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UNIFORM DIFFERENTIABILITY, UNIFORM ABSOLUTE CONTINUITY AND THE VITALI-HAHN-SAKS THEOREM

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1. **Introduction.** In the following paper we continue the study of the relationship between differentiability of the norm in spaces of measures and absolute continuity which was initiated in [3] and pursued in [4], [8], and [22]. The first portion of this discussion is concerned with the nature of smooth points in abstract L-spaces and characterizations of uniform differentiability—uniform absolute continuity. Our results are obtained by using elementary norm-lattice properties and without using the deep theorem of Kakutani [20] on the concrete representation of L-spaces. Only briefly in the concluding section of the paper do we make use of this representation theorem. We do, however, use some of the terminology and preliminary lemmas of [20]. In particular, we shall refer to units rather than weak order units [26], [29]. The interested reader may want to refer to Schaefer [29] or Peressini [26] for results on units in the setting of ordered topological vector spaces. We note that a concept related to that of units, namely the notion of cyclic vectors and cyclic subspaces, is discussed in Lindenstrauss and Tzafriri [24].

As a corollary of our discussion of uniform differentiability, we obtain the following geometrical characterization of weak compactness in the space of bounded finitely additive measures defined on a ring: (*) A subset K of ba$(\mathscr{R})$ is conditionally weakly compact if and only if there is a point $\mu \in$ ba$(\mathscr{R})$ so that the derivative of the norm at μ in the direction ν exists uniformly for $\nu \in K$. In keeping with the theme of this paper, the preceding result is stated in terms of uniform differentiability—uniform absolute continuity.

The question of whether condition (*) characterizes weak compactness in the space $L^1(\mu, X)$ of Bochner integrable functions (as well as in arbitrary Banach spaces) is also discussed in some detail. While neither implication is valid in general, an easy uniform differentiability result for compact subsets of arbitrary Banach spaces is established. This result is used to produce a new proof of the classical Vitali-Hahn-Saks Theorem, a major theorem long of interest to measure theorists and functional analysts and the focal point for the next section of the paper. We note

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here that Diestel and Uhl **[12]** have recently given a very thorough discussion of the Vitali-Hahn-Saks theorem. In **[12]** the authors point out that the version of the Vitali-Hahn-Saks theorem in Brooks and Jewett **[7]** may be reduced to the scalar case and use a result of Rosenthal as a principal tool in their development of this and related results. In this section, we produce a lattice theoretic proof of a theorem due independently to Brooks **[6]** and Drewnowski **[14]**—a result which we feel has not been properly appreciated—to produce a simplified proof of the Vitali-Hahn-Saks theorem without the strong boundedness assumption present in **[7]**. The equivalence of the so called Brooks-Jewett Theorem and the Phillips Lemma is also established.

In the concluding section of the paper, we characterize those L-spaces with units (producing, in the process, a simple argument to show that an order interval is weakly compact in an L-space), and we discuss a question raised by preceding results.

Some useful terminology and notation follow. All Banach spaces are defined over the real field. If $\mathscr{R}$ is a ring of sets and X is a Banach space, then $\mathrm{ba}(\mathscr{R}, X)$ is the Banach space of all bounded, finitely additive X-valued set functions defined on $\mathscr{R}$ equipped with the supremum norm, i.e., if $\mu \in \mathrm{ba}(\mathscr{R}, X)$, then $\|\mu\| = \sup\{\|\mu(A)\| : A \in \mathscr{R}\}$. The countably additive members of $\mathrm{ba}(\mathscr{R}, X)$ are denoted by $\mathrm{ca}(\mathscr{R}, X)$. If μ is a finitely additive X-valued set function defined on $\mathscr{R}$, then μ is said to be strongly additive (s-additive) if $\mu(A_i) \to 0$ whenever (A_i) is a disjoint sequence from $\mathscr{R}$; the totality of all s-additive X-valued set functions defined on $\mathscr{R}$ is denoted by $\mathrm{sa}(\mathscr{R}, X)$. It should be noted that $\mathrm{sa}(\mathscr{R}, X)$ forms a closed linear subspace of the Banach space $\mathrm{ba}(\mathscr{R}, X)(\|\mu\| = \sup\{\|\mu(A)\| : A \in \mathscr{R}\})$ and that $\sum \mu(A_i)$ converges unconditionally in X whenever (A_i) is a disjoint sequence from $\mathscr{R}$ and $\mu \in \mathrm{sa}(\mathscr{R}, X)$. (An application of the Orlicz-Pettis Theorem yields the unconditional convergence.) The members of $\mathrm{sa}(\mathscr{R}, X)$ have been referred to as exhaustive or strongly bounded measures **[5]**, **[28]**, **[15]**. If X is the scalar field, then the notation is shortened to $\mathrm{ba}(\mathscr{R})$ and $\mathrm{ca}(\mathscr{R})$, and these spaces are equipped with the equivalent total variation norm, i.e., if $\mu \in \mathrm{ba}(\mathscr{R})$, then $\|\mu\| = \sup\{|\mu|(A) : A \in \mathscr{R}\}$, where $|\mu|(A) = \sup\{\sum |\mu(B_i)| : B_i \in \mathscr{R}$ for each i and (B_i) forms a finite partition of $A\}$. A subset K of $\mathrm{sa}(\mathscr{R}, X)$ is uniformly exhaustive (or uniformly s-additive) if $\mu(A_i) \to^i 0$ uniformly for $\mu \in K$ whenever (A_i) is a disjoint sequence form $\mathscr{R}$.

If X is a Banach space and $x, y \in X$, the $D^+(x, y)$ and $D^-(x, y)$ are defined, respectively, to be $\lim_n(\|nx + y\| - \|nx\|)$ and $\lim_n(\|nx\| - \|nx - y\|)$, and the norm is said to be Gateaux differentiable at x in the direction y if $-D^+(x, y) = D^+(x, -y)$, i.e.,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. This limit is denoted by $D(x, y)$. Since the norm function is convex, the existence of this limit is equivalent to the inequality $D^+(x, y) \leqq D^-(x, y)$; furthermore, we note that for all positive integers n we have the inequality

$$\|nx\| - \|nx - y\| \leqq D^-(x, y) \leqq D^+(x, y) \leqq \|nx + y\| - \|nx\|.$$

If $x \neq 0$, then x is said to be a smooth point if $D(x, y)$ exists for each $y \in X$. X is said to be smooth if each vector of norm one is smooth. Equivalently, the non-zero point x is a smooth point if and only if there is a unique $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$, **[11**, Chapter 2]. The non-zero point x is said to be Frechet smooth or strongly smooth if $D(x, y)$ exists uniformly for $y \in X$, $\|y\| \leqq 1$, i.e., the norm is Frechet differentiable at x. If x is a Banach lattice, then X is said to be an abstract L-space if the following conditions are satisfied:

$$\text{(i)} \quad \text{if } x \wedge y = 0, \text{ then } \|x + y\| = \|x - y\|$$

and

$$\text{(ii)} \quad \text{if } x, y \geqq 0, \text{ then } \|x + y\| = \|x\| + \|y\|.$$

We note that if $\mu, \nu \in \mathrm{ba}(\mathscr{R})$, then $\mu \wedge \nu$ is defined set-wise as follows:

$$\mu \wedge \nu(A) = \inf\{\mu(B) + \nu(A \backslash B) : B \subseteqq A, B \in \mathscr{R}\}.$$

(For the details, see p. 162 of Dunford and Schwartz **[16]**.) Furthermore $n|\mu| \wedge |\nu|$ converges in norm (as $n \to \infty$) to the absolutely continuous part of $|\nu|$ with respect to $|\mu|$ **[2]**.

Also, if τ is a topology on the Banach space X and $K \subseteqq X$, then K is said to be conditionally τ-compact if the τ-closure of K is τ-compact.

2. **Differentiability and Weak Compactness.** Let X be an L-space. According to Kakutani **[20]**, the totality of all non-negative elements of X is called the unit ideal, and X is said to have a unit if the unit ideal is principal, i.e., there exists $0 \leqq x \in X$ such that $\{y \in X : y \geqq 0\} = \{y \in X : y \prec x\}$, where the notation $y \prec x$ means that $y \geqq 0$ and if $u \geqq 0$ and $x \wedge u = 0$, then $y \wedge u = 0$. Consequently we shall say that x is a unit for X if $u \geqq 0$ and $x \wedge u = 0$ imply that $u = 0$. The following argument presents a geometrical characterization of units in L-spaces.

THEOREM 2.1. *The element x of the L-space X is a smooth point if and only if $|x|$ is a unit.*

PROOF. The first assertion is that x is a unit if and only if $D(x, y)$ exists for each $y \geqq 0$. To see this, begin by noting that if $x > 0$ and $y \geqq 0$, then

$$\lim_{n \to \infty} (\|nx + y\| - \|nx\|) = \|y\|$$

since the norm is additive on positive elements. And if x is a unit, then $nx \wedge y \to y$, e.g., see Lemma 3.9 of Kakutani [20].
Hence

$$\|nx - (nx \wedge y)\| - \|nx\| \to D^+(x, -y).$$

But

$$\|nx - (nx \wedge y)\| = \|nx\| - \|nx \wedge y\|,$$

and consequently

$$D^+(x, -y) = \lim_{n\to\infty} - \|nx \wedge y\| = -\|y\|.$$

Therefore $D(x, y)$ exists.

Conversely, suppose $D(x, y)$ exists for each $y \leqq 0$ and let u be a positive element so that $x \wedge u = 0$. Then $D^+(x, u) = \|u\|$ and $D^+(x, -u) = -\|u\|$. But $nx \wedge u = 0$ for each n. Thus $\|nx - u\| - \|nx\| = \|nx+u\| - \|nx\|$, and $\|u\| = -\|u\|$, i.e., $u = 0$.

Next suppose that x is an arbitrary smooth point in X; the claim is that $|x| = x^+ + x^-$ is a unit, where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Suppose then that $u \geqq 0$ and $|x| \wedge u = 0$. Now $(nx^+ + u) \wedge nx^- = (nx^+ \vee u) \wedge nx^- = (nx^+ \wedge nx^-) \vee (nx^- \wedge u) = 0$.

Therefore

$$\begin{aligned} D^+(x, u) &= \lim_{n\to\infty}\|nx + u\| - \|nx\| \\ &= \lim_{n\to\infty}\|nx^+ - nx^- + u\| - \|nx^+ - nx^-\| = \|u\|. \end{aligned}$$

Similarly,

$$\begin{aligned} nx^+ \wedge (u + nx^-) &= nx^+ \wedge (u \vee nx^-) \\ &= (nx^+ \wedge u) \vee (nx^+ \wedge nx^-) = 0, \end{aligned}$$

and therefore

$$\begin{gathered} \|nx^+ - nx^- - u\| - \|nx\| = \\ \|nx^+ + nx^- + u\| - \|nx\| \to \|u\| = D^+(x, -u). \end{gathered}$$

It follows that $u = 0$, and the claim is verified.

The final assertion is that if x is a unit, then $D(x, u)$ exists for arbitrary $u \in X$. One may obtain this result by observing that $\{y\colon D(x, y) \text{ exists}\}$ is a subspace—or one may proceed essentially as above. Write u as $u^+ - u^-$ and note that

$$\left\|\frac{n}{2}x + u^+\right\| - \left\|\frac{n}{2}x\right\| \to D^+(x, u^+),$$

$$\left\|\frac{n}{2}x - u^-\right\| - \left\|\frac{n}{2}x\right\| \to -D^+(x, u^-)$$

$$\frac{n}{2}x \wedge u^+ \to u^+,$$

$$\frac{n}{2}x \wedge u^- \to u^-,$$

and

$$\begin{aligned} D^+(x, u) &= \lim_n(\|nx + u^+ - u^-\| - \|nx\|) \\ &= \lim_n\left[\left\|\frac{n}{2}x + u^+\right\| - \left\|\frac{n}{2}x\right\| + \left\|\frac{n}{2}x - u^-\right\| - \left\|\frac{n}{2}x\right\|\right] \\ &= D^+(x, u^+) - D^+(x, u^-). \end{aligned}$$

Similarly,

$$D^+(x, -u) = D^+(x, u^-) - D^+(x, u^+),$$

i.e., $D(x, u)$ exists. The result follows by Theorem 2.4, infra.

COROLLARY 2.2. *If X is a weakly compactly generated L-space, then X has a unit.*

PROOF. By a result of Asplund **[1]**, every weakly compactly generated Banach space has an abundance of smooth points. By the preceding theorem, the absolute value of each smooth point is a unit.

Our next theorem characterizes uniform Gateaux differentiability in the L-space X. The following elementary lemma will prove helpful in establishing this result.

LEMMA 2.3. *Suppose that X is an L-space and $x, y \in X$. Then*

$$|x + y| + |x - y| = 2(|x| \vee |y|) = 2|x| + 2(|y| - |x| \wedge |y|).$$

PROOF. We show that $|x| + |y| = |x + y| \vee |x - y|$, which is an equivalent form of the first equality. Note that

$$\begin{aligned} |x| + |y| &= (x \vee -x) + |y| \\ &= (x + |y|) \vee (-x + |y|) \\ &= [x + (y \vee -y)] \vee [-x + (y \vee -y)] \\ &= (x + y) \vee (x - y) \vee (-x + y) \vee (-x - y) \\ &= [(x + y) \vee (-x - y)] \vee [(x - y) \vee (-x + y)] \\ &= |x + y| \vee |x - y|. \end{aligned}$$

The second equality in the conclusion of the lemma follows since

$$|x| + |y| = |x| \vee |y| + |x| \wedge |y|.$$

REMARK. Since the norm of X is additive on positive elements,

$$\|x + y\| + \|x - y\| = 2\| \, |x| \vee |y| \, \| \\ = 2\|x\| + 2\| \, |y| - |x| \wedge |y| \, \|.$$

THEOREM 2.4. *Suppose that X is an L-space, $K \subseteqq X$, and $f \in X$. Then the following three statements are equivalent.*

(1) *$D(f, g)$ exists uniformly for $g \in K$;*

(2) *$D(|f|, |g|)$ exists uniformly for $g \in K$;*

(3) *$|nf| \wedge |g| \to^n |g|$ uniformly for $g \in K$.*

PROOF. We show that (1) and (3) are equivalent; it then follows immediately that (2) and (3) are equivalent because of the absolute values which appear in (3). Since

$$- \|nf - g\| + \|nf\| \\ \leqq D^-(f, g) \leqq D^+(f, g) \\ \leqq \|nf + g\| - \|nf\|$$

for all positive integers n, then $D(f, g)$ exists uniformly for $g \in K$ if and only if

$$\|nf + g\| + \|nf - g\| - 2\|nf\| \xrightarrow{n} 0$$

uniformly for $g \in K$. By the preceding lemma,

$$\|nf + g\| + \|nf - g\| - 2\|nf\| \\ = 2\| \, |g| - |nf| \wedge |g| \, \|,$$

and the stated equivalences follow.

REMARK. As a result of the implication (1) implies (3) in Theorem 2.4, it follows that if $D(f, g)$ exists uniformly for $g \in K$, then K must be norm bounded in the L-space X, This boundedness result will be established later for arbitrary Banach spaces.

As a corollary of Theorem 2.4, we obtain the characterization of weak compactness in $\mathrm{ba}(\mathscr{R})$ which was mentioned in the introduction. We recall that a classical result states that a subset K of $\mathrm{ba}(\mathscr{R})$ is conditionally weakly compact if and only if K is bounded and uniformly absolutely continuous with respect to some measure $\xi \in \mathrm{ba}(\mathscr{R})$, e.g., see Dunford and Schwartz **[16**, IV. 9.12] for a proof when $\mathscr{R}$ is an algebra. Consequently, we state the following theorem in terms of differentiability—absolute continuity.

THEOREM 2.5. *Let K be a subset of* $\mathrm{ba}(\mathscr{R})$. *Then K is bounded and there is an element $\mu \in \mathrm{ba}(\mathscr{R})$ so that $\nu \ll \mu$ uniformly for $\nu \in K$ if and only if there is an element $\mu \in \mathrm{ba}(\mathscr{R})$ so that $D(\mu, \nu)$ exists uniformly for $\nu \in K$.*

PROOF. Suppose that $\mu \in \mathrm{ba}(\mathscr{R})$ and $D(\mu, \nu)$ exists uniformly for $\nu \in K$. Hence K is bounded and

$$\| |\nu| - |n\mu| \wedge |\nu| \| \xrightarrow{n} 0$$

uniformly for $\nu \in K$. Let $\varepsilon > 0$, and choose n such that

$$\| |\nu| - |n\mu| \wedge |\nu| \| < \frac{\varepsilon}{2}$$

for all $\nu \in K$. We assert that if $|\mu|(A) < \varepsilon/2n$, then $|\nu|(A) < \varepsilon$ for all $\nu \in K$. Suppose not. Then there exist $A \in \mathscr{R}$ and $\nu \in K$ such that $|\mu|(A) < \varepsilon/2n$ and $|\nu|\,(A) \geqq \varepsilon$. Then

$$\begin{aligned} \frac{\varepsilon}{2} &> \| |\nu| - |n\mu| \wedge |\nu| \| \\ &\geqq |\nu|(A) - |n\mu| \wedge |\nu|(A) \\ &\geqq \varepsilon - |n\mu| \wedge |\nu|(A). \end{aligned}$$

Therefore $|n\mu| \wedge |\nu|\,(A) \geqq \varepsilon/2$, an impossibility since $|n\mu|\,(A) < \varepsilon/2$. Hence the asserted uniform absolute continuity follows.

Conversely, suppose K is bounded and $\nu \ll \mu$ uniformly for $\nu \in K$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that if $|\mu|(A) < \delta$, then $|\nu|(A) < \varepsilon/2$ for each $\nu \in K$. Further, choose a positive integer n so that

$$(**) \qquad n > \frac{\sup\{\|\nu\| : \nu \in K\} - \frac{\varepsilon}{2}}{\delta}.$$

Now let $S \in \mathscr{R}$ and let $A \subseteqq S$, $A \in \mathscr{R}$ so that

$$|n\mu| \wedge |\nu|(S) > |n\mu|(A) + |\nu|(S \backslash A) - \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} (|\nu| - |n\mu| \wedge |\nu|)(S) &= |\nu|(S) - |n\mu| \wedge |\nu|(S) \\ &< |\nu|(S) - |n\mu|(A) - |\nu|(S \backslash A) + \frac{\varepsilon}{2} \\ &= |\nu|(A) - |n\mu|(A) + \frac{\varepsilon}{2}. \end{aligned}$$

If $|\nu|\,(A) < \varepsilon/2$, then

$$|\nu|(A) - |n\mu|(A) + \frac{\varepsilon}{2} < \varepsilon.$$

And if $|\nu|(A) \geqq \varepsilon/2$, then $|\mu|(A) \geqq \delta$, and

$$|\nu|(A) - |n\mu|(A) + \frac{\varepsilon}{2} < \varepsilon$$

from (**) above. Thus

$$\||\nu| - |n\mu| \wedge |\nu|\| \leqq \varepsilon$$

for each $\nu \in K$, and the desired uniformity follows from Theorem 2.4.

We note that a version of Theorem 2.5, with an argument closer in spirit to the proofs frequently given for *weak compactness* in spaces of countably additive measures (e.g. Dunford and Schwartz **[16]**, IV.8 and IV.9), is contained in Theorem 2.2 of Brooks and Lewis **[8]**, a result which gives a characterization of weakly compact operators on spaces of continuous functions.

REMARK. While the preceding results (especially 2.4 and 2.5) make it clear that $D(\mu, \nu)$ exists if and only if $D(|\mu|, |\nu|)$ exists in ba$(\mathscr{R})$ (in fact in any L-space), this equivalence is not transparent when merely considering the norm in ba$(\mathscr{R})$ and not using the interpretation of differentiability in terms of absolute continuity.

We now present two example of a conditionally weakly compact subset K of a Banach space and a smooth point f so that $D(f, g)$ does not exist uniformly for $g \in K$. In the first example (which is quite simple), K consists of a bounded, pointwise convergent sequence in $C[0, 1]$. The second example, due essentially to Lindenstrauss **[23]**, is more delicate. It consists of an equivalent renorming of ℓ^2 and a smooth point x with respect to the new norm so that $D(x, y)$ does not exist uniformly for all y so that $\|y\| \leqq 1$. We include the details of this example since they are not in **[23]** and to our knowledge are not otherwise readily available. These examples show that condition (*) of the introduction cannot characterize weak compactness for arbitrary Banach spaces; in particular (*) does not characterize weak compactness for the space $L^1(\mu, X)$ of Bochner integrable functions—even if X is reflexive (a problem suggested to us by Professor D. R. Lewis). For if μ is a probability measure, then X is contained isometrically in $L^1(\mu, X)$.

EXAMPLE 2.6. Let $X = C[0, 1]$, and let $f(x) = x$, $0 \leqq x \leqq 1$. Then f is a smooth point in X; for if μ is a regular Borel measure of norm 1 on $[0, 1]$ for which $\int f d\mu = 1$, then $\mu(\{1\}) = 1$. Define the sequence (f_k), $k = 3, 4, \ldots$ in $C[0, 1]$ as follows:

$$f_k(x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq 1 - \dfrac{2}{k} \\ kx + 2 - k & \text{for } 1 - \dfrac{2}{k} \leqq x \leqq 1 - \dfrac{1}{k} \\ -kx + k & \text{for } 1 - \dfrac{1}{k} \leqq x \leqq 1. \end{cases}$$

Then the sequence (f_k) is bounded and pointwise convergent to 0; hence (f_k) is conditionally weakly compact. Note that $D(f, f_k) = 0$ for each k. Next observe that if λ is positive, then

$$\frac{\|f+\lambda f_k\|-1}{\lambda} \geqq \frac{(f+\lambda f_k)\left(1-\frac{1}{k}\right)-1}{\lambda} = 1-\frac{1}{k\lambda}.$$

Clearly for λ fixed, $1 - 1/k\lambda \to 1$ as $k \to \infty$, and $D(f, f_k)$ does not exist uniformly.

EXAMPLE 2.7. Let $\|\cdot\|_2$ denote the usual norm on ℓ^2, and let (e_n) denote the usual orthonormal basis for ℓ^2. For $n = 2, 3, \ldots$, set $v_n = (1 - 1/n)e_1 + e_n$, and for $x \in \ell^2$ define $\|x\|$ to be $\sup_n\{\|x\|_2, |(x, v_n)|\}$, where $(\cdot, \cdot)$ denotes the usual inner product. Note that $\|\cdot\|$ is an equivalent norm on ℓ^2 since $\|x\|_2 \leqq \|x\| \leqq 2\|x\|_2$. We claim that e_1 is a smooth point with respect to $\|\cdot\|$. Let $\varepsilon > 0$, $z \in \ell^2$, and put

$$p_0 = \frac{1}{\varepsilon}\|z\|_2^2 + 4\|z\| + \sup\{0, n(z_n - \varepsilon)\colon n = 2, 3, \ldots\},$$

where z_n is the n-th coordinate of z. (Since $z_n \to 0$, this supremum is finite.) Suppose that $p > p_0$. Since

$$z_1 + \left(\frac{n}{n-1}\right)z_n \leqq (z_1^2 + z_n^2)^{1/2}\left(1 + \left(\frac{n}{n-1}\right)^2\right)^{1/2} \leqq 4(z_1^2 + z_n^2)^{1/2} \leqq 4\|z\|$$

for $n \geqq 2$, we have that

$$p_0 + z_1 + \left(\frac{n}{n-1}\right)z_n \geqq 0$$

for $n = 2, 3, \ldots$. Thus

$$p + z_1 + \left(\frac{n}{n-1}\right)z_n > 0$$

for $n \geqq 2$, and $p + z_1 > 0$. Also,

$$p + z_1 \geqq p - \|z\| > p_0 - \|z\| \geqq n(z_n - \varepsilon),$$

and this implies that

$$-\frac{1}{n}(p + z_1) < \varepsilon - z_n.$$

Therefore we have

$$(\#)\qquad \begin{aligned} |(pe_1 + z, v_n)| &= \left(1 - \frac{1}{n}\right)(p + z_1) + z_n \\ &= p + z_1 - \left(\frac{p + z_1}{n}\right) + z_n \\ &< p + z_1 + \varepsilon \end{aligned}$$

for $p > p_0$, $n \geqq 2$. Furthermore,

$$\|pe_1 + z\|_2 - (p + z_1) = \frac{\|pe_1 + z\|_2^2 - (p + z_1)^2}{\|pe_1 + z\|_2 + (p + z_1)}$$
$$\leqq \frac{\sum z_n^2}{\frac{3}{4}p + \frac{3}{4}p},$$

$(p + z_1 \geqq p - \|z\| \geqq 3/4\, p)$. And

$$\frac{\sum z_n^2}{\frac{3}{2}p} \leqq \frac{\|z\|_2^2}{p} < \varepsilon.$$

Therefore

$$\|pe_1 + z\|_2 < p + z_1 + \varepsilon,$$

and we have that

$$\|pe_1 + z\| \leqq p + z_1 + \varepsilon$$

for large values of p from (#), the preceding inequality, and the definition of $\|\cdot\|$. Thus, for large positive integers n,

$$\|ne_1 + z\| - \|ne_1\| \leqq z_1 + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $D^+(e_1, z) < z_1$ and (replacing z with $-z$) $D^+(e_1, -z) \leqq -z_1$. Combining these inequalities, we see that

$$(D^+(e_1, z) \leqq z_1 \leqq -D^+(e_1, -z) = D^-(e_1, z);$$

thus $D(e_1, z)$ exists.

Now we make use of Lemma 2.1 in Chapter 2 of Diestel **[11]**: *If* e_1 *is Frechet smooth,* $(x_n^*) \subseteq (\ell^2, \|\cdot\|)^*$, $\|x_n^*\| = 1$, *and* $(x_n^*, e_1) \to 1$, *then* $\|x_n^* - e_1\| \to 0$ *in* $(\ell^2, \|\cdot\|)^*$. But the norm of each v_n as a linear functional on $(\ell^2, \|\cdot\|)$ is one, $(v_n, e_1) \to 1$, and the norm of $v_n - e_1$ as a continuous linear function is at least as large as 1 for $n = 2, 3, \ldots$. Therefore, e_1 is not Frechet smooth.

We remark that the converse implication is also false in general. Specifically, if x is a Banach space such that X^* is separable (e.g., c_0), then by the Kadec-Klee renorming theorem **[11**, Chapter 4], X has an equivalent Frechet differentiable norm. Consequently, if c_0 is thus renormed and μ is a probability measure, then c_0 (equivalently renormed) is isometrically contained in $L^1(\mu, c_0)$. Therefore, if S is the unit ball in c_0 and x is a point of norm one in c_0, then $D(x, y)$ exists uniformly for $y \in S$, but S is not conditionally weakly compact.

Another question now arises. Although (2.5) was a characterization of weak compactness in spaces of measures, the theorem literally investigates the connection between uniform Gateaux differentiability of the norm and uniform absolute continuity. Therefore, if fabv($\mathscr{R}$, X) is the Banach space of all X-valued finitely additive measures on $\mathscr{R}$ endowed with the total variation norm, one might ask (and we do) if boundedness and uniform absolute continuity is equivalent to uniform differentiability of the norm. One half of this question has already been answered negatively by examples 2.6 and 2.7. More specifically, if (S, Σ, μ) is a probability measure space and (f_k) is the sequence of continuous functions defined in 2.6, then the measures in $L^1(\mu, C[0, 1])$ associated with (f_k) are $(\mu \cdot f_k)$, where $\mu \cdot f_k(A) = \mu(A)f_k$, $A \in \Sigma$. Hence $\mu \cdot f_k \ll \mu \cdot f$ uniformly in k, where $f(x) = x$, $0 \leqq x \leqq 1$, but $D(\mu \cdot f, \mu \cdot f_k)$ does not exist uniformly.

Before exploring the converse of this implication, we point out that some positive results are known in this area for pairs of measures. For example, in Theorem 3.2 of [3] it is shown that if X is a smooth space with the Radon-Nikodym property and $\mu, \nu \in \mathrm{cabv}(\Sigma, X)$, where Σ is a σ-algebra, then $D(\mu, \nu)$ exists if $\nu \ll \mu$. And in [3] it is shown that if X is an arbitrary Banach space, Σ is an algebra of sets, and $\mu, \nu \in \mathrm{fabv}(\Sigma, X)$, then $\nu \ll \mu$ if $D(\mu, \nu)$ exists.

We remark that the smootheness assumption in the result mentioned above is essential. For if x and y belong to the Banach space X, $\|x\| = \|y\| = 1$, and $D(x, y)$ does not exist, then $\mu \cdot y \ll \mu \cdot x$, where $\mathscr{R}$ is a ring of sets and μ is a non-zero member of ba($\mathscr{R}$). But

$$\frac{\|\mu\cdot x + t\mu\cdot y\| - \|\mu\cdot x\|}{t} = \|\mu\|\left(\frac{\|x + ty\| - \|x\|}{t}\right),$$

and consequently $D(\mu\cdot x, \mu\cdot y)$ does not exist.

We conclude our discussion of the question raised above with the following result. As above, X is a Banach space and $\mathscr{R}$ is a ring of sets.

THEOREM 2.8. *If* $\mu \in \mathrm{fabv}(\mathscr{R}, X)$, $K \subseteqq \mathrm{fabv}(\mathscr{R}, X)$, *and* $D(\mu, \nu)$ *exists uniformly for* $\nu \in K$, *then* $\nu \ll \mu$ *uniformly for* $\nu \in K$.

PROOF. In order to establish the uniform absolute continuity of K with respect to μ, it clearly suffices to establish the uniform absolute continuity of $|K| = \{|\nu| : \nu \in K\}$ with respect to $|\mu|$.

And, in view of Theorem 2.5, it will suffice to show that $D(|\mu|, |\nu|)$ exists uniformly for $\nu \in K$, where $|\nu|$ is the total variation of ν. Therefore, as was observed in the proof of Theorem 2.4, it will suffice to show that

$$\lim_n \big\|n|\mu| + |\nu|\big\| + \big\|n|\mu| - |\nu|\big\| - 2\big\|n|\mu|\big\| = 0$$

uniformly for $\nu \in K$. And, since

$$\lim_n \|n\mu + \nu\| + \|n\mu - \nu\| - 2\|n\mu\| = 0$$

uniformly for $\nu \in K$ (this is the uniformity asserted in the hypothesis), the theorem will follow if we show that

$$\|\xi + \eta\| + \|\xi - \eta\| \geqq \||\xi| + |\eta|\| + \||\xi| - |\eta|\|$$

for $\xi, \eta \in \text{fabv}(\mathscr{R}, X)$.

Let $\varepsilon > 0$, let $A \in \mathscr{R}$, and let $(A_i)_{i=1}^n$ be a partition of A into pairwise disjoint members of $\mathscr{R}$ such that

$$\begin{aligned}\sum\|\xi(A_i)\| &+ \|\eta(A_i)\| \\ &+ \sum\left||\xi|(A_i) - |\eta|(A_i)\right| + \varepsilon \\ &\quad > \||\xi| + |\eta|\| + \||\xi| - |\eta|\|.\end{aligned}$$

Then for each i let (A_{ij}) be a finite $\mathscr{R}$-partition of A_i such that

$$\begin{aligned}\sum_i \sum_j\|\xi(A_{ij})\| &+ \|\eta(A_{ij})\| \\ &+ \sum_i\left|\sum_j\|\xi(A_{ij})\| - \|\eta(A_{ij})\|\right| + \varepsilon \\ &\quad > \||\xi| + |\eta|\| + \||\xi| - |\eta|\|.\end{aligned}$$

Now fix i and note that

$$\sum_j\|\xi(A_{ij})\| + \|\eta(A_{ij})\| + \left|\sum\|\xi(A_{ij})\| - \|\eta(A_{ij})\|\right|$$

equals $2\sum_j\|\xi(A_{ij})\|$ or $2\sum_j\|\eta(A_{ij})\|$.

But

$$\begin{aligned}\max\{2\sum_j\|\xi(A_{ij})\|, 2\sum_j\|\eta(A_{ij})\|\} \\ \leqq \sum_j\|(\xi + \eta)A_{ij}\| + \sum_j\|(\xi - \eta)A_{ij}\|,\end{aligned}$$

$i = 1, 2, \ldots$. Thus,

$$\begin{aligned}\|\xi + \eta\| + \|\xi - \eta\| + \varepsilon &\geqq \sum_i \sum_j\|(\xi + \eta)\, A_{ij}\| + \|(\xi - \eta)A_{ij}\| + \varepsilon \\ &\geqq \sum_i \sum_j\|\xi(A_{ij})\| + \|\eta(A_{ij})\| + \sum_i\Big|\sum_j\|\xi(A_{ij})\| \\ &\quad - \|\eta(A_{ij})\|\Big| + \varepsilon > \||\xi| + |\eta|\| + \||\xi| - |\eta|\|.\end{aligned}$$

Since ε was arbitrary,

$$\|\xi + \eta\| + \|\xi - \eta\| \geqq \||\xi| + |\eta|\| + \||\xi| - |\eta|\|,$$

and the theorem is proved.

We conclude this section with a proposition which shows that uniform differentiability does imply boundedness for arbitrary Banach spaces; the reader may want to compare the argument with the proof of implication (1) implies (3) in Theorem 2.4.

PROPOSITION 2.9. *Suppose that X is a Banach space, $K \subseteqq X$, $x \in X$, and $D(x, y)$ exists uniformly for $y \in K$. Then K is bounded.*

PROOF. Choose $\delta > 0$ so that if $y \in K$ and $0 < |t| \leqq \delta$, then

$$\left|\frac{\|x + ty\| - \|x\|}{t} - D(x, y)\right| < 1.$$

Put $t = -\delta$. Then

$$D(x, y) \leqq 1 + \frac{\|x - \delta y\| - \|x\|}{-\delta} \leqq 1 + \frac{\|x\|}{\delta}$$

for each $y \in K$. But if $t = \delta$, then

$$\frac{\|x + \delta y\| - \|x\|}{\delta} \leqq 1 + D(x, y) \leqq 1 + 1 + \frac{\|x\|}{\delta},$$

and

$$\|x + \delta y\| \leqq 2\delta + 2\|x\|.$$

Therefore

$$\|\delta y\| \leqq 2\delta + 3\|x\|$$

for all $y \in K$, and it follows that K is bounded.

3. **Vitali-Hahn-Saks Theorem.** In this section we give a proof of the classical Vitali-Hahn-Saks Theorem using the relationships between differentiability of the norm in spaces of measures and absolute continuity established in (2.5). Then we discuss the Vitali-Hahn-Saks Theorem due to Brooks and Jewett [7].

Although we have seen examples of a smooth point x and a weakly compact set K such that $D(x, y)$ does not exist uniformly for $y \in K$, we are able to obtain the following result.

LEMMA 3.1. *If K is a conditionally compact subset of the Banach space X and x is a smooth point, then $D(x, y)$ exists uniformly for $y \in K$.*

PROOF. Suppose that K is conditionally compact, x is a smooth point, and $D(x, y)$ does not exist uniformly for $y \in K$. Then there exist $\varepsilon > 0$, a sequence $(x_n) \subseteqq K$, and a sequence t_n of numbers such that $t_n \to 0$ and

$$\left|\frac{\|x + t_n x_n\| - \|x\|}{t_n} - D(x, x_n)\right| > \varepsilon$$

for each n. Without loss of generality, we may (and shall) assume that $x_n \to y$. Therefore $D(x, x_n) \to D(x, y)$, and

$$\left|\frac{\|x + t_n x_n\| - \|x\|}{t_n} - D(x, y)\right| > \frac{3\varepsilon}{4}$$

for sufficiently large n. But since

$$\left|\frac{\|x + t_n x_n\| - \|x\|}{t_n} - \frac{\|x + t_n y\| - \|x\|}{t_n}\right| \xrightarrow{n} 0,$$

then

$$\left| \frac{\|x + t_n y\| - \|x\|}{t_n} - D(x, y) \right| > \frac{\varepsilon}{2}$$

for sufficiently large n, and we have the desired contradiction.

As a corollary to the lemma, we demonstrate a new proof of the classical Vitali-Hahn-Saks theorem. Since the Banach space version of the theorem reduces easily to the scalar case, we state and prove the theorem for real valued measures. Before proceeding to this proof, however, we point out that if (μ_n) is a sequence of countably additive measures on the σ-ring $\mathscr{R}$, $0 \leqq \mu \in \mathrm{ca}(\mathscr{R})$, and $\mu_n \ll \mu$ for each n, but (μ_n) is not uniformly absolutely continuous with respect to μ, then one can produce an $\varepsilon > 0$, a disjoint sequence (A_i) from $\mathscr{R}$, and a subsequence (μ_{n_i}) of (μ_n) so that $|\mu_{n_i}(A_i)| > \varepsilon$ for each i. For if we do not have uniform absolute continuity, then there is a $\delta > 0$, a sequence $(B_n) \subseteqq \mathscr{R}$, and a subsequence (μ_n') of (μ_n) such that $\mu(B_n) \to 0$ and $|\mu_n'(B_n)| > \delta$ for each n. Let $n_1 = 1$ and choose n_2 sufficiently large that $|\mu_n'(A)| < \varepsilon/4$ for $A \in \mathscr{R}$, $A \subseteqq B_{n_2}$. Then choose n_3 sufficiently large that $|\mu_{n_1}'(A)| < \varepsilon/8$ and $|\mu_{n_2}'(A)| < \varepsilon/4$ for $A \in \mathscr{R}$, $A \subseteqq B_{n_3}$. Continue this process inductively. If we define A_{n_i} to be $B_{n_i} \backslash (\bigcup_{j>i} B_{n_j})$ and set $\mu_{n_i} = \mu_{n_i}'$, then $|\mu_{n_i}(A_{n_i})| \geqq \delta/2$ and (A_{n_i}) is a disjoint sequence.

THEOREM 3.2. *Suppose that* $\mathscr{R}$ *is a* σ*-ring*, $(\mu_n) \subseteqq \mathrm{ca}(\mathscr{R})$, $(\mu_n(A))$ *converges for each* $A \in \mathscr{R}$, *and* λ *is a non-negative* (*possibly infinite*) *countably additive measure on* $\mathscr{R}$ *such that* $\mu_n \ll \lambda$ *for each n. Then* $\mu_n \ll \lambda$ *uniformly in n.*

PROOF. Let

$$\mu = \sum \frac{|\mu_n|(\cdot)}{(1 + \|\mu_n\|)2^n}.$$

Since $\mu \ll \lambda$, it clearly will suffice to show that $\mu_n \ll \mu$ uniformly in n. Suppose this is not the case. Then by the observation preceding this theorem, there is an $\varepsilon > 0$, a subsequence (μ_{n_i}) of (μ_n) and a disjoint sequence (A_i) of elements from $\mathscr{R}$ such that $|\mu_{n_i}(A_i)| > \varepsilon$, $i = 1, 2, \ldots$. We now consider the sequence $\{x_i\} \subseteqq \ell^1$, $x_i = (x_{ik})$, where $x_{ik} = \mu_{n_i}(A_k)$, and the point $x = (x_k) \in \ell^1$, where $x_k = \mu(A_k)$. The point $x = (x_k)$ is a smooth point in ℓ^1 since $x_k > 0$ for each k, and the sequence (x_n) is weakly convergent in ℓ^1 since (μ_n) is set-wise convergent. Thus the sequence (x_n) is conditionally weakly compact in ℓ^1 and hence conditionally compact in ℓ^1 (because weak and norm convergence of sequences coincide in ℓ^1). By Lemma 3.1, $D(x, x_n)$ exists uniformly, and by Theorem 2.5 $x_n \ll x$ uniformly in n. Therefore $x_{ii} \to^i 0$, and we have a contradiction. The theorem follows.

We next state a result whose importance and relationship to the Vitali-Hahn-Saks (= VHS) Theorem will set the tone for the remainder of this section.

THEOREM 3.3. *Suppose that $\mathscr{R}$ is a ring of sets, K is a uniformly exhaustive subset of* $\mathrm{ba}(\mathscr{R})$, *and* $0 \leqq \lambda$ *is a finitely additive (possibly infinite) measure on $\mathscr{R}$ such that $\mu \ll \lambda$ for each $\mu \in K$. Then $\mu \ll \lambda$ uniformly for $\mu \in K$.*

Versions of this theorem were established independently by L. Drewnowski **[14]** and by James K. Brooks **[6]** in a paper in which (3.3) was used to significantly improve the classical Vitali Convergence Theorem.

However, as noted earlier, we feel that the scope of (3.3) has not been fully realized—even by the authors who established the result. For example, on p. 167 of **[6]**, Professor Brooks states that the proof of the abstract version of the VHS Theorem due to Brooks and Jewett is more delicate than the proof of (3.3) and requires, in addition, a generalization of the Phillips Lemma. (Indeed, the proof of (3.3) in **[6]** is simpler than the difficult and ingenious proof of the VHS Theorem in [7].) In fact, we shall show that the VHS Theorem without a strong boundedness assumption follows rather easily from (3.3) and the classical Phillips Lemma. And in **[14]** Professor Drewnowski uses (3.3) to establish the equivalence of two theorems (3.5 and 3.6) originally established in Brooks and Jewett [7]. We feel that (3.3) may be a deeper result than either of these two results.

Two technical aspects of the Brooks-Drewnowski Theorem (3.3) seem noteworthy: (1) $\mathscr{R}$ is not assumed to be a σ-ring; (2) K is not assumed to be norm-bounded. Perhaps the fact that the uniform absolute continuity conclusion can be obtained without the assumption that $\mathscr{R}$ is a σ-ring is somewhat surprising because one can construct counter-examples to the Vitali-Hahn-Saks Theorem when $\mathscr{R}$ is a ring. Furthermore, the relaxation of the boundedness assumption on K means that standard weak compactness techniques usually applied in discussions of $\mathrm{ba}(\mathscr{R})$ are not likely to be very beneficial.

Rather than presenting a proof of (3.3) as stated, we shall adapt Drewnowski's argument and give an L-space interpretation of the result. If μ and ν are non-negative members of $\mathrm{ba}(\mathscr{R})$, then we define $P_\mu(\nu)$ to be $\lim_n n\mu \wedge \nu$; if ν is arbitrary, we define $P_\mu(\nu)$ to be $P_\mu(\nu^+) - P_\mu(\nu^-)$. The function P_μ is a bounded linear projection on $\mathrm{ba}(\mathscr{R})$; let $\mathscr{O} = \{P_\mu\colon \mu \geqq 0\}$. The reader may consult Kakutani **[20]** for a discussion of some of the properties of the operator P_μ. We note that if $\mu_1 \leqq \mu_2$, then $P_{\mu_2} - P_{\mu_1} \in \mathscr{O}$; specifically, $P_{\mu_2} - P_{\mu_1} = P_z$, where $z = \mu_2 - P_{\mu_1}(\mu_2)$. Further, two projections P_{μ_1} and P_{μ_2} are said to be disjoint if $P_{\mu_1}P_{\mu_2}(= P_{\mu_2}P_{\mu_1} = P_{\mu_1\wedge\mu_2})$ is zero, and a subset K of $\mathrm{ba}(\mathscr{R})$ is said to be uniformly additive if $\|P_i(u)\| \to 0$ uniformly for $u \in K$ whenever (P_i) is a pairwise disjoint sequence from $\mathscr{O}$. Also, a subset K of $\mathrm{ba}(\mathscr{R})$ is said to be $\mathscr{O}$-continuous relative to an element $\mu \in \mathrm{ba}(\mathscr{R})$ if $P_i(k) \to 0$ for each $k \in K$ whenever (P_i) is a sequence from $\mathscr{O}$ such that $P_i(\mu) \to 0$. (The proof of the equivalence of uniform additivity and uniform exhaustiveness is technical and will be postponed until the concluding section of the paper. A close look at the definition of $\mathscr{O}$-continuity will convince the reader that it implies absolute continuity; since this implication is the one required in the VHS Theorem, we omit the proof that these two notions are equivalent.) We remark that the preceding definition and the following result may be stated in terms of an abstract L-space; however, to maintain more immediate contact with (3.3), we have chosen to state them in $\mathrm{ba}(\mathscr{R})$.

THEOREM 3.3L. *If K is a uniformly additive subset of* $\mathrm{ba}(\mathscr{R})$, $\mu \in \mathrm{ba}(\mathscr{R})$, *and K is $\mathscr{O}$-continuous with respect to μ, then K is uniformly $\mathscr{O}$-continuous with respect to μ.*

PROOF. Suppose that $\mu \in \mathrm{ba}(\mathscr{R})$, K is a uniformly additive subset of $\mathrm{ba}(\mathscr{R})$, and K is $\mathscr{O}$-continuous but not uniformly $\mathscr{O}$-continuous with respect to μ. Hence there is a sequence (ξ_i) of positive members of $\mathrm{ba}(\mathscr{R})$, an $\varepsilon > 0$, and a sequence (ν_i) from K such that

$$\text{(a)} \quad \sum \|P_{\xi_i}(\mu)\| < \infty$$

and

$$\text{(b)} \quad \|P_{\xi_i}(\nu_i)\| > 2\varepsilon$$

for each i. Let n_1 be a positive integer so that if $n \geqq n_1$, then

$$\|P_{\xi_n} - P_{\xi_n \wedge (\vee_{k=1}^{n_1} \xi_k)}(u)\| < \frac{\varepsilon}{2}$$

for each $u \in K$. (Since the preceding difference of projections is an element of $\mathscr{O}$, the assumption that no such n_1 exists leads to the construction of a disjoint sequence from $\mathscr{O}$ which produces a contradiction of the uniform additivity of K.) Let $z_1 = \bigvee_{k=1}^{n_1} \xi_k$; then

$$\|P_{z_1 \wedge \xi_k}(\nu_k)\| > 2\varepsilon - \frac{\varepsilon}{2}$$

for $k \geqq n_1$. Let $\sigma_1 = z_1 \wedge \xi_{n_1}, \sigma_2 = z_1 \wedge \xi_{n_1+1}, \ldots$. Let $n_2(>n_1)$ be a positive integer so that if $n \geqq n_2$, then

$$\|P_{\sigma_n} - P_{\sigma_n \wedge (\vee_{k=1}^{n_2} \sigma_k)}(u)\| < \frac{\varepsilon}{4}$$

for each $u \in K$. Then there is a sequence (γ_n) from K so that

$$\|P_{\sigma_n \wedge (\vee_{k=1}^{n_2} \sigma_k)}(\gamma_n)\| > 2\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{4}$$

for $n \geqq n_2$.

Continue inductively to manufacture a sequence (τ_k) of positive measures and a sequence (ϕ_k) from K such that

$$(1) \quad \tau_{k+1} \leqq \tau_k \qquad \text{for each } k,$$
$$(2) \quad \|P_{\tau_k}(\phi_k)\| > \varepsilon \qquad \text{for each } k,$$

and

$$(3) \quad \|P_{\tau_k}(\mu)\| \to 0.$$

The third property follows from (a) above, i.e.

$$\|P_{z_1}(\mu)\| \leqq \sum_{k=1}^{n_1} \|P_{\xi_k}(\mu)\|.$$

Now use (3) and the fact that K is $\mathcal{O}$-continuous relative to μ to select subsequences $(P_{\tau_{k_i}})$and (ϕ_{k_i}) such that

$$\|(P_{\tau_{k_i}} - P_{\tau_{k_{i+1}}})(\phi_{k_i})\| > \frac{\varepsilon}{2}.$$

But $(P_{\tau_{k_i}} - P_{\tau_{k_{i+1}}})$ is a disjoint sequence of projections from $\mathcal{O}$. Thus we have contradicted the uniform additivity of K; the theorem follows.

We now state the classical Phillips Lemma [**27**], [**9**, p. 36], a result which we use in the proof of the next theorem. The relationship between the Phillips Lemma and uniform exhaustivity results will be discussed subsequently.

LEMMA 3.4 (PHILLIPS). *If $\mathcal{P}$ is the class of all subsets of the natural numbers and (μ_n) is a sequence in* ba$(\mathcal{P})$ *such that $\mu_n(A) \xrightarrow{n} 0$ for each $A \in \mathcal{P}$, then $\sum_{k\in\Delta} \mu_n(\{k\}) \xrightarrow{n} 0$ uniformly for $\Delta \in \mathcal{P}$.*

As we have remarked, the following result (in its present generality) is due to Brooks and Jewett [**7**]. However, these authors assumed that the measures ν_n in the statement of the theorem are exhaustive. Also, Diestel and Uhl [**12**] give a very comprehensive account of this theorem. But in [**12**] the control measure ν in the following statement is assumed to be bounded, hence exhaustive. Therefore exhaustivity is also present (implicity) in the Vitali-Hahn-Saks Theorem of [**12**]. Other authors have also overlooked the fact that the VHS Theorem may be obtained without strong boundedness, e.g. Faires [**17**] and Oberle [**25**].

THEOREM 3.5 (VITALI-HAHN-SAKS). *Suppose that $\mathcal{R}$ is a σ-ring and X is a Banach space. If ν is a non-negative (perhaps infinite) finitely additive measure on $\mathcal{R}$ and (ν_n) is a sequence from* ba$(\mathcal{R}, X)$ *so that $(\nu_n(A))$ converges in X for each $A \in \mathcal{R}$ and $\nu_n \ll \nu$ for each n, then $\nu_n \ll \nu$ uniformly in n.*

PROOF. We consider first the case when $\nu_n(A) \to 0$ for each A. Let (x_n^*) be a sequence from the unit ball of X^*. Then we claim that $(x_n^*\nu_n)$ must be uniformly exhaustive. For suppose not. Then we may assume without loss of generality that $\varepsilon > 0$, (A_n) is a disjoint sequence from $\mathscr{R}$, and $x_n^*\nu_n(A_n) > \varepsilon$. For $\Delta \in \mathscr{P}$, put

$$\mu_n(\Delta) = x_n^*\nu_n\,(\bigcup_{i\in\Delta} A_i);$$

then $\mu_n(\Delta) \to 0$ for each $\Delta \in \mathscr{P}$. Hence

$$\lim_n \sum_{k\in\Delta} \mu_n(\{k\}) = 0$$

uniformly for $\Delta \in \mathscr{P}$. But this is impossible since

$$\mu_n(\{n\}) = x_n^*\nu_n(A_n) > \varepsilon$$

for each n, and we have verified the claim. Now suppose that (ν_n) fails to be uniformly absolutely continuous with respect to ν. Then there is an $\varepsilon > 0$, a sequence (A_i) in $\mathscr{R}$ such that $\nu(A_i) \to 0$, and a subsequence (ν_{n_i}) of (ν_n) so that

$$\|\nu_{n_i}(A_i)\| > \varepsilon$$

for each i. Let $x_1^* \in X^*$, $\|x_1^*\| \leqq 1$, be chosen such that

$$x_1^*\nu_{n_i}(A_i) > \varepsilon. \tag{1}$$

However, $(x_1^*\nu_{n_i})$ must be uniformly exhaustive by the claim above, and $x_1^*\nu_{n_i} \ll \nu$ for each i. Therefore by (3.3), $x_1^*\nu_{n_i} \ll \nu$ uniformly in i, and we contradict (1).

We now consider the general case in which we merely assume that $(\nu_n(A))$ converges for each A. Again, we deny the uniform absolute continuity. Then, using the termwise absolute continuity, we may assume that $\varepsilon > 0$, $(A_n) \subseteqq \mathscr{R}$, $\nu(A_n) \to 0$, $\|\nu_n(A_n)\| > \varepsilon$ for each n, and $\|\nu_n(A_k)\| < \varepsilon/2$ for $k > n$. Let $\xi_n = \nu_{n+1} - \nu_n$, $n = 1, 2, \ldots$. Therefore $\xi_n(A) \to 0$ for each A, and $\xi_n \ll \nu$ uniformly in n by the preceding paragraph. But this is impossible because $\nu(A_n) \to 0$ and $\|(\nu_{n+1} - \nu_n)(A_{n+1})\| > \varepsilon/2$ for each n. The theorem follows.

We note that the preceding proof shows that an application of the Phillips Lemma yields the following result.

THEOREM 3.6 (BROOKS and JEWETT [7]). *If $\mathscr{R}$ is a σ-ring and (ν_n) is a sequence of X-valued exhaustive set functions such that* $\lim \nu_n(A)$ *exists for each $A \in \mathscr{R}$, then (ν_n) is uniformly exhaustive.*

We shall follow Drewnowski [**15**] and refer to this result as the Brooks-Jewett Theorem. In Diestel and Uhl [**12**], this theorem is termed the Vitali-Hahn-Saks-Nikodym Theorem and is discussed in detail. In the following paragraphs, we demonstrate that the Brooks-Jewett Theorem implies the

Phillips Lemma. The interested reader might compare this implication with those established in Drewnowski **[15]**.

Suppose then that (μ_n) is a sequence from ba$(\mathscr{P})$ such that $\mu_n(A) \to 0$ for each $A \in \mathscr{P}$. Then we claim that

$$\sum_{k \in \Delta} \mu_n(\{k\}) \xrightarrow{n} 0$$

for each $\Delta \in \mathscr{P}$. For suppose not. Then we may (and shall) assume that $\varepsilon > 0$, $\Delta \in \mathscr{P}$, and

$$\sum_{k \in \Delta} |\mu_n(\{k\})| > 2\varepsilon$$

for each n. Then clearly Δ must be infinite. Let (N_i) be a strictly increasing sequence of positive integers, and let (μ_{n_i}) be a subsequence of (μ_n) so that

$$\sum_{k \in \Delta \cap [N_{i-1}+1, N_i]} |\mu_{n_i}(\{k\})| > \varepsilon$$

for each i. Choose $B_i \subseteqq \Delta \cap [N_{i-1} + 1, N_i]$ such that $|\mu_{n_i}(B_i)| > \varepsilon/2$. But this contradicts the Brooks-Jewett Theorem, and the claim follows.

Now suppose the uniformity statement in the conclusion of the Phillips Lemma fails for (μ_n). Then, without loss of generality, we suppose that $\varepsilon > 0$ and (Δ_n) is a sequence of sets from $\mathscr{P}$ so that

$$\left| \sum_{k \in \Delta_n} \mu_n(\{k\}) \right| > \varepsilon$$

for each n. By the preceding paragraph, we know that there is an integer j such that

$$\sum_{k \in \Delta_1} |\mu_t(\{k\})| < \frac{\varepsilon}{4}$$

for $t \geqq j$. For convenience, suppose one such j is 2. Then

$$\sum_{k \in \Delta_2 \setminus \Delta_1} |\mu_2(\{k\})| > \frac{3\varepsilon}{4}$$

Again, using the preceding claim, we may suppose that

$$\sum_{k \in \Delta_2} |\mu_t(\{k\})| < \frac{\varepsilon}{8}$$

for $t \geqq 3$. Hence

$$\left| \sum_{k \in \Delta_3 \setminus \Delta_1 \cup \Delta_2} \mu_3(\{k\}) \right| \geqq \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{8}.$$

Continue in this fashion inductively, i.e., we suppose that

$$\left| \sum_{k \in \Delta_n \setminus (\bigcup_{t=1}^{n-1} \Delta_t)} \mu_n(\{k\}) \right| \geqq \frac{\varepsilon}{2}.$$

Now $\Delta_1, \Delta_2\backslash\Delta_1, \ldots, \Delta_n\backslash(\bigcup_{t=1}^{n-1}\Delta_t), \ldots$ is a disjoint sequence of sets from $\mathscr{P}$. For each n, choose

$$B_n \subseteqq \Delta_n\backslash\left(\bigcup_{t=1}^{n-1}\Delta_t\right)$$

so that

$$|\mu_n(B_n)| = \left|\sum_{k\in B_n}\mu_n(\{k\})\right| > \frac{\varepsilon}{4}.$$

Since (B_n) is necessarily disjoint, we contradict the fact that (μ_n) is uniformly exhaustive. Consequently, the Brooks-Jewett Theorem does imply the Phillips Lemma.

As we remarked, authors have recently studied various relationships that exist between the Brooks-Jewett Theorem and the VHS Theorem (as well as the Nikodym Theorem and the Nikodym Boundedness Theorem). Specifically, we note the papers of Faires **[17]**, Labuda **[21]**, and Drewnowski **[14]**, **[15]**. In **[15]** Drewnowski gives the following characterization of exhaustive set functions.

THEOREM 3.7. *If $\mathscr{R}$ is a σ-ring, X is a Banach space, and μ is a finitely additive X-valued set function defined on $\mathscr{R}$, then μ is exhaustive if and only if each disjoint sequence has a subsequence so that μ is countably additive on the σ-algebra generated by the subsequence.*

We note that Bob Huff **[19]** has given a proof of the Brooks-Jewett Theorem based on (3.7). In fact, one may combine (3.7) with the techniques used in the proof of (3.2) to obtain a short argument for the Brooks-Jewett Theorem. We omit the details and proceed to yet another characterization of exhaustive measures (a characterization based on (3.7) and a result of Diestel **[10]**). If C is a collection of subsets of a set S, then $\sigma(C)$ is the σ-ring generated by C.

THEOREM 3.8. *Suppose that $\mathscr{R}$ is a σ-ring, X is a Banach space, and μ is a bounded, finitely additive X-valued measure defined on $\mathscr{R}$. Then μ is exhaustive if and only if each disjoint sequence (A_n) from $\mathscr{R}$ has a subsequence (A_{n_i}) so that $\mu(\sigma(A_{n_i}))$ is separable.*

PROOF. Suppose μ is exhaustive and (A_n) is a disjoint sequence from $\mathscr{R}$. Use Drewnowski's Theorem to extract a subsequence $A_{n_i} = B_i$, $i = 1, 2, \ldots$, so that μ is countably additive on $\sigma(\{B_i\}_{i=1}^{\infty})$. Since μ is countably additive when restricted to $\sigma(\{B_i\})$, we know that $\hat{\mu}(C_i) \downarrow 0$ whenever $C_i \downarrow \varnothing$, e.g., see Gould **[13]**. Let S be the rational span of $\{\mu(B_i): i = 1, 2, \ldots\}$, let $\varepsilon > 0$, and let $B \in \sigma(\{B_i\})$. Then

$$B \cap \left(\bigcup_{i\geqq n} B_i\right) \downarrow \varnothing$$

as $n \to \infty$; hence

$$\hat{\mu}(B \cap (\bigcup_{i \geqq n} B_i)) \downarrow 0.$$

Choose N such that

$$\hat{\mu}(B \cap (\bigcup_{i > n} B_i)) < \varepsilon$$

for each $n \geqq N$. Recalling that (B_i) is a disjoint sequence and consequently, for each i, $B \cap B_i$ is B_i or $\emptyset$, then

$$\mu(B \cap (\bigcup_{i=1}^{N} B_i)) \in S,$$

and

$$\|\mu(B) - \mu(B \cap (\bigcup_{i=1}^{N} B_i))\| < \varepsilon.$$

The converse follows immediately from the theorem of Diestel **[10]** which asserts that a bounded measure on a σ-ring with separable range is exhaustive.

4. **Concluding Remarks.** We conclude this paper with some remarks and a question which we feel are pertinent to the material discussed above.

First, we note that if one uses the fundamental theorem of Kakutani **[20]** which represents the arbitrary L-space X as an $L^1(\mu)$-space, then Corollary 2.2 characterizes those L-spaces with units. For if X is an L-space with unit μ, then $x \wedge nu \to x$ for each $x \geqq 0$. Hence X is the closed linear span of the order interval $[0, u]$. And the interval $[0, u]$ is weakly compact by (2.5) since $D(u, x)$ exists uniformly for $x \in [0, u]$. In fact,

$$\|nu + x\| + \|nu - x\| - 2\|nu\| = 0$$

for each n and each $x \in [0, u]$.

Our question involves the nature of smooth points in the space fasv $(\mathscr{R}, B(E, F))$ consisting of all finitely additive set functions with finite semivariation defined on the ring $\mathscr{R}$ with values in the space of continuous linear transformation (=operators) from the Banach space E to the Banach space F. An element of $m \in \text{fasv}(\mathscr{R}, B(E, F))$ is endowed with the following norm: $\|m\| = \sup\{\|\sum m(A_i)x_i\| : (A_i)$ is a finite disjoint collection of members of $\mathscr{R}$ and $\|x_i\| \leqq 1$ for each $i\}$. The space fasv$(\mathscr{R}, B(E, F))$ arises naturally when one studies the representation of operators on spaces of continuous Banach-valued functions, e.g., see Dinculeanu **[13]**, especially §19. We remark that while the semivariation norm differs from both the supremum and the total variation norms in general, if $F =$ real numbers then the semivariation and total variation norms agree. (Again we note that Dinculeanu [13] is an excellent source of information about spaces of measures endowed with this norm.)

Given the relationships established earlier between differentiability of the norm in spaces of measures and absolute continuity, as well as an awareness of the space fasv($\mathscr{R}$, $B(E, F)$), it seems that one would naturally be interested in investigating necessary and sufficient conditions for the existence of $D(\mu, \nu)$ for $\mu, \nu \in$ fasv($\mathscr{R}$, $B(E, F)$). This problem seems somewhat intractable. We reproduce an example (from [3]) which points out some of the difficulties.

Lex X be two-dimensional Hilbert space, and let Σ be an algebra of sets such that ba(Σ) contains disjoint positive members μ_1 and μ_2 of unit norm. The norm in ba(Σ, X) is calculated with respect to the operators from the real numbers to X. Then $(0, \mu_2)$ is not absolutely continuous with respect to $(\mu_1, 0)$, yet $D((\mu_1, 0), (0, \mu_2))$ exists because

$$\|(\mu_1, 0) + t(0, \mu_2)\| = (1 + t^2)^{1/2}.$$

On the other hand,

$$(\mu_1, \mu_1) \ll (\mu_1, \mu_2),$$
$$D^+((\mu_1, \mu_2), (\mu_1, \mu_1)) = 2^{1/2},$$

and

$$D^-((\mu_1, \mu_2), (\mu_1, \mu_1)) = 0.$$

We now turn to the equivalence of uniform additivity and uniform exhaustiveness mentioned in the discussion preceding Theorem 3.3L. Suppose K is a uniformly additive subset of ba($\mathscr{R}$) which fails to be uniformly exhaustive. Then there is a disjoint sequence (A_i) from $\mathscr{R}$, a sequence (μ_i) from K, and an $\varepsilon > 0$ such that $|\mu_i|(A_i) > \varepsilon$ for all i. Let ν_i denote the restriction of $|\mu_i|$ to A_i. Then (P_{ν_i}) is a disjoint sequence of projection operators, and $\|P_{\nu_i}(u)\| \xrightarrow{i} 0$ uniformly for $u \in K$. But

$$\|P_{\nu_i}(\mu_i)\| = |\mu_i|(A_i) > \varepsilon,$$

and we have a contradiction.

Conversely, we suppose that K is a uniformly exhaustive subset of ba($\mathscr{R}$) which fails to be uniformly additive. Therefore there is an $\varepsilon > 0$, a disjoint sequence (P_i) in $\mathscr{O}$, and a sequence (μ_i) from K such that $\|P_i(\mu_i)\| < \varepsilon$ for each i. Let $\nu_i = |P_i(\mu_i)|$; since the projections were disjoint, it follows that $\nu_i \wedge \nu_j = 0$ if $i \neq j$. The desired contradiction will be obtained when we establish the following lemma. The reader may notice some similarity between this lemma and discussion immediately preceding (3.2). (For technical reasons involving the Stone isomorphism [16, Chapter 1], we assume that Σ is an algebra; minor modifications amy be made if Σ is a ring.)

LEMMA 4.1. *If $\varepsilon > 0$ and (μ_n) is a disjoint sequence of positive members of* ba(Σ) *such that $\|\mu_n\| > \varepsilon$ for all n, then there is a $\delta > 0$, a disjoint sequence (A_i) from Σ, and a subsequence (μ_{n_i}) of (μ_n) so that $\mu_{n_i}(A_i) > \delta$ for all i.*

PROOF. Suppose that (μ_n) and ε are as in the hypothesis. By passing to the Stone algebra Σ_1, taking the countably additive extension of each measure to the σ-algebra Σ_2 generated by the Stone algebra, and then restricting to the Baire sub-σ-algebra $\mathscr{B}$ of Σ_2, we may (and shall) assume that (μ_n) is a disjoint sequence of positive measures on the Baire σ-algebra of a totally disconnected compact Hausdorff space S and that $\|\mu_n\| > \varepsilon$ for each n. Since the measures are countably additive and $\mathscr{B}$ is a σ-algebra, we produce a disjoint sequence (A_n) from $\mathscr{B}$ so that μ_n is concentrated on A_n, $n = 1, 2, \ldots$. Let F be a closed set contained in A_1 so that $\mu_1(F) > \varepsilon$ and $(C_i^k)_{i=1}^{\infty}$ be an increasing sequence of closed sets lying in A_k such that $\mu_k(C_i^k) \to^i \mu_k(A_k)$, $k = 2, 3, \ldots$. (Recall that Baire measures are regular.) For $k \geqq 2$ and $i \geqq 1$, let f_i^k be a continuous function on S so that $0 \leqq f_i^k \leqq 1$, $f_i^k(F) = 0$, and $f_i^k(C_i^k) = 1$ (Tietze Extension Theorem). Define g_k to be $\Sigma_i f_i^k/2^i$; then $g_k(F) = 0$ and $g_k(\bigcup_i C_i^k) = 0$. Now put $g = \sum_k g_k/2^k$, and let $Z(g)$ be the zero set of g. Since g is continuous, $Z(g)$ is a closed G_δ subset of S; also, $\mu_i(Z(g)) = 0$ for $i > 1$. Let (U_n), $n = 0, 1, \ldots$, be a decreasing sequence of clopen sets ($U_0 = S$) so that $\bigcap U_n = Z(g)$. (Such a sequence may be found since S has a base of clopen sets.)

We next consider the following statement:

(*) There is a positive integer n so that for infinitely many i, $\mu_i(S \backslash U_n) > 3\varepsilon/4$.

Suppose (*) is false. Then there is a positive integer n_1 so that if $i \geqq n_1$, then $\mu_i(U_1) > \varepsilon/4$. Choose $k_1 > 1$ such that

$$\mu_{n_1}(U_1 \backslash U_{k_1}) > \frac{\varepsilon}{4}.$$

Then pick $n_2 > n_1$ so that if $i \geqq n_2$, then

$$\mu_i(S \backslash U_{k_1}) \leqq \frac{3\varepsilon}{4}.$$

Thus

$$\mu_{n_2}(U_{k_1}) > \frac{\varepsilon}{4},$$

and we choose $k_2 > k_1$ such that

$$\mu_{n_2}(U_{k_1} \backslash U_{k_2}) > \frac{\varepsilon}{4}.$$

Continue inductively to manufacture sequences $(U_{k_i} \backslash U_{k_{i+1}})$ and $(\mu_{n_{i+1}})$ so that

$$\mu_{n_{i+1}}(U_{k_i} \backslash U_{k_{i+1}}) > \frac{\varepsilon}{4}$$

for all i. We now use the Stone isomorphism on the sequence $(U_{k_i} \backslash U_{k_{i+1}})$ of clopen sets to obtain the desired sequences in Σ.

Now suppose (*) is true. Let n_1 be the first positive integer so that

$$T = \{i: \mu_i(S \backslash U_{n_1}) > \frac{3\varepsilon}{4}\}$$

is finite. Let $k_1 = 1$, and let $k_2 = \inf(T)$.

$$\mu_{k_1}(U_{n_1}) > \frac{\varepsilon}{2},$$

$$\mu_{k_2}(S \backslash U_{n_1}) > \frac{\varepsilon}{2},$$

and each of U_{n_1} and $S \backslash U_{n_1}$ is clopen. We then repeat the preceding construction with respect to the measures (μ_i), $i \in T$, the σ-algebra $\mathscr{B} \cap [S \backslash U_{n_1}]$, and (*) with $3\varepsilon/4$ replaced by $5\varepsilon/8$. Continuing this process, we obtain the desired conclusion.

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