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SOME EXAMPLES OF ALGEBRAIC DEGENERACY AND HYPERBOLIC MANIFOLDS

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1. **Introduction.** Let D be an algebraic curve in the complex projective space $\mathbf{P}^2$ of dimension 2. We shall call a non-constant holomorphic mapping from the complex line $\mathbf{C}$ to the manifold $\mathbf{P}^2$-D a *holomorphic curve in* $\mathbf{P}^2$-D. A holomorphic curve f in $\mathbf{P}^2$-D is called *algebraically degenerate* if the image $f(\mathbf{C})$ lies in an algebraic curve in $\mathbf{P}^2$. It is conjectured by M. L. Green [**2**, **3**, **4**] that *for any D with normal crossings and of degree d at least 4, any holomorphic curve in* $\mathbf{P}^2$-D *is algebraically degenerate.*

We shall first give some examples of algebraic degeneracy in the case of $d = 4$ (Examples 1, 2 and 3 in section 2).

Next using a result of N. Toda [7] we shall give an example of D for which there is no holomorphic curve in $\mathbf{P}^2$-D (Theorem in section 3). Consequently we shall have an example of a complete hyperbolic manifold of the form $\mathbf{P}^2$-D where D is non-singular (Proposition in section 4).

2. **Examples of Algebraic Degeneracy.** We shall give three new examples of D and f where D is an algebraic curve in $\mathbf{P}^2$ with degree 4 and f is a holomorphic curve in $\mathbf{P}^2$-D and the image $f(\mathbf{C})$ lies in an algebraic curve.

In what follows we use (z_0, z_1, z_2) for the homogeneous coordinate system of $\mathbf{P}^2$. In the following examples we use k for an arbitrary non-constant entire function.

EXAMPLE 1. Let D be defined by $(z_0^2 + z_1^2)^2 + (z_0^2 + z_2^2)^2 = 0$ and f be defined by $(1 + i, (1 + i)(e^k - e^{-k})/\sqrt{2}, e^k - ie^{-k})$. Then the image $f(\mathbf{C})$ lies in the conic $z_0^2 + z_1^2 + z_2^2 - \sqrt{2}\, z_1 z_2 = 0$.

EXAMPLE 2. Let D be defined by $z_0(z_0^3 + z_1^3 + z_2^3) = 0$ and f be defined by $(9e^{4k}, -9e^{4k} + 3e^k, -9e^{3k} + 1)$. Then the image $f(\mathbf{C})$ lies in the quartic $9z^0 z_2^3 = (-2z_0 + z_1)^3 (z_0 + z_1)$.

Let D be as in Example 2. A trivial example of f is defined by $(1, k, \sqrt[3]{-1}\, k)$. Then the image $f(\mathbf{C})$ lies in the line $\sqrt[3]{-1}\, z_1 = z_2$.

EXAMPLE 3. Let D be the Fermat curve $z_0^4 + z_1^4 + z_2^4 = 0$ and f be de-

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fined by ($\sqrt[4]{2}$ ($\sin^2 k - \cos^2 k + i \sin k \cos k$), α ($\sin^2 k + 2i \sin k \cos k$), $\alpha(-\cos^2 k + 2i \sin k \cos k)$) where $\alpha^2 = i$. Then the image $f(\mathbf{C})$ lies in the conic $iz_0^2/\sqrt{2} - z_1^2 - z_2^2 + z_1 z_2 = 0$.

3. **Main Theorem.** We shall consider an algebraic curve D_ε^d in $\mathbf{P}^2$ of even degree d, with a parameter ε of non-zero complex number, defined by the equation

$$z_0^d + z_1^d + z_2^d + \varepsilon (z_0 z_1)^{d/2} + \varepsilon (z_0 z_2)^{d/2} = 0.$$

By calculation we have

(i) D_ε^d is non-singular if and only if ε^2 is not 2 nor 4,

(ii) D_ε^d is reducible if $\varepsilon^2 = 2$ or if $\varepsilon^2 = 4$ and d is divisible by 4.

Our main result is the following:

THEOREM. *Let D_ε^d be as above. Suppose that D_ε^d satisfies one of the conditions*

(1) $\varepsilon^2 \neq 4$ *and* $d \geqq 30$,

(2) $\varepsilon^2 = 2$ *and* $d \geqq 14$.

Then there is no holomorphic curve in $\mathbf{P}^2$-D_ε^d.

In proving the theorem we shall use two lemmas.

LEMMA 1. *Let P, Q be polynomials of one variable such that $P(0) = 0$, $Q(0) = 0$ and n_i ($i = 1, \ldots, r$)($r \geqq 0$) be positive integers. Suppose*

$$\sum_{i=1}^{r} \frac{1}{n_i} < \frac{1}{\|P\| + \|Q\| + r - 1}$$

where $\|P\|$, $\|Q\|$ are the numbers of the monomials included in P, Q respectively. Then for any entire solution $(g_0, \ldots, g_r)$ of the functional equation

$$P(e^{g_0}) + Q(e^{-g_0}) + \sum_{i=1}^{r} g_i^{n_i} = 1,$$

at least one of the g_i is constant.

PROOF. If $P = Q = 0$ the assertion is a corollary of Theorem 1 in [7]. Generally we can find a positive integer n_0 such that the inequality

$$\frac{\|P\| + \|Q\|}{n_0} + \sum_{i=1}^{r} \frac{1}{n_i} < \frac{1}{\|P\| + \|Q\| + r - 1}$$

holds. Then we can apply the result for the case of $P = Q = 0$.

LEMMA 2. *Any entire solution of the functional equation $g_0^2 + g_1^2 = 1$ is of the form $g_0 = (e^h + e^{-h})/2$, $g_1 = (e^h - e^{-h})/2i$ where h is an entire function.*

PROOF OF THEOREM. We take any holomorphic mapping $f: \mathbf{C} \to \mathbf{P}^2 - D^d_\varepsilon$. We shall show that f is a constant mapping. Now f is written by (f_0, f_1, f_2) where f_i are entire functions not vanishing at the same time. By assumption we have

$$f_0^d + f_1^d + f_2^d + \varepsilon(f_0 f_1)^{d/2} + \varepsilon(f_2 f_0)^{d/2} = e^h$$

where h is an entire function. Considering $f_i e^{-h/d}$ in stead of f_i we may assume

$$f_0^d + f_1^d + f_2^d + \varepsilon(f_0 f_1)^{d/2} + \varepsilon(f_0 f_2)^{d/2} = 1. \tag{3.1}$$

PART (1): Suppose $\varepsilon^2 \neq 4$ and $d \geqq 30$. Applying Lemma 1 to the functional equation

$$g_1^d + g_2^d + g_3^d + g_4^{d/2} + g_5^{d/2} = 1,$$

we have by (3.1) and $d > 28$ that at least one of the functions $f_0, f_1, f_2, f_0 f_1, f_0 f_2$ is constant. In each case we examine as follows.

(a) $f_0 = c$(const.): By Lemma 1 we may assume $c^d = 1$. Then

$$(f_1^{d/2} + \varepsilon'/2)^2 + (f_2^{d/2} + \varepsilon'/2)^2 = \varepsilon'^2/2 (\neq 0)$$

where $\varepsilon' = \varepsilon c^{d/2}$. By Lemmas 2 and 1 we have that f_1 and f_2 are constant.

(b) f_1 or $f_2 = c$ (const.): By symmetry we may assume $f_1 = c$. Suppose $c^d \neq 1$. We may assume by Lemma 1 that $f_0 f_2 = c_1$ (const.) and $c_1 \neq 0$. Writing $f_0 = e^h$, $f_2 = c_1 e^{-h}$ where h is an entire function and applying Lemma 1, we have that h is constant and so are f_0 and f_2. If $c^d = 1$ (3.1) implies

$$(f_2^{d/2} + \varepsilon f_0^{d/2}/2)^2 + \varepsilon''(f_0^{d/2} + \varepsilon'/2\varepsilon'')^2 = \varepsilon'^2/4\varepsilon''$$

where $\varepsilon' = \varepsilon c^{d/2}$, $\varepsilon'' = 1 - \varepsilon^2/4$ ($\neq 0$, by assumption (1)). By the same argument as (a) we can show that f_0 and f_2 are constant.

(c) $f_0 f_1$ or $f_0 f_2 = c$ (const.): By symmetry we may assume $f_0 f_1 = c$. By (a) and (b) we may assume $c \neq 0$. Denote $f_0 = e^h$, $f_1 = ce^{-h}$ where h is an entire function. If $\varepsilon c^{d/2} \neq 1$, applying Lemma 1 we have that f_2 or $e^h f_2$ or h is constant. Then f_0, f_1 and f_2 are constant. If $\varepsilon c^{d/2} = 1$ we have

$$1 + (ce^{-2h})^d + (f_2 e^{-h})^d + \varepsilon(f_2 e^{-h})^{d/2} = 0.$$

By Lemma 1 we have that h or $f_2 e^{-h}$ is constant. Then f_0, f_1 and f_2 are constant.

We have proved part (1) of the theorem.

PART (2): Suppose $\varepsilon^2 = 2$ and $d \geqq 14$. By a linear change of the coordinate system D^d_ε is reduced to the reducible curve defined by

$$(z_0^{d/2} + z_1^{d/2})^2 + (z_0^{d/2} + z_2^{d/2})^2 = 0.$$

With respect to the new coordinate system the functional equation (3.1) is

$$(f_0^{d/2} + f_1^{d/2})^2 + (f_0^{d/2} + f_2^{d/2})^2 = 1.$$

By Lemma 2 we have

$$(1 + i) f_0^{d/2} + f_1^{d/2} + i f_2^{d/2} = e^h, \tag{3.2}$$

$$(1 - i) f_0^{d/2} + f_1^{d/2} - i f_2^{d/2} = e^{-h} \tag{3.3}$$

where h is an entire function. Since $d > 12$, applying Lemma 1 to (3.2) we obtain that at least one of $f_i e^{-2h/d}$ is constant. As we can do the same argument for the others we may assume $f_0 e^{-2h/d} = c$ (const.). If $(1 + i) c^{d/2} \neq 1$, by Lemma 1 we have that $f_1 e^{-2h/d}$ and $f_2 e^{-2h/d}$ are constant, hence f is a constant mapping. Suppose $(1 + i) c^{d/2} = 1$. Eliminating f_0 and f_1 from (3.3) we have

$$-ie^{2h} - 2i\,(f_2 e^{2h/d})^{d/2} = 1.$$

By Lemma 1 we have that h and $f_2 e^{2h/d}$ are constant. Hence f_0, f_1 and f_2 are constant. We have proved part (2) of the theorem.

4. **An Example of a Complete Hyperbolic Manifold.** From the theorem in the previous section and Theorem 2 in [5] we obtain the

PROPOSITION. *Let D_ε^d be as in the theorem. Suppose that D_ε^d satisfies one of the conditions*

(1) *ε^2 is not 2 nor 4 and $d \geqq 30$,*

(2) *$\varepsilon^2 = 2$ and $d \geqq 14$.*

Then $\mathbf{P}^2$-D_ε^d is a complete hyperbolic manifold in the sense of Kobayashi [6].

We have an example of a complete hyperbolic manifold of the form $\mathbf{P}^2$-D where D is non-singular ((1) in the proposition).

If $\varepsilon^2 = 2$ and $d = 4$, $\mathbf{P}^2$-D_ε^d is not a hyperbolic manifold (Example 1 in the section 2).

By the use of the theorem in [1] we obtain another proof of part of the proposition:

For sufficiently small ε, D_ε^d is non-singular and $\mathbf{P}^2$-D_ε^d is a complete hyperbolic manifold provided $d \geqq 50$.

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