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COMPLETENESS OF RELATIVE HOLOMORPHS OF ABELIAN GROUPS

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1. **Introduction.** A complete group is a group with trivial center and all its automorphisms are inner. These are exactly the properties needed of a group to ensure that it is a direct factor of every group containing it as a normal subgroup. Groups that are direct factor of their holomorph are either complete or a direct product of a cyclic group of order 2 and a complete group without a subgroup of index 2 [5].

The symmetric groups S_n, $n \neq 2$, $n \neq 6$, are the best known examples of complete groups [7]. If G is a direct product of non-abelian simple groups, then Aut G, the automorphism group of G, is complete [8]. Similarly, if G is finite with trivial center, then the tower of automorphism groups of G is finite and its final term is complete [10]. Many examples of complete groups are given as the holomorph of certain groups [5].

In this paper we get necessary and sufficient conditions for a relative holomorph of an abelian group A by a group Φ of automorphisms of A to be complete. Of course, these conditions also tell when the holomorph of a (necessarily) abelian group is complete. An infinite number of new infinite complete groups are then determined. There is considerable overlap in this paper to that of Rose [6]. However this paper considers the general case of relative holomorphs, whereas Rose restricts his attention to the finite case.

2. **Notation.** In this section, we list most of the notation subsequently used that may have some ambiguity. We also give appropriate references for definitions in some cases.

A	—	a group written additively, usually abelian
a, b, c	—	elements of A
Aut G	—	the group of automorphisms of a group G
Φ	—	a subgroup $\neq 1$ of Aut A
α, β	—	elements of Φ
fix Φ	—	all elements $a \in A$ such that $\alpha a = a$ for all $\alpha \in \Phi$
$A \times_\theta \Phi$	—	the relative holomorph of A by Φ [9]
α_a	—	the inner automorphism of A defined by $\alpha_a(b) = -a+b+a$
$\mathscr{Z}(G)$	—	the center of a group G

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F, F'	—	elements of Aut $(A \times_\theta \Phi)$
$\mathscr{N}(\Phi)$	—	the normalizer of Φ in Aut A **[9]**
η, λ	—	elements of $\mathscr{N}(\Phi)$
$\bar{\eta}$	—	for $\eta \in \mathscr{N}(\Phi)$, $\bar{\eta}(\alpha) = \eta\alpha\eta^{-1}$ defines $\bar{\eta} \in$ Aut Φ
$Z^1_\gamma(\Phi, A)$	—	all crossed homomorphisms from Φ to A with respect to the morphism $\gamma : \Phi \to$ Aut A **[4]**
$B^1_\gamma(\Phi, A)$	—	all principal crossed homomorphisms of $Z^1_\gamma(\Phi, A)$
$H^1_\gamma(\Phi, A)$	—	the first cohomology group when A is abelian
f, g	—	elements of $Z^1_\gamma(\Phi, A)$.

3. **The Main Results.** Since complete groups must have trivial centers, we include this usefuls reult.

PROPOSITION 3.1. $\mathscr{Z}(A \times_\theta \Phi) = \{(a, \alpha_a) \mid a \in \text{fix } \Phi, \alpha_a \in \mathscr{Z}(\Phi)\}$.

PROOF. For $(a, \alpha) \in \mathscr{Z}(A \times_\theta \Phi)$, we have

$$(a + \alpha b, \alpha\beta) = (b + \beta a, \beta\alpha)$$

for all $(b, \beta) \in A \times_\theta \Phi$. Hence $\alpha \in \mathscr{Z}(\Phi)$, and if we set $\beta = 1$, we get $a + \alpha b = b + a$, or $\alpha = \alpha_a$. Now set $b = 0$, and we get $\beta a = a$ for all $\beta \in \Phi$, and so $a \in \text{fix } \Phi$.

Conversely, take $a \in \text{fix } \Phi$ with $\alpha_a \in \mathscr{Z}(\Phi)$. For arbitrary $(b, \beta) \in A \times_\theta \Phi$, we have

$$\begin{aligned}(a, \alpha_a)(b, \beta) = (a + \alpha_a b, \alpha_a \beta) = (b + a, \beta\alpha_a) &= (b + \beta a, \beta\alpha_a)\\ &= (b, \beta)(a, \alpha_a).\end{aligned}$$

COROLLARY 3.2. $\mathscr{Z}(A \times_\theta \Phi) = \{(0, 1)\}$ *if and only if* $0 \neq a \in \text{fix } \Phi$ *implies* $\alpha_a \notin \mathscr{Z}(\Phi)$.

COROLLARY 3.3. *If A is abelian, then* $\mathscr{Z}(A \times_\theta \Phi) = \{(0, 1)\}$ *if and only if* $\text{fix } \Phi = \{0\}$.

PROPOSITION 3.4. *If A is abelian and A is a characteristic subgroup of $A \times_\theta \Phi$, then*

$$\text{Aut } (A \times_\theta \Phi) = \{F \mid F(a, \alpha) = (\eta a + f\alpha, \bar{\eta}\alpha), (\eta, f) \in \mathscr{N}(\Phi) \times Z^1_{\bar{\eta}}(\Phi, A)\}$$

and distinct pairs $(\eta, f) \in \mathscr{N}(\Phi) \times Z^1_{\bar{\eta}}(\Phi, A)$ *yield distinct* $F \in \text{Aut}(A \times_\theta \Phi)$.

NOTE. We are identifying A with $\{(a, 1) \mid a \in A\} \lhd A \times_\theta \Phi$.

PROOF. For $\eta \in \mathscr{N}(\Phi)$, $f \in Z^1_{\bar{\eta}}$, define $F(a, \alpha) = (\eta a + f\alpha, \bar{\eta}\alpha)$. Then

$$\begin{aligned}F[(a, \alpha)(b, \beta)] &= F(a + \alpha b, \alpha\beta)\\ &= (\eta(a + \alpha b) + f(\alpha\beta), \bar{\eta}(\alpha\beta))\\ &= (\eta a + \eta(\alpha b) + f(\alpha) + \bar{\eta}(\alpha) f(\beta), \bar{\eta}(\alpha\beta))\end{aligned}$$

$$= (\eta a + f(\alpha) + \bar{\eta}(\alpha)\,[\eta(b) + f(\beta)],\ \bar{\eta}(\alpha)\bar{\eta}(\beta))$$
$$= F(a, \alpha)\, F(b, \beta).$$

Here we have used the identity $\eta(\alpha b) = \bar{\eta}(\alpha)\eta(b)$.

If $(0, 1) = F(a, \alpha)$, then $\alpha = 1$ since $\bar{\eta} \in \operatorname{Aut} \Phi$. Hence $f\alpha = 0$ and $\eta(a) = 0$. But $\eta \in \mathscr{N}(\Phi)$ implies $a = 0$. We conclude F is a monomorphism. For $(b, \beta) \in A \times_\theta \Phi$,

$$F(\eta^{-1}(b - f(\bar{\eta}^{-1}(\beta))), \bar{\eta}^{-1}(\beta)) = (b, \beta),$$

so $F \in \operatorname{Aut}(A \times_\theta \Phi)$. Certainly distinct pairs (η, f) yield distinct F.

Conversely, suppose we start with $F \in \operatorname{Aut}(A \times_\theta \Phi)$. Since A is characteristic in $A \times_\theta \Phi$, we have $F(a, 1) = (\tau a, 1)$ for some monomorphism $\tau\colon A \to A$. Also, $F(0, \alpha) = (f\alpha, s\alpha)$ where $s\colon \Phi \to \Phi$ is an epimorphism and $f \in Z^1_s(\Phi, A)$. (Here we compute with $(a, \alpha) = (a, 1)(0, \alpha)$ and $(0, \alpha\beta) = (0, \alpha)(0, \beta)$.) From

$$(0, \alpha)\,(a, 1) = (\alpha a, \alpha) = (\alpha a, 1)\,(0, \alpha)$$

one gets

$$(f\alpha + s(\alpha)\tau(a), s\alpha) = (\tau(\alpha a) + f(\alpha), s\alpha)$$

and consequently

$$s(\alpha)\tau(a) = \tau(\alpha a). \tag{3.1}$$

If $s(\alpha) = 1$, then $\tau a = \tau(\alpha a)$ for all $a \in A$. Hence $\alpha = 1$, and s is an automorphism.

If $F^{-1}(a, \alpha) = (\tau' a + f'\alpha, s'\alpha)$, then

$$(a, 1) = F \circ F^{-1}(a, 1) = F(\tau' a, 1) = (\tau\tau' a, 1),$$

and we conclude that $\tau \in \operatorname{Aut} A$.

From (3.1) we get $s(\alpha) = \tau \cdot \alpha \cdot \tau^{-1}$, and so $\tau \in \mathscr{N}(\Phi)$ and $s = \bar{\tau}$. Finally $F(a, \alpha) = F[(a, 1)(0, \alpha)] = (\tau a + f\alpha, \bar{\tau}\alpha)$. Certainly distinct $F \in \operatorname{Aut}(A \times_\theta \Phi)$ yield distinct pairs (τ, f).

Proposition 3.5. *For an abelian group A and $\tau \in \Phi$, the map*

$$M_\tau\colon Z^1_{\bar{\tau}}(\Phi, A) \to Z^1_1(\Phi, A)$$

defined by $M_\tau(f) = \tau^{-1} \circ f$ is an isomorphism and induces an isomorphism between $B^1_{\bar{\tau}}(\Phi, A)$ and $B^1_1(\Phi, A)$ by restricting M_τ to $B^1_{\bar{\tau}}(\Phi, A)$.

Proof. Certainly M_τ is a morphism. From $0 = M_\tau(f) = \tau^{-1} \circ f$, we have $f = 0$ since $\tau \in \operatorname{Aut} A$. For $\bar{f} \in Z^1_1$, $\tau \circ \bar{f} \in Z^1_{\bar{\tau}}$ and $M_\tau(\tau \circ \bar{f}) = \bar{f}$. If $f(\alpha) = (1 - \bar{\tau}(\alpha))\, a$ for some $a \in A$, then $\tau^{-1} \circ f(\alpha) = (1 - \alpha)\tau^{-1}(a)$, so $M_\tau(f) \in B^1_1$. As just noted, $\bar{f}(\alpha) = (1 - \alpha)\, \bar{a}$ in Z^1_1 is the image of $f(\alpha) = (1 - \bar{\tau}(\alpha))\tau(\bar{a})$ in $Z^1_{\bar{\tau}}$.

PROPOSITION 3.6. *If A is abelian and* fix $\Phi = \{0\}$, *then*

$$\eta^*: B^1_{\bar{\eta}}(\Phi, A) \to A$$

defined by $\eta^*(f) = a$, *where* $f(\alpha) = (1 - \bar{\eta}(\alpha))a$, *is an isomorphism.*

PROOF. If $f(\alpha) = (1 - \bar{\eta}(\alpha))a = (1 - \bar{\eta}(\alpha))a'$, then $a - a' = \bar{\eta}(\alpha)(a - a')$. Since $\bar{\eta} \in \operatorname{Aut} \Phi$, $a - a' \in \operatorname{fix} \Phi = \{0\}$. Hence η^* is well defined. Trivially η^* is a morphism. For $b \in A$, $g(\alpha) = (1 - \bar{\eta}(\alpha))b$ defines $g \in B^1_{\bar{\eta}}$ and $\eta^*(g) = b$. It is direct that η^* is injective, hence η^* is an isomorphism.

We are now in a position to prove our main result.

THEOREM 3.7. *Let A be an abelian group. If $A \times_\theta \Phi$ is complete, then* 1) fix $\Phi = \{0\}$; 2) *A is a characteristic subgroup of $A \times_\theta \Phi$*; 3) $H^1_1(\Phi, A) = \{0\}$, *and* 4) $\mathcal{N}(\Phi) = \Phi$. *Conversely, if* i) fix $\Phi = \{0\}$; ii) *A is a characteristic subgroup of $A \times_\theta \Phi$, and* iii) $H^1_1(\Phi, A) = \{0\}$, *then* $\operatorname{Aut}(A \times_\theta \Phi)$ *and $A \times_\theta \mathcal{N}(\Phi)$ are isomorphic. If, in addition,* iv) $\mathcal{N}(\Phi) = \Phi$, *then $A \times_\theta \Phi$ is complete.*

NOTE. Condition ii) is somewhat unsatisfactory since it is a property of $\operatorname{Aut}(A \times_\theta \Phi)$ used to prove a property of $\operatorname{Aut}(A \times_\theta \Phi)$. This objection will be partially eliminated by theorem 3.9.

PROOF OF 3.7. If $A \times_\theta \Phi$ is complete, $\mathcal{Z}(A \times_\theta \Phi) = \{(0, 1)\}$, and so fix $\Phi = \{0\}$ by 3.3. Hence 1) is true.

For $F \in \operatorname{Aut}(A \times_\theta \Phi)$, we get $F(a, 1) = (\tau a, na)$, $F(0, \alpha) = (f\alpha, s\alpha)$, where $n: A \to \Phi$ and $s: \Phi \to \Phi$ are morphims and $f \in Z^1_s(\Phi, A)$. Also

$$F(a, \alpha) = (\tau a + n(a)f(\alpha), n(a)s(\alpha)). \tag{3.2}$$

Since F is an inner automorphism, there is a $(b, \beta) \in A \times_\theta \Phi$ such that

$$F(a, \alpha) = (b, \beta)(a, \alpha)(b, \beta)^{-1} = (b + \beta a - \beta\alpha\beta^{-1} b, \beta\alpha\beta^{-1}). \tag{3.3}$$

Hence $n(a)s(\alpha) = \beta\alpha\beta^{-1}$ and so $n(a) \equiv 1$ and $F(a, 1) = (\tau a, 1)$. This gives 2).

Since A is characteristic in $A \times_\theta \Phi$, 3.4 applies. If $\eta \in \mathcal{N}(\Phi)$, and $f \in Z^1_{\bar{\eta}}(\Phi, A)$, then $F(a, \alpha) = (\eta a + f\alpha, \eta(\alpha))$ defines an $F \in \operatorname{Aut}(A \times_\theta \Phi)$, and so, as in the above paragraph,

$$\eta a + f\alpha = b + \beta a - \beta\alpha\beta^{-1}b$$

for some $(b, \beta) \in A \times_\theta \Phi$. Setting $a = 0, f\alpha = (1 - \bar{\beta}(\alpha))b$ follows. From $\alpha = 1, \eta = \beta$ follows. Hence 3) and 4) hold.

We now consider the converse. From fix $\Phi = \{0\}$ and 3.3, we conclude $\mathcal{Z}(A \times_\theta \Phi) = \{(0, 1)\}$. Since A is characteristic in $A \times_\theta \Phi$, theorem 3.4 tells us how to construct $\operatorname{Aut}(A \times_\theta \Phi)$. For $F \in \operatorname{Aut}(A \times_\theta \Phi)$, there is a unique pair $(\eta, f) \in \mathcal{N}(\Phi) \times Z^1_{\bar{\eta}}$. Now $H^1_1(\Phi, A) = (0)$, so $Z^1_1(\Phi, A) = B^1_1(\Phi, A)$. From 3.5 we get $H^1_{\bar{\eta}}(\Phi, A) = (0)$ and $Z^1_{\bar{\eta}}(\Phi, A) = B^1_{\bar{\eta}}(\Phi, A)$. By 3.6, the unique pair (η, f) gives the unique pair $(\eta^*(f), \eta) \in A \times \mathcal{N}(\Phi)$. Thus $F \leftrightarrow (\eta^*(f), \eta)$. If $F' \leftrightarrow (\lambda^*(g), \lambda)$, $f(\alpha) = (1 - \eta(\alpha))a$, and $g(\alpha) = (1 - \lambda(\alpha))b$, then

$$F \circ F'(x, \alpha) = (\eta\lambda(x) + (\eta g + f\bar{\lambda})(\alpha), \bar{\eta}\bar{\lambda}(\alpha)).$$

But

$$\begin{aligned} [\eta g + f\bar{\lambda}](\alpha) &= \eta[b - \bar{\lambda}(\alpha)b] + (a - \bar{\eta}\circ\bar{\lambda}(\alpha)a) \\ &= a + \eta b - \bar{\eta}\bar{\lambda}(\alpha)\eta(b) - \bar{\eta}\bar{\lambda}(\alpha)(a) \\ &= (a + \eta b) - \bar{\eta}\bar{\lambda}(\alpha)(a + \eta(b)) \\ &= (1 - \bar{\eta}\bar{\lambda}(\alpha))(a + \eta(b)). \end{aligned}$$

Hence

$$F \circ F' \leftrightarrow (a + \eta b, \eta\lambda) = (a, \eta)(b, \lambda) = (\eta^*(f), \eta)(\lambda^*(g), \lambda) \in A \times_\theta \mathcal{N}(\Phi).$$

This gives an isomorphism between $\mathrm{Aut}(A \times_\theta \Phi)$ and $A \times_\theta \mathcal{N}(\Phi)$.

If $\mathcal{N}(\Phi) = \Phi$, then it is direct to see that each $F \in \mathrm{Aut}(A \times_\theta \Phi)$ is inner. That is,

$$\begin{aligned} F(a, \alpha) = (\beta a + (1 - \bar{\beta}(\alpha))b, \bar{\beta}\alpha) &= (b + \beta a - \beta\alpha\beta^{-1}b, \beta\alpha\beta^{-1}) \\ &= (b, \beta)(a, \alpha)(b, \beta)^{-1}. \end{aligned}$$

Hence $A \times_\theta \Phi$ is complete.

REMARKS 3.8. As pointed out to the author by A. Fröhlich, it is informative to look at the condition $H^1_1(\Phi, A) = (0)$ in another way. We have the semidirect product $A \times_\theta \Phi$, which can be thought of as a split short exact sequence

$$0 \longrightarrow A \longrightarrow A \times_\theta \Phi \underset{\zeta}{\overset{\mu}{\rightleftarrows}} \Phi \longrightarrow 1,$$

where $\mu(a, \alpha) = \alpha$ and ζ is a morphism such that $\mu \cdot \zeta = 1$. Now $\zeta(\alpha) = (h(\alpha), \alpha)$, and so

$$(h(\alpha\beta), \alpha\beta) = \zeta(\alpha\beta) = \zeta(\alpha)\zeta(\beta) = (h(\alpha), \alpha)(h(\beta), \beta) = (h(\alpha) + \alpha h(\beta), \alpha\beta).$$

Hence, $h \in Z^1_1(\Phi, A)$. Since $H^1_1(\Phi, A) = (0)$, there is a $c \in A$ such that $h(\alpha) = (1 - \alpha)c$. Consequently,

$$\zeta(\alpha) = ((1 - \alpha)c, \alpha) = (c, \alpha)(-c, 1) = (c, 1)(0, \alpha)(c, 1)^{-1}.$$

Consequently, up to conjugation by elements of A, there is only one way to define the required splitting map ζ.

This leads to an observation that will be used later. *The condition* $H_1^1(\Phi, A) = (0)$ *is equivalent to the condition that any complement to* A *in* $A \times_\theta \Phi$ *is a conjugate of* Φ *in* $A \times_\theta \Phi$ *by an element of* A. This is exactly Satz 17.3b), Kapitel *I* in **[2]**, or it can easily be proved directly.

The following theorem is useful for constructing examples of complete groups, when used in conjunction with 3.7.

THEOREM 3.9. *In addition to* A *being abelian, suppose one of the following is true*:

i) Φ *is regular (i.e.,* $A \times_\theta \Phi$ *is a Frobenius group), and* A *is finite*;
ii) Hom $(A, \Phi) = \{0\}$;
iii) Φ *is abelian, and* $1\text{-}\alpha \in \operatorname{Aut} A$ *for some* $\alpha \in \Phi$;
iv) Φ *has exactly two orbits,* $\{0\}$ *and* $A \backslash \{0\}$;
v) $(|A|, |\Phi|) = 1$.

Then A *is characteristic in* $A \times_\theta \Phi$. *Moreover, if* v) *is true, then* $H_1^1(\Phi, A) = \{0\}$ *also.*

PROOF. Theorem V. 8.3 of **[2]** gives our result if i) is valid. If $F \in \operatorname{Aut}(A \times_\theta \Phi)$ and $F(a, 1) = (\tau(a), \eta(a))$ as in (3.2), then $\eta: A \to \Phi$ is a morphism. If ii) is valid, then $\eta(a) \equiv 1$, and so $F(a, 0) = (\tau a, 1)$ and A is characteristic in $A \times_\theta \Phi$. Otherwise,

$$\begin{aligned}(f(\alpha) + s(\alpha)\tau(a), s(\alpha)\eta(a)) &= (f(\alpha), s(\alpha))(\tau(a), \eta(a))\\ &= f[(0, \alpha)(a, 1)] = f[(\alpha a, 1)(0, \alpha)]\\ &= (\tau(\alpha a) + \eta(\alpha a)f(\alpha), \eta(\alpha a)s(\alpha)).\end{aligned}$$

Consequently,

$$s(\alpha)\eta(a) = \eta(\alpha a)s(\alpha). \tag{3.4}$$

If iii) is true, then (3.4) reduces to

$$\eta(a) = \eta(\alpha a)$$

for all $\alpha \in \Phi$ and all $a \in A$. Let $b \in A, \alpha \in \Phi$ such that $1 - \alpha \in \operatorname{Aut} A$ and $a \in A$ such that $b = (1 - \alpha)a$. Then

$$\eta(b) = \eta[(1 - \alpha)a] = \eta(a)\eta(-\alpha a) = \eta(a)\eta(\alpha a)^{-1} = \eta(a)\eta(a)^{-1} = 1.$$

Again, this shows that A is characteristic in $A \times_\theta \Phi$.

If iv) is true, then A is a minimal normal subgroup of $A \times_\theta \Phi$. For, if $(0, 1) \neq (b, 1) \in B \subset A$, then $(a, \alpha)(b, 1)(a, \alpha)^{-1} = (\alpha b, 1)$.

If B were another minimal normal subgroup of $A \times_\theta \Phi$, then $A \cap B = \{(0, 1)\}$, so the elements of B commute with the elements of A. Hence, B is contained in the centralizer $C_{A\times_\theta\Phi}(A)$ of A. Let $(b, \beta) \in C_{A\times_\theta\Phi}(A)$. Then

$$(a, 1)(b, \beta) = (b, \beta)(a, 1)$$

for all $(a, 1)$. From this it follows that

$$\beta a = a$$

for all $a \in A$. Hence $\beta = 1$. Consequently, $C_{A \times_\theta \Phi}(A) = A$, and so we have the contradiction that $B \subseteqq A$. We conclude that A is the only minimal normal subgroup of $A \times_\theta \Phi$. If A were not characteristic in $A \times_\theta \Phi$, then there would be more than one minimal normal subgroup.

Suppose now that v) is true. Then ii) is true and we have A characteristic in $A \times_\theta \Phi$. By the Schur-Zassenhaus theorem, [7, Th. 7.15] and [9, Th. 9.3.2], every extension of A by Φ is a semidirect product and any two complements of A in $A \times_\theta \Phi$ are conjugate. The remark before theorem 3.9 allows us to conclude that $H_1^1(\Phi, A) = \{0\}$.

For an interesting paper on the automorphism group of relative holomorphs, the reader is referred to a paper by Hsu [1].

4. **Applications.** Let A be a vector space over a field K, and let Φ be all left multiplications by elements of $K^* = K\backslash\{0\}$. Then fix $\Phi = \{0\}$ if $|K| \neq 2$. Hence $\mathscr{Z}(A \times_\theta \Phi) = \{(0, 1)\}$. Certainly 3.9 iii) is valid, so A is characteristic in $A \times_\theta \Phi$.

Suppose $f \in Z_1^1(\Phi, A)$. Then

$$f(\alpha) + \alpha f(\beta) = f(\alpha\beta) = f(\beta\alpha) = f(\beta) + \beta f(\alpha)$$

for all $\alpha, \beta \in \Phi = F^*$. Fix $\beta \neq 1$. Then

$$f(\alpha) = (1 - \alpha)[f(\beta)(1 - \beta)^{-1}]$$

and so $f \in B_1^1(\Phi, A)$ and $H_1^1(\Phi, A) = \{0\}$.

If the dimension of A over K is greater than 1, then $\mathscr{N}(\Phi)$ contains Φ properly. So we will restrict our attention to $A = K^+$, the additive group of K; i.e., A is 1-dimensional.

For $\tau \in \operatorname{Aut} K$, the group of all field automorphisms of K, we have $\tau(ab) = \tau(a)\tau(b)$. So if $\tau \in \Phi$, we would have $\tau^2 = \tau$, hence $\tau = 1$. Hence $\Phi \cap \operatorname{Aut} K = \{1\}$. For $\tau \neq 1$ and $\alpha \in \Phi$, $\tau(\alpha b) = \tau(\alpha)\tau(b)$. Hence $\tau(\alpha) = \tau \cdot \alpha \cdot \tau^{-1}$ and $\tau \in \mathscr{N}(\Phi)$. (Here we are identifying elements of Φ as left multiplication mappings with elements of F^*.) Conversely, if $\tau \in \mathscr{N}(\Phi)\backslash\Phi$, then $\hat{\tau}(a) = \tau \cdot a \cdot \tau^{-1}$ defines $\hat{\tau} \in \operatorname{Aut} K$. In summary we have the following theorem.

THEOREM 4.1. *Let A be a vector space over a field K of more than two elements, and let Φ be the automorphisms of A given by multiplying by non-zero elements of K. Then* $\operatorname{Aut}(A \times_\theta \Phi) \cong A \times_\theta \mathscr{N}(\Phi)$ *and* $\mathscr{Z}(A \times_\theta \Phi) = \{(0, 1)\}$. *Hence $A \times_\theta \Phi$ is complete if and only if A is* 1*-dimensional and K has only one field atuomorphism.*

Let $K^+ \times_\theta K^*$ denote the group obtained by extending the additive group K^+ of K by the multiplicative group K^* of K with the operation $(a, \alpha)(b, \beta) = (a + \alpha b, \alpha\beta)$, and let Aut K denote the field automorphisms of K. Then we have the following corollary.

COROLLARY 4.2. *If* $|K| \neq 2$, *then the short exact sequence*

$$1 \to K^+ \times_\theta K^* \to \mathrm{Aut}(K^+ \times_\theta K^*) \to \mathrm{Aut}\ K \to 1$$

is a split exact sequence.

PROOF. In 4.1, $A = K^+$ and $\Phi = K^*$, and

$$\mathrm{Aut}(K^+ \times_\theta K^*) \cong K^+ \times_\theta \mathscr{N}(K^*).$$

Hence the insertion $1 \to K^+ \times_\theta K^* \to K^+ \times_\theta \mathscr{N}(K^*)$ gives us part of our desired result.

Define ρ: $K^+ \times_\theta \mathscr{N}(K^*) \to \mathrm{Aut}\ K$ by $\rho(a, \tau) = \hat{\tau}$ where $\hat{\tau}(a) = \tau \cdot a \cdot \tau^{-1}$ if $a \neq 0$, $\hat{\tau}(0) = 0$. (Again, we identify elements $a \in K^*$ with left multiplication mappings of K^+, and since $\tau \in \mathscr{N}(K^*)$, $\tau \cdot a \cdot \tau^{-1}$ can be identified with an element of K^* if $a \in K^*$.) It is direct to see that ρ is a morphism and it is easy to see that the kernel of ρ is $K^+ \times_\theta K^*$. Hence our sequence is short exact.

The map γ: Aut $K \to K^+ \times_\theta \mathscr{N}(K^*)$ defined by $\gamma(\tau) = (0, \tau)$ is a morphism since Aut $K \subseteqq \mathscr{N}(K^*)$. We have $\rho\gamma = 1$, hence the short exact sequence splits.

COROLLARY 4.3. *The normalizer* $\mathscr{N}(K^*)$ *of* K^* *in* Aut K^+ *is a semidirect product of* K^* *by* Aut K; *i.e.*,

$$1 \to K^* \to \mathscr{N}(K^*) \to \mathrm{Aut}\ K \to 1$$

is split exact.

PROOF. The maps ρ and γ in the proof of 4.2 must be modified in the obvious way.

As a consequence of 4.1 and 4.2, the group $K^+ \times_\theta K^*$, for $|K| > 2$, is complete if and only if there is only the identity automorphism of the field K. Fields with only one field automorphism include all the prime fields and all the real closed fields [**3**, theorem XI. 3]. The real closed fields seem to provide new examples of complete groups. Also the following class of examples seem to provide new examples of complete groups.

Let **Q** be the rationals, so **Q** is left fixed by any automorphism of a field containing **Q**. Let A be any subset of rational numbers. To each $a \in A$ let n_a be an odd positive integer and let α_a be the real n_a-th root of a. If $B = \{\alpha_a | a \in A\}$, then $\mathbf{Q}(B)$ is a field with exactly one automorphism, hence $\mathbf{Q}(B)$ yields a complete group.

Suppose f is an automorphism of $\mathbf{Q}(B)$. Then $f(r) = r$ for all $r \in \mathbf{Q}$. For $\alpha_a \in B$, $\alpha_a^{n_a} - a = 0$, hence $0 = f(0) = f(\alpha_a)^{n_a} - a$ and so

$$f(\alpha_a) \in \{\alpha_a, \alpha_a\xi, \alpha_a\xi^2, \dots, \alpha_a\xi^{n_a-1}\}$$

where $\xi = \cos(2\pi/n_a) + i\sin(2\pi/n_a)$. Since n_a is odd and since $f(\alpha_a)$ is real, it must be that $f(\alpha_a) = \alpha_a$. Since f fixes each element of $\mathbf{Q} \cup B$, f fixes each element of $\mathbf{Q}(B)$, and so $f = 1$.

If K is a field with more than two elements, then

$$\operatorname{Aut}(K^+ \times_\theta K^*) \cong K^+ \times_\theta (K^* \times_\theta \operatorname{Aut} K).$$

In **[6**, Coro 5.6], John Rose has shown that this group is complete if K is also finite. The referee of this paper has shown that this group is complete also when K is infinite. His proof is an application of our main theorem, theorem 3.7, and it is essentially given blow.

THEOREM 4.4. *Let K be a field of more than two elements. Then $\operatorname{Aut}(K^+ \times_\theta K^*)$ is complete.*

PROOF. We have that

$$\operatorname{Aut}(K^+ \times_\theta K^*) \cong K^+ \times_\theta (K^* \times_\theta \operatorname{Aut} K).$$

Let $\Phi = K^* \times_\theta \operatorname{Aut} K$, so K^* is a subgroup of Φ, hence fix $\Phi = \{0\}$. By theorem 3.9 iv), K^+ is characteristic in $K^+ \times_\theta \Phi$. So by theorem 3.7, we only need to prove that $H_1^1(\Phi, K^+) = \{0\}$ and $\mathscr{N}(\Phi) = \Phi$.

In order to show that $\mathscr{N}(\Phi) = \Phi$, we will first show that K^* is characteristic in $K^* \times_\theta \operatorname{Aut} K$. If K^* were not characteristic in $\Phi = K^* \times_\theta \operatorname{Aut} K$, then Φ would have a normal abelian subgroup L such that $L \nleqq K^*$. So $K^* < K^* L$ and so $K^*L = K^*B$ where $1 \neq B = (\operatorname{Aut} K) \cap K^*L$.

Let F be the subfield of K consisting of all elements fixed by every member of B. Then $F^* = \mathscr{Z}(K^*B)$. Now K^* and L are abelian normal subgroups of Φ, so $K^* \cap L \leqq \mathscr{Z}(K^*B)$. Hence

$$\begin{aligned}(k^*\ell)(k_1^*\ell_1)(K^* \cap L) &= [k^*(K^* \cap L)][\ell k_1^*(K^* \cap L)][\ell_1(K^* \cap L)] \\ &= [k^*(K^* \cap L)][k_1^*\ell(K^* \cap L)][\ell_1(K^* \cap L)] \\ &= (k_1^*\ell_1)(k^*\ell)(K^* \cap L)\end{aligned}$$

for all $k^*, k_1^* \in K^*$, all $\ell, \ell_1 \in L$, since $k^*\ell(K^* \cap L) = \ell k^*(K^* \cap L)$. This shows that $K^*L/K^* \cap L$ is abelian.

Let $1 \neq \tau \in B$. For $x \in K^*$,

$$[(K^* \cap L)(x^{-1}\tau)][(K^* \cap L)(x1)] = [(K^* \cap L)(x1)][(K^* \cap L)(x^{-1}\tau)].$$

Hence $(K^* \cap L)(x^{-1}\tau(x))\tau = (K^* \cap L)(1\tau)$, and so

$$x^{-1}\tau(x) \in K^* \cap L \leqq \mathscr{Z}(K^*B) = F^*.$$

Thus for each $x \in K$, there is an $a \in F$ such that $\tau(x) = ax$. Considering K as a vector space over F, then τ is a linear transformation of K onto K such that every element of K is an eigenvector of τ. So τ is a scalar transformation, and since $\tau(1) = 1$, we have $\tau = 1$, a contradiction. Thus we conclude that K^* is characteristic in $K^* \times_\theta \mathrm{Aut}\, K$.

Since K^* is characteristic in $K^* \times_\theta \mathrm{Aut}\, K$, $f \in \mathscr{N}(K^* \times_\theta \mathrm{Aut}\, K)$ implies $f \in \mathscr{N}(K^*)$. We have seen in corollary 4.3 that

$$\mathscr{N}(K^*) \cong K^* \times_\theta \mathrm{Aut}\, K.$$

Hence $\mathscr{N}(\Phi) = \Phi$.

There remains to prove that $H_1^1(\Phi, K^+) = \{0\}$. As shown in the remarks of 3.8, this is equivalent to showing that if L is any complement to K^+ in $K^+ \times_\theta \Phi$, then L is a conjugate of Φ by an element of K^+.

In the following argument it is useful to note that

$$K^+ \times_\theta (K^* \times_\theta \mathrm{Aut}\, K) \cong (K^+ \times_\theta K^*) \times_\theta \mathrm{Aut}\, K$$

and so we represent elements of G by $(a^+, a^*, \alpha) \in K^+ \times K^* \times \mathrm{Aut}\, K$ where multiplication is defined by

$$(a^+, a^*, \alpha)(b^+, b^*, \beta) = (a^+ + a^*\alpha(b^+), a^*\alpha(b), \alpha\beta)$$

and

$$(a^+, a^*, \alpha)^{-1} = (\alpha^{-1}(a^{*-1})\alpha^{-1}(-a^+), \alpha^{-1}(a^{*-1}), \alpha^{-1}).$$

Subgroups K^+, K^*, Aut K are identified by elements of the form $(a^+, 1, 1)$, $(0, a^*, 1)$, and $(0, 1, \alpha)$, respectively.

Let L be a complement to K^+ in $K^+ \times_\theta \Phi$. Then $L \cap (K^+ \times_\theta K^*)$ is a complement to K^+ in $K^+ \times_\theta K^*$. Now $H_1^1(K^*, K^+) = \{0\}$ by an argument like that at the beginning of §4. Hence K^* and $L \cap (K^+ \times_\theta K^*)$ are conjugate in $K^+ \times_\theta K^*$ by an element of K^+.

So we can, and do, replace L by a conjugate by an element of K^+.

Since $K^+ \times_\theta K^*$ is normal in $(K^+ \times_\theta K^*) \times_\theta \mathrm{Aut}\, K$, and $L \subseteqq \mathscr{N}(K^*)$, and $\mathscr{N}(K^*) = K^* \times_\theta \mathrm{Aut}\, K$, we conclude that $L = K^* \times_\theta \mathrm{Aut}\, K$ and hence $H_1^1(\Phi, K^+) = \{0\}$. This completes the proof of the theorem.

The development so far provides an infinite number of new complete groups, but they are all infinite. We will now see that the theory also provides an infinite number of finite complete groups. These are special cases of those given by Rose [**6**. Th. 5.2].

Let us study the requirements of theorem 3.7. We need *a* Φ such that $\mathscr{N}(\Phi) = \Phi$. If Q is a Sylow subgroup and $\Phi \supset \mathscr{N}(\mathrm{Q})$, then $\mathscr{N}(\Phi) = \Phi$.

So, as candidates for Φ, let $\Phi = \mathcal{N}(Q)$ for some Sylow subgroup Q. Let $F = GF(p^r)$, the Galois field of p^r elements, with $p^r > 3$ and p an odd prime. Let $q \neq p$ also be an odd prime where $q \mid (p^{2r} - 1)$. Since $q \mid |GL(2, p^r)|$, there is a Sylow q-subgroup Q of $GL(2, p^r)$. Let $\Phi = \mathcal{N}(Q)$. Since $\mathcal{Z}(GL(2, p^r)) \subseteqq \Phi$ we have fix $\Phi = \{0\}$. There remains to show that $A = F^+ \oplus F^+$ is characteristic in $A \times_\theta \Phi$ and the $H_1^1(\Phi, A) = \{0\}$. By theorem 3.9, it is enough to show $(|A|, |\Phi|) = 1$, where $|G|$ denotes the order of a group G.

Now $SL(2, p^r)$ contains all elements of order p in $GL(2, p^r)$. Since $|SL(2, p^r)| = (p^{2r} - 1)p^r$, we know that $SL(2, p^r)$ has elements of order a positive power of q from $GL(2, p^r)$. Now consider $PSL(2, p^r)$. We have that $|PSL(2, p^r)| = \frac{1}{2}(p^{2r} - 1)p^r$, so $q \mid |PSL(2, p^r)|$. Now $|A| = p^{2r}$, and if $p \mid |\Phi|$, then some element of order p in $PSL(2, p^r)$ would be in the normalizer of the Sylow q-subgroup of $PSL(2, p^r)$. That this cannot be is an immediate consequence of Satz 8.10, Hilfssatz 8.22, and Hauptsatz 8.27 in Kapitel II of [2]. Hence, $A \times_\theta \Phi$ is complete.

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