, Further, let $\zeta: L' \to L$ be a surjective mapping preserving up-directed suprema and down-directed infima. Then the interval topology on L is Hausdorff.*

PROOF. It is well known that the IV topology on a completely distributive lattice is Hausdorff (see [**10**]). Now apply (4.5).

PROPOSITION 4.7. *Let L and L' be two complete lattices, and let $\zeta: L \to L'$ be a monotone mapping. If the* IV *topologies on L and L' resp. are Hausdorff,* *then ζ is continuous if and only if ζ preserves up-directed suprema and down-directed infima.*

PROOF. Every supremum of an up-directed family is the limit of the family, indexed by itself. Hence every continuous map preserves up-directed suprema and down directed infima. Conversely, let $\mathscr{F}$ be an ultrafilter on L and let $\mathscr{F}' = \zeta(\mathscr{F})$. Then the proof of (4.5) shows that

$$\begin{aligned}\lim \zeta(\mathscr{F}) &= \operatorname{limsup} \zeta(\mathscr{F}) \leqq \zeta(\operatorname{limsup} \mathscr{F}) = \zeta(\lim \mathscr{F}) \\ &\leqq \operatorname{liminf} \zeta(\mathscr{F}) = \lim \zeta(\mathscr{F}),\end{aligned}$$

i.e., $\lim \zeta(\mathscr{F}) = \zeta(\lim \mathscr{F})$. This is equivalent to the continuity of ζ.

Corollary (4.6) gives rise to the question whether every complete lattice with the property that the IV topology is Hausdorff is a quotient of a completely distributive lattice under a mapping preserving upwards directed suprema and down-directed infima, i.e., a continuous and monotone mapping. We shall prove later on that this is at least true for meet (or join) continuous lattices, and we will see in a moment that this is true for distributive lattices. But we do not know the answer in general.

Let us first recall some facts from the Priestley duality for distributive lattices.

THEOREM 4.8. (Priestley). *Let* LD *be the category of all distributive lattices with* 0 *and* 1 *together with all lattice homomorphisms preserving* 0 *and* 1 *as morphisms, and let* PZ *be the category of all zero-dimensional compact* (T_2) *ordered spaces with the property that two different points can be separated by a clopen upper set, together with all order preserving continuous mappings as morphisms. Then* LD *and* PZ *are dually equivalent under the contravariant functors* $\mathrm{LD}(-,2)\colon \mathrm{LD} \to \mathrm{PZ}$, *where for a lattice* L, $\mathrm{LD}(L, 2)$ *denotes the set of all* LD*-morphisms from* L *into* 2, *equipped with the pointwise ordering and the topology induced from the product topology of the discrete space* 2, *and* $\mathrm{PZ}(-,2)\colon \mathrm{PZ} \to \mathrm{LD}$, *where for a* PZ*-object* X, $\mathrm{PZ}(X, 2)$ *is the lattice of all continuous and order preserving mappings into the discrete two element chain ordered by the pointwise ordering. The natural transformations* $\eta\colon 1_{\mathrm{LD}} \to \mathrm{PZ}(\mathrm{LD}(-,2),2)$ *and* $\varepsilon\colon 1_{\mathrm{PZ}} \to \mathrm{LD}(\mathrm{PZ}(-,2),2)$ *are given by*

$$\begin{aligned}\eta_L\colon L &\to \mathrm{PZ}(\mathrm{LD}(L,2),2) \\ &\mapsto \hat{a},\ \hat{a}(\zeta) = \zeta(a)\end{aligned}$$

and

$$\begin{aligned}\varepsilon_p\colon P &\to \mathrm{LD}(\mathrm{PZ}(P,2),2) \\ p &\mapsto \hat{p}, \hat{p}(\zeta) = \zeta(p).\end{aligned}$$

As a consequence we have the following proposition.

PROPOSITION 4.9. (see [1]). *Let X be a partially ordered set. Then the free distributive lattice generated by X is isomorphic to* $\mathrm{PZ}(\tilde{X}, 2)$, *where $\tilde{X}$ denotes the set of all order preserving maps $\zeta\colon X \to 2$, equipped with the topology of pointwise convergence and the pointwise ordering.*

In the sequel, for a partially ordered set X the algebraic lattice $\tilde{X}$ is always endowed with the CL topology.

PROPOSITION 4.10. *Let P be a* PZ-*object and let $\zeta\colon P \to 2$ be a monotone mapping. Then*

$$\begin{aligned}\zeta &= \sup\{\inf\{\psi\colon \psi \in \mathrm{PZ}(P, 2), \psi \geqq \chi\}\colon \chi \leqq \zeta, \chi^{-1}(1) \textit{ is closed}\}\\ &= \inf\{\sup\{\psi\colon \psi \in \mathrm{PZ}(P, 2), \psi \leqq \chi\}\colon \chi \geqq \zeta, \zeta^{-1}(1) \textit{ is open}\}.\end{aligned}$$

Furthermore, all occurring suprema and infima are directed.

PROOF. First, let $\chi\colon P \to 2$ be a monotone mapping such that $\chi^{-1}(1)$ is closed. Then $\chi^{-1}(1) = \bigcap\{U\colon \chi^{-1}(1) \subseteqq U, U$ a clopen upper set$\}$ by the definition of PZ-objects and an easy compactness argument. Hence $\chi = \inf\{\chi_U\colon \chi^{-1}(1) \subseteqq U, U$ a clopen upper set$\} = \inf\{\psi\colon \psi \in \mathrm{PZ}(P, 2), \psi \geqq \chi\}$, where χ_U denotes the characteristic function of U, and this infimum is up-directed. Next, let $\zeta\colon P \to 2$ be monotone. Then $\zeta^{-1}(1) = \bigcup\{A\colon \zeta^{-1}(1) \supseteqq A, A$ a closed upper set$\}$. Hence $\zeta = \sup\{\chi_A\colon \zeta^{-1}(1) \supseteqq A = \uparrow A, A$ is closed$\} = \sup\{\chi\colon \chi \leqq \zeta, \chi^{-1}(1)$ is closed$\}$, where again χ_A denotes the characteristic function of A, and this supremum is also directed. This proves the first equality. The proof of the second equality is similar.

Now let us return to generalized continuous lattices.

PROPOSITION 4.11. *Let L be a* GCL *and let $(I, \leqq)$ be a down-directed index set. For each $i \in I$ let $D_i \subseteqq L$ be an up-directed subset of L such that $i \leqq j$ implies $D_i \subseteqq D_j$. Then the equation holds in L.*

$$\bigwedge_{i\in I} \bigvee D_i = \bigvee_{\alpha\in\prod_{i\in I} D_i} \bigwedge_{i\in I} \alpha(i)$$

PROOF. The inequality "$\geqq$" holds in every complete lattice. Conversely, let $F \in L$ be a finite subset that guards the left hand side from below. Then for every $i \in I$ there is a $d \in D_i$ such that $d \in \uparrow F$. Hence we can find an $f \in F$ such that $\{i \in I\colon$ there is a $d \in D_i$ with $f \leqq d\}$ is cofinal in I. This certainly implies $\{i \in I\colon$ there is a $d \in D_i$ with $f \leqq d\} = I$. Hence there is an $\alpha \in \prod_{i\in I} D_i$ such that $\bigwedge_{i\in I} \alpha(i) \geqq f \in F$. But then the right hand side is contained in $\uparrow F$. As F is an arbitrary finite subset which guards $\bigwedge_{i\in I} \bigvee D_i$ from below, the proof is complete by (1.4).

PROPOSITION 4.12. *Let L be a* GCL *and let $\tilde{L}$ denote the* PZ*-object of all monotone mappings ζ: $L \to 2$. Further, let λ: $\mathrm{PZ}(\tilde{L}, 2) \to L$ be an $\wedge$-homomorphism and let $\tilde{\tilde{L}}$ be the set of all monotone mappings ζ: $\tilde{L} \to 2$. Then the mapping λ_*: $\tilde{\tilde{L}} \to L$ defined by*

$$\lambda_*(\zeta) = \sup\{\inf\{\lambda(\psi): \psi \in \mathrm{PZ}(\tilde{L}, 2), \psi \geqq \chi\}: \chi \leqq \zeta, \chi^{-1}(1) \textit{ is closed}\}$$

preserves down-directed infima. Dually, if L^{op} is a GCL *and if λ: $\mathrm{PZ}(\tilde{L}, 2) \to L$ is a $\vee$-homomorphism, then λ^*: $\tilde{\tilde{L}} \to L$ defined by*

$$\lambda^*(\zeta) = \inf\{\sup\{\lambda(\psi): \psi \in \mathrm{PZ}(L, 2), \psi \leqq \chi\}: \chi \geqq \zeta, \zeta^{-1}(1) \textit{ is open}\}$$

preserves up directed suprema.

PROOF. Let $I \subseteqq \tilde{\tilde{L}}$ be down-directed and for every $\zeta \in I$ let

$$D_\zeta = \{\inf\{\lambda(\psi): \psi \in \mathrm{PZ}(L, 2), \psi \geqq \chi\}: \chi \leqq \zeta, \chi^{-1}(1) \text{ closed}\}.$$

Then $\sup D_\zeta = \lambda_*(\zeta)$ and $\zeta \leqq \zeta'$ implies $D_\zeta \subseteqq D_{\zeta'}$. Hence we have

$$\inf\{\lambda_*(\zeta): \zeta \in I\} = \bigwedge_{\zeta \in I} \bigvee D_\zeta = \bigwedge_{\alpha \in \prod D_\zeta} \bigvee_{\zeta \in I} \alpha(\zeta)$$

by (4.11). We want to show that

$$\bigvee_{\alpha \in \prod D_\zeta} \bigwedge_{\zeta \in I} \alpha(\zeta) \leqq \lambda_*(\inf I).$$

Let $\alpha \in \prod_{\zeta \in I} D_\zeta$. Then we have to show that

$$\bigwedge_{\zeta \in I} \alpha(\zeta) \leqq \lambda_*(\inf I) = \{\sup\{\inf \lambda(\psi): \psi \in \mathrm{PZ}(\tilde{L}, 2), \psi \geqq \chi\}: \chi \leqq \inf I, \chi^{-1}(1) \text{ is closed}\}.$$

Now $\alpha(\zeta) = \inf\{\lambda(\psi): \psi \in \mathrm{PZ}(\tilde{L}, 2), \psi \geqq \chi_\zeta\}$ for some $\chi_\zeta \leqq \zeta$, $\chi_\zeta^{-1}(1)$ closed. Choose $\chi_0 = \inf\{\chi_\zeta: \zeta \in I\}$. Then $\chi_0 \leqq \inf I$ and

$$\chi_0^{-1}(1) = \bigcap_{\zeta \in I} \chi_\zeta^{-1}(1)$$

is closed. Hence it is enough to show that $\bigwedge_{\zeta \in I} \alpha(\zeta) \leqq \lambda(\psi)$ for every $\psi \in \mathrm{PZ}(\tilde{L}, 2)$, $\psi \geqq \chi_0$. But for every $\psi \in \mathrm{PZ}(\tilde{L}, 2)$, $\psi^{-1}(1)$ is clopen and

$$\psi^{-1}(1) \supseteqq \chi_0^{-1}(1) = \bigcap_{\zeta \in I} \chi_\zeta^{-1}(1).$$

Therefore we can find finitely many $\zeta_1, \ldots, \zeta_n \in I$ such that

$$\chi_{\zeta_1}^{-1}(1) \cap \cdots \cap \chi_{\zeta_n}^{-1}(1) \subseteqq \psi^{-1}(1),$$

using the compactness of $\tilde{L}$. Again the compactness of $\tilde{L}$ yields elements $\psi_1, \ldots, \psi_n \in \mathrm{PZ}(\tilde{L}, 2)$ such that $\chi_{\zeta_i} \leqq \psi_i$ and $\psi_1 \wedge \cdots \wedge \psi_n \leqq \psi$. But now the fact that λ is an $\wedge$-homomorphism allows us to conclude

$$\bigwedge_{\zeta \in I} \alpha(\zeta) \leqq \lambda(\psi_1) \wedge \ldots \wedge \lambda(\psi_n) = \lambda(\psi_1 \wedge \cdots \wedge \psi_n) \leqq \lambda(\psi).$$

This proves $\inf \lambda_*(I) \leqq \lambda_*(\inf I)$. The other inequality is always true. The second claim can be proven dually.

PROPOSITION 4.13. *If L and L^{op} are* GCL *and if λ: $\mathrm{PZ}(\tilde{L}, 2) \to L$ is a lattice homomorphism, then $\lambda_*: \tilde{\tilde{L}} \to L$ and $\lambda^*: \tilde{\tilde{L}} \to L$ agree.*

PROOF. We first show that for every $\zeta \in \tilde{\tilde{L}}$ the inequality $\lambda_*(\zeta) \leqq \lambda^*(\zeta)$ holds. Indeed, this inequality is true for every complete lattice L and every monotone map λ: $\mathrm{PZ}(\tilde{L}, 2) \to L$. Let $\chi \leqq \zeta \leqq \gamma$ such that $\chi^{-1}(1)$ is closed and $\gamma^{-1}(1)$ is open. Then we have to show that

$$\inf\{\lambda(\phi): \phi \in \mathrm{PZ}(\tilde{L}, 2), \chi \leqq \phi\} \leqq \sup\{\lambda(\phi): \phi \in \mathrm{PZ}(\tilde{L}, 2), \phi \leqq \gamma\}.$$

But $\chi \leqq \gamma$ implies $\chi^{-1}(1) \subseteqq \gamma^{-1}(1)$. Now the compactness of L and the fact that L has enough clopen upper sets yield an open and closed upper set U such that $\chi^{-1}(1) \subseteqq U \subseteqq \gamma^{-1}(1)$. Then the characteristic function χ_U of U satisfies $\chi \leqq \chi_U \leqq \gamma$ and hence

$$\inf\{\lambda(\phi): \phi \in \mathrm{PZ}(\tilde{L}, 2), \chi \leqq \phi\} \leqq \lambda(\chi_U) \leqq \sup\{\lambda(\phi): \phi \in \mathrm{PZ}(L, 2), \phi \leqq \gamma\}.$$

To show the converse inequality, let $I = \{\chi \in \tilde{\tilde{L}}: \chi \leqq \zeta, \chi^{-1}(1) \text{ is closed}\}$, $J = \{\gamma \in \tilde{\tilde{L}}; \zeta \leqq \gamma, \gamma^{-1}(1) \text{ is open}\}$. For every $\chi \in I$ let $D_\chi = \{\lambda(\phi): \phi \in \mathrm{PZ}(\tilde{L}, 2), \chi \leqq \phi\}$ and for every $\gamma \in J$ let $D_\gamma = \{\lambda(\phi): \phi \in \mathrm{PZ}(\tilde{L}, 2), \phi \leqq \gamma\}$. Then I is up-directed, J is down-directed and for $\chi, \chi' \in I$, $\chi \leqq \chi'$ implies $D_\chi \subseteqq D_{\chi'}$ as well as for $\gamma, \gamma' \in I$, $\gamma \leqq \gamma'$ implies $D_\gamma \subseteqq D_{\gamma'}$. Moreover, by (4.11) we have

$$\lambda_*(\zeta) = \bigvee_{\chi \in I} \bigwedge D_\chi = \bigwedge_{\alpha \in \Pi D_\chi} \bigvee_{\chi \in I} \alpha(\chi)$$

and

$$\lambda^*(\zeta) = \bigwedge_{\gamma \in J} \bigvee D_\gamma = \bigvee_{\alpha \in \Pi D_\gamma} \bigwedge_{\gamma \in J} \alpha(\gamma).$$

Fix $\alpha \in \Pi D_\chi$ and $\beta \in \Pi D_\gamma$. Then we have to show that

$$\bigwedge_{\gamma \in J} \beta(\gamma) \leqq \bigvee_{\chi \in I} \alpha(\chi).$$

But for every $\gamma \in J$ we have $\beta(\gamma) = \lambda(\phi_\gamma)$ for some $\gamma \geqq \phi_\gamma \in \mathrm{PZ}(\tilde{L}, 2)$ and for every $\chi \in I$ we have $\alpha(\chi) = \lambda(\phi_\chi)$ for some $\chi \leqq \phi_\chi \in \mathrm{PZ}(\tilde{L}, 2)$. Moreover,

$$\begin{aligned}\bigcap\{\phi_\gamma^{-1}(1): \gamma \in J\} \subseteqq \bigcap\{\gamma^{-1}(1): \gamma \in J\} = \zeta^{-1}(1) &= \bigcup\{\chi^{-1}(1): \chi \in I\} \\ &\subseteqq \bigcup\{\phi_\chi^{-1}(1): \chi \in I\}.\end{aligned}$$

Because both $\phi_\gamma^{-1}(1)$ and $\phi_\chi^{-1}(1)$ are clopen for all $\gamma \in J$, $\chi \in I$, and because L is compact, we can find $\gamma_1, \ldots, \gamma_n \in J$ and $\chi_1, \ldots, \chi_m \in I$ such that

$$\phi_{\gamma_1}^{-1}(1) \cap \cdots \cap \phi_{\gamma_n}^{-1}(1) \subseteqq \phi_{\chi_1}^{-1}(1) \cup \cdots \cup \phi_{\chi_m}^{-1}(1),$$

i.e., $\phi_{\gamma 1} \wedge \cdots \wedge \phi_{\gamma_n} \leqq \phi_{\chi_1} \vee \cdots \vee \phi_{\chi_m}$. Now the fact that λ is a lattice homomorphism yields

$$\begin{aligned}\bigwedge_{\gamma \in J} \beta(\gamma) &\leqq \beta(\gamma_1) \wedge \cdots \wedge \beta(\gamma_n)\\ &= \lambda(\phi_{\gamma 1}) \wedge \cdots \wedge \lambda(\phi_{\gamma_n}) = \lambda(\phi_{\gamma 1}) \wedge \cdots \wedge \phi(_{\gamma_n})\\ &\leqq \lambda(\phi_{\chi_1} \vee \cdots \vee \phi_{\chi_m}) = \lambda(\phi_{\chi_1}) \vee \cdots \vee \lambda(\phi_{\chi_m})\\ &= \alpha(\chi_1) \vee \cdots \vee \alpha(\chi_m) \leqq \bigvee_{\chi \in I} \alpha(\chi).\end{aligned}$$

Now we can prove a partial converse of corollary (4.6).

THEOREM 4.14. *Let L be a distributive lattice such that L and L^{op} are* GCL. *Then*

(i) *There exists a completely distributive lattice L' and a surjective mapping ζ: $L' \to L$ preserving up-directed suprema and down-directed infima. Moreover, L' can be chosen to be $\tilde{\tilde{L}}$ and ζ: $\tilde{\tilde{L}} \to L$ can be chosen to be an extension of the canonical map λ:* PZ $(\tilde{L}, 2) \to L$, *which lifts the identity* id: $L \to L$ *(Recall that* PZ$(\tilde{L}, 2)$ *is a free lattice generated by the partially ordered set L). In this case, ζ is a lattice homomorphism on a dense sublattice of L'.*

(ii) *The* CL *topology on L and L^{op} and the interval topology on L agree. In particular the interval topology on L is Hausdorff.*

PROOF. Let $L' = L$ and let λ: PZ$(\tilde{L}, 2) \to L$ be the canonical map which lifts id: $L \to L$. Then λ is a lattice homomorphism. Let $\zeta = \lambda_* = \lambda^*$. Then for $\phi_0 \in$ PZ$(\tilde{L}, 2)$ we have

$$\begin{aligned}\zeta(\phi_0) &= \{\sup\{\inf\{\lambda(\phi)\colon \chi \leqq \phi \in \mathrm{PZ}(\tilde{L}, 2)\}\colon \chi \leqq \phi_0, \chi^{-1}(1) \text{ is closed}\}\\ &= \inf\{\lambda(\phi)\colon \phi_0 \leqq \phi \in \mathrm{PZ}(\tilde{L}, 2)\} = \lambda(\phi_0),\end{aligned}$$

hence ζ extends λ. Moreover, ζ preserves up-directed suprema and down-directed infima by (4.12). Further, PZ$(\tilde{L}, 2)$ is dense in $\tilde{\tilde{L}}$ by (4.10), because directed suprema and infima are limits in the CL topology. This proves (i). As $\tilde{\tilde{L}}$ is a completely distributive lattice, (ii) is an immediate consequence of (4.6) and (4.2).

COROLLARY 4.15. *Let L be a distributive lattice. Then the following conditions are equivalent.*

(i) *The* IV *topology on L is Hausdorff.*

(ii) *L is a quotient of a completely distributive lattice under a mapping preserving directed suprema and infima.*

5. **Characterization of generalized continuous lattices by the lattice of scott open subsets.** It occurs regularly that every property of a complete lattice L can be characterized by a stronger property of the lattice $O(L)$

of Scott open subset of L (see **[2]**). In this section we focus our attention on this phenomenon.

Let L be a complete lattice, $O(L)$ (resp. $C(L)$) the lattice of all Scott open (resp. Scott closed) subsets of L and $U(L)$ (resp. $D(L)$) the (completely distributive) lattice of all upper sets (resp. lower sets) of L. Then we have a mapping

$$\begin{aligned} {}^{od}\colon U(L) &\to O(L) \\ A &\mapsto A^{od} \end{aligned}$$

where A^{od} denotes the largest Scott open subset contained in A. The mapping ^{od} is a kernel operator which preserves arbitrary infima. Dually, we have a mapping

$$\begin{aligned} {}^{-d}\colon D(L) &\to C(L) \\ A &\mapsto \bar{A}^{d} \end{aligned}$$

where $\bar{A}^d$ is the smallest Scott closed subset containing A. Clearly, ${}^{-d}$ is a hull operator preserving arbitrary suprema. Note that ${}^{od}\colon U(L) \to O(L)$ (resp. ${}^{-d}\colon D(L) \to C(L)$) is the left adjoint (resp. right adjoint) to the inclusion map.

THEOREM 5.1. *Let L be a complete lattice. Then the following conditions are equivalent.*

(i) *L is a* GCL.

(ii) ${}^{od}\colon U(L) \to O(L)$ *preserves up-directed suprema.*

(ii′) ${}^{-d}\colon D(L) \to C(L)$ *preserves down-directed infima.*

(iii) $(O(L), \bigcap)$ *is a continuous lattice and the* CL-*topology agrees with the* IV *topology, i.e., the* IV *topology is Hausdorff.*

(iii′) $(C(L), \bigcup)$ *is a continuous lattice and the* CL-*topology agrees with the* IV *topology.*

PROOF. Clearly (ii) and (ii′) as well as (iii) and (iii′) are equivalent.

(i) $\Rightarrow$ (ii′). For $A \subseteqq L$ let $\bar{A}$ be the closure of A in the CL-topology. Then for $A \in D(L)$ we have $\bar{A}^d = \downarrow\bar{A}$ by (2.9). Let $\{A_i : i \in I\}$ be a down-directed net of lower sets and let $a \in \bigcap \bar{A}_i^d$. Then for every Scott open neighborhood U of a we have $U \cap A_i \neq \emptyset$. As $\{A_i\colon i \in I\}$ is down-directed, the set $\{U \cap A_i\colon i \in I \text{ and } a \in U, U \text{ Scott open}\}$ forms a filter base. Let $\mathscr{F}$ be an ultrafilter containing this base. Then $A_i \in \mathscr{F}$ for every $i \in I$ and hence for every $M \in \mathscr{F}$ we have $A_i \cap M \neq \emptyset$. This implies $\inf M \leqq \inf(A_i \cap M) \in A_i$, i.e., $\inf M \in A_i$ for every $M \in \mathscr{F}$ and every $i \in I$. But this means $\inf M \in \bigcap A_i$ for every $M \in \mathscr{F}$. Therefore we can conclude $\lim \mathscr{F} = \liminf \mathscr{F} \in \overline{\bigcap A_i^d}$. On the other hand, $\mathscr{F}$ contains all Scott open neighborhoods of a. If $\lim \mathscr{F}$ would not be greater or equal to a, then we could find a finite subset $F \ll a$ with $\downarrow \lim \mathscr{F} \cap F = \emptyset$. Because $\uparrow F = \bigcup\{\uparrow f : f \in F\} \in \mathscr{F}$, we can find an $f \in F$ such that $\uparrow f \in \mathscr{F}$. But then $\lim \mathscr{F} = \operatorname{liminf} \mathscr{F} \geqq f$, a contradiction. Hence $a \leqq \lim \mathscr{F} \in \bigcap \overline{A_i^d}$. This proves $\bigcap \bar{A}_i^d \subseteqq \bigcap A_i^d$. The other inclusion is obvious.

(ii′) ⇒ (iii′) follows from (5.6) and (5.3), if we let $L' = D(L)$ and $\zeta = {}^{-d}$.

(iii′) ⇒ (i). The mapping $a \mapsto \downarrow a: L \to C(L)$ preserves arbitrary infima and up-directed suprema and $(C(L), \bigcap)$ is a GCL by (4.3). Now the result follows from (3.7).

6. CL-**Quotients of completely distributive lattices, hypercontinuous lattices.** Recall from (0.3) that a complete lattice L is a continuous lattice if and only if for each point $x \in L$ we have $x = \operatorname{liminf} \mathfrak{A}_x$, where $\mathfrak{A}_x$ denotes the filter of Scott open neighborhoods of x. If one replaces Scott open neighborhoods by neighborhoods which are open in the upper topology, one gets the following definition.

DEFINITION 6.1. A complete lattice L is called *hypercontinuous* provided that for every point $x \in L$ we have $x = \operatorname{liminf} \mathscr{L}_x$, where $\mathscr{L}_x$ denotes the neighborhood filter in the upper topology of L.

Clearly, every hypercontinuous lattice is a continuous lattice (Note that $\mathscr{L}_x \subseteqq \mathfrak{A}_x$ implies $x \leqq \operatorname{liminf} \mathscr{L}_x \leqq \operatorname{liminf} \mathfrak{A}_x \leqq x$). Before we state our main theorem on hypercontinuous lattices, let us give two definitions.

DEFINITION 6.2. Let L be a complete lattice. Then the bi-Scott topology on L is the topology generated by the Scott open sets of L and L^{op}.

DEFINITION 6.3. Let L be a complete lattice and $U \subseteqq L$ be a Scott open filter. Then we define $\operatorname{Spec}(U) = \{y : y \text{ is maximal in } L \backslash U\}$. If $k \in L$ is a compact element, we let $\operatorname{Spec}(k) = \operatorname{Spec}(\uparrow k)$.

Note that for every Scott open filter U we have $\operatorname{Spec}(U) \subseteqq \operatorname{IRR}(L)$, where $\operatorname{IRR}(L)$ denotes the set of meet irreducible elements of L, i.e., the set of those elements of L which cannot be written down as an infimum of two strictly larger elements. (see [**6**])

Theorem 6.4. *Let L be a complete lattice. Then the following conditions are equivalent*

(o) *L is a hypercontinuous lattice.*

(i) *L is a* CL-*quotient of a completely distributive lattice.*

(ii) *L is a continuous lattice and the* IV *topology and the* CL *topology agree.*

(iii) *L is meet continuous and the* IV *topology is Hausdorff.*

(iv) *L is a continuous lattice, L^{op} is a* GCL, *and the bi-Scott topology agrees with the* CL *topology.*

(v) *L is a continuous lattice and for open lower sets U, $V \subseteq L$ we have $\bar{U} \subseteq V$ if there are finitely many $a_1, \ldots, a_n \in V$ such that $U \subseteq \downarrow a_1 \cup \cdots \cup \downarrow a_n$.*

(vi) *L is a continuous lattice and for Scott open filters F_1, $F_2 \subseteq L$ with $\bar{F}_1 \subseteq F_2$ there exists a finite subset $A \subseteq \mathrm{Spec}(F_1)$ such that $\mathrm{Spec}(F_2) \subseteq \downarrow A$.*

(vii) *L is a meet continuous and generalized bicontinuous lattice.*

PROOF. (i) $\rightarrow$ (ii) is clear by (4.6) and (4.3).

(ii) $\Rightarrow$ (iii) is easy.

(iii) $\Rightarrow$ (iv). If the IV topology is Hausdorff, then L and L^{op} are GCL by (4.3). Moreover, every meet continuous GCL is a continuous lattice by (2.6), hence L is a continuous lattice. Furthermore, it follows from (5.3) that the bi-Scott topology is coarser than the CL-topology. A standard argument using the way below relation shows that the bi-Scott topology is Hausdorff. Hence the bi-Scott topology is a compact Hausdorff topology coarser than the (compact Hausdorff) CL-topology. Therefore both topologies agree.

(iv) $\Rightarrow$ (v). If the bi-Scott topology agrees with the CL topology, then the CL-topology on L^{op} is coarser than the CL topology on L. If L^{op} is a GCL, then the CL topology on L^{op} is Hausdorff, hence the compactness of both topologies yields that both topologies agree. This implies that the open down-sets of L in the CL-topology are exactly the Scott open sets of L^{op}. Furthermore $\bar{U} \subseteq V$ is equivalent to $U \ll V$ in $O(L^{op})$. Because ${}^{od}: U(L^{op}) \rightarrow O(L^{op})$ is a CL-morphism by (5.1), its right adjoint i: $O(L^{op}) \rightarrow U(L^{op})$, which is given by the inclusion map, preserves the way below relation. Hence $\bar{U} \subseteq V$ implies $U \ll V$ in $U(L^{op})$. But this is the case if and only if $U \subseteq \downarrow F$ for some finite subset $F \subseteq V$.

(v) $\Rightarrow$ (vi). Let F_1 and F_2 be Scott open filters with $\bar{F}_1 \subseteq F_2$ and set $U = L\backslash\bar{F}_2$, $V = L\backslash\bar{F}_1$. Then we have $F_2 \subseteq \mathring{\bar{F}}_2$, hence $\bar{U} = (L\backslash\bar{F}_2)^- = L\backslash\mathring{\bar{F}}_2 \subseteq L\backslash F_2 \subseteq L\backslash\bar{F}_1 = V$. Therefore we can find finitely many points $a_1, \ldots, a_n \in L\backslash\bar{F}_1$ such that $\bar{U} \subseteq \downarrow a_1 \cup \cdots \cup \downarrow a_n$. For each $a_i \in L\backslash\bar{F}_1 \subseteq L\backslash F_1$ use Zorn's lemma to pick a $b_i \in \mathrm{Spec}(F_1)$ with $a_i \leq b_i$. Then $L\backslash F_2 \subseteq \downarrow b_1 \cup \cdots \cup \downarrow b_n$, i.e., $\mathrm{Spec}(F_2) \subseteq \downarrow b_1 \cup \cdots \cup \downarrow b_n$.

(vi) $\Rightarrow$ (vii). By the assumption (vi), L is clearly meet continuous and a GCL. Moreover, if $b \nleq a$, pick open filters F_1, F_2 such that $\bar{F}_1 \subseteq F_2$, $b \in F_1$, $a \notin \bar{F}_2$. (We can do this by [9]). Then we can find a finite subset $A \subseteq \mathrm{Spec}\ F_1$ such that $\mathrm{Spec}\ F_2 \subseteq \downarrow A$. Clearly $\uparrow b \cap A = \emptyset$ and $L\backslash\bar{F}_2 \subseteq L\backslash F_2 \subseteq \downarrow\mathrm{Spec}\ F_2 \subseteq \downarrow A$. But now $A \ll a$ in L^{op}, because for a down-directed set D, $\inf D \leq a$ implies $\inf D^{\cdot} = \lim D \in L\backslash\bar{F}_2$. Therefore $d \in L\backslash\bar{F}_2 \subseteq \downarrow A$ for some $d \in D$. Hence L^{op} is a GCL. Finally, the CL-topologies on L and L^{op} agree. Let $\mathscr{F}$ be an ultrafilter on L. Then $\liminf \mathscr{F} = \limsup \mathscr{F}$, because otherwise we would have $\liminf \mathscr{F} \ngeq \limsup \mathscr{F}$. Now the above proof yields open filters F_1, F_2 such that $\bar{F}_1 \subseteq F_2$, $\limsup \mathscr{F}$

$\in F_1$, liminf $\mathscr{F} \notin \bar{F}_2$ and a finite subset $A \in \operatorname{Spec} F_1$ with $L \backslash \bar{F}_2 \subseteq \downarrow A$. As liminf $\mathscr{F}$ = lim $\mathscr{F}$ in the CL-topology on L and as lim $\mathscr{F} \in L\backslash \bar{F}_2$, we have $L\backslash F_2 \in \mathscr{F}$, hence $\downarrow A \in \mathscr{F}$. Because A is finite, we can find an $a \in A$ such that $\downarrow a \in \mathscr{F}$. Therefore limsup $\mathscr{F} \in \downarrow a$, a contradiction to $F_1 \cap \downarrow a = \varnothing$.

(vii) $\Rightarrow$ (i). Let $L' = D(L)$ and let $\zeta\colon L' \to L$ be defined by $\zeta(A) = \inf(L\backslash A)$. Then ζ is surjective and preserves arbitrary infima. Moreover, $\inf(L\backslash A) = \inf\uparrow(\overline{L\backslash A})$, where $^-$ is the closure in the CL-topology. As L is a continuous lattice by (2.6), the mapping $U \mapsto \inf(L\backslash U)$ from the lattice of open lower sets into L preserves up-directed suprema by **[4]**. Further, by (5.1) the mapping $U \mapsto U^{od} = L\backslash\uparrow(L\backslash U)^-$ from the lattice of all lower sets into the lattice of open lower sets preserves up-directed suprema. Because ζ is the composition of these two mappings, ζ will preserve up-directed suprema.

(o) $\Rightarrow$ (ii). We show that every Scott open neighborhood of a point $x \in L$ contains a neighborhood in the upper topology. So let $x \in U$ be Scott open. As x = liminf $\mathscr{L}_x$, we can find a $V \in \mathscr{L}_x$ such that inf $V \in U$. But this implies $V \subseteq U$, as desired. We now know that the upper topology and the Scott topology agree. As the IV topology is generated by the upper topology and the lower topology and the CL topology is generated by the Scott topology and the lower topology, the latter topologies agree, too.

(v) $\Rightarrow$ (o). It is enough to prove that for every point $x \in L$ the filter $\mathfrak{U}_x$ of Scott open neighborhoods is the same as the filter $\mathscr{L}_x$ of neighborhoods of x in the upper topology. So let W be a Scott open neighborhood of x. Choose Scott open neighborhoods U and V of x such that $\bar{V} \subseteq U$, $\bar{U} \subseteq W$. Then $U' = L\backslash\bar{U}$ and $V' = L/\bar{V}$ are open lower sets satisfying $\bar{U}' \subseteq V'$. By hypothesis (v) we can find a finite subset $F \subseteq V'$ such that that $U' \subseteq \downarrow F$. Now $U' = L\backslash\bar{U} \subseteq \downarrow F$ implies $L\backslash\downarrow F \subseteq \bar{U} \subseteq W$, i.e., $L\backslash\downarrow F$ is open in the upper topology and contained in W. Moreover, $L\backslash\downarrow F$ contains x, because otherwise we would have $x \in \downarrow F \subseteq L\backslash\bar{V}$, contradicting $x \in V$. This proves $\mathfrak{U}_x \subseteq \mathscr{L}_x$. The other inclusion holds in every complete lattice. This completes the proof.

COROLLARY 6.5. *Let L be an algebraic lattice. Then the following statements are equivalent.*

(a) *Conditions* (o) $\Rightarrow$ (vii) *of* (6.4) *hold.*

(b) *For every compact element* $K \in L$ *the set* Spec(K) *is finite.*

PROOF. If (a) holds, (b) follows easily from condition (vi) in (6.3). Conversely, under the assumption (b), the proof of condition (vii) of (6.3) is an easy modification of the proof (vi) $\Rightarrow$ (vii) in (6.3).

Recall that a complete lattice L is called bicontinuous, if L and L^{op}

are both continuous lattices and if the CL-topologies on L and L^{op} coincide. The following corollary is now an easy consequence of (2.6) and (6.3).

COROLLARY 6.6. *Let L be a complete meet and join continuous lattice. Then the following conditions are equivalent.*

(i) *L is bicontinuous.*

(ii) *The interval topology on L is Hausdorff.*

(iii) *L is a* CL-*quotient of a completely distributive lattice.*

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