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APPROXIMATING MAPS INTO FIBER BUNDLES BY HOMEOMORPHISMS

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1. **Introduction.** By a *manifold* we will mean either a finite-dimensional topological n-manifold or a Q-manifold, i.e., a manifold modeled on the Hilbert cube Q. Let $p: E \rightarrow B$ be a fiber bundle and let $f: M \rightarrow E$ be a map, where M, E and B are all manifolds. In this paper we will be interested in the following general question: *When is f homotopic to a homeomorphism $h: M \rightarrow E$ so that ph is close to pf?* Our main results in this direction are Theorem 1, which concerns Q-manifolds, and Theorem 3, which concerns n-manifolds. In Theorem 2 we apply Theorem 1 to the problem of approximating approximate fibrations of Q-manifolds by fiber bundle projections. Theorems 4 and 5 are applications of Theorem 3. Theorem 4 is a new proof of Goad's result on approximate fibrations of n-manifolds [10] and Theorem 5 is a new proof of the codimension 2 tubular neighborhood theorem of Kirby-Siebenmann for n-manifolds [13].

In order to state our results we will need some definitions. All spaces in this paper will be locally compact, separable and metric, unless otherwise stated, and a *proper* map is a map for which preimages of compacta are compact. If α is an open cover of a space Y, then a proper map $f: X \rightarrow Y$ is an *α-equivalence* if there is a map $g: Y \rightarrow X$ so that (1) fg is α-homotopic to the identity and (2) gf is $f^{-1}(\alpha)$-homotopic to the identity. Statement (1) means that the track of each point of Y under the homotopy $fg \simeq id$ lies in some element of α, and (2) means that the track of each point of X under the homotopy $gf \simeq id$ lies in some element of $f^{-1}(\alpha) = \{f^{-1}(U) \mid U \in \alpha\}$. It follows from [11] that if X and Y are ANRs and f is a CE map (i.e., f is proper, onto, and preimages of points have trivial shape), then f is an α-equivalence, for every α. Here is our main result for Q-manifolds.

THEOREM 1. *For each open cover α of a Q-manifold B there is an open cover β of B so that if $p: E \rightarrow B$ is a fiber bundle, with fiber a compact* ANR *for which π_1 of each component is free abelian, then any $p^{-1}(\beta)$-equivalence from a Q-manifold M to E is $p^{-1}(\alpha)$-homotopic to a homeomorphism.*

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Note that the ANR Theorem of Edwards [3, p. 106] implies that E is also a Q-manifold. The α-Approximation Theorem of [8] proves Theorem 1 for the case in which the fiber of $E \to B$ is a point. In general, it is not possible to completely remove the fundamental group condition on the fiber F of $E \to B$, for there are compact Q-manifolds F and homotopy equivalences $f: M \to F$ which are not homotopic to homeomorphisms [3, p. 86].

Recall from [6] that a proper map $p: E \to B$ is an *approximate fibration* provided that given any lifting problem with prescribed initial lift,

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "F"] & B \end{tikzcd}$$

then for any open cover α of B there is a map $\tilde{F}: F \times [0, 1] \to E$ such that $\tilde{F}_0 : X \to E$ agrees with f and $p\tilde{F}$ is α-close to F. Approximate fibrations were introduced in $\{6\}$ as a generalization of the notion of a Hurewicz fibration. It was shown there that if B is path connected, then any two fibers must have the same shape. Also, it follows from [11] that CE maps of ANRs are approximate fibrations.

In [3, p. 105] it is shown that any CE map between Q-manifolds can be approximated by homeomorphisms. This leads to the following general question: *When can an approximate fibration between Q-manifolds be approximated by fiber bundle projections?* In the following result we use Theorem 1 to give some partial answers to this question.

THEOREM 2. *Let $p: E \to B$ be an approximate fibration between Q-manifolds and assume that the fibers are connected. Then p can be approximated by fiber bundle projections provided that any one of the following four conditions is satisfied.*

1. *The fibers of p are shape equivalent to S^1.*
2. *B is homotopy equivalent to a 1-dimensional polyhedron and $\check{\pi}_1$ of each fiber is free or free abelian.*

(*$\check{\pi}_i$ denotes the Cech homotopy groups introduced in* [1].)

3. *B is homotopy equivalent to a 2-dimensional polyhedron and $\check{\pi}_1$ of each fiber vanishes.*
4. *B is homotopy equivalent to an n-dimensional polyhedron, $n \geqq 3$, and $\check{\pi}_i$ of each fiber vanishes, $1 \leqq i \leqq n$.*

In the remainder of this paper we will be concerned with topological n-manifolds. Here is our main result which parallels Theorem 1.

THEOREM 3. *For each open cover α of an n-manifold B, $n \geqq 6$, there is an open cover β of B so that if $p : E \to B$ is a fiber bundle with fiber S^1, then any $p^{-1}(\beta)$-equivalence from an $(n+1)$-manifold M to E, which is already a homeomorphism from ∂M to ∂E, is $p^{-1}(\alpha)$-homotopic* rel ∂M *to a homeomorphism.*

The finite-dimensional α-Approximation Theorem of [5] proves Theorem 3 for the case in which the fiber of $E \to B$ is a point. While Theorem 3 appears to be only a slight generalization of the result from [5], it in fact gives new proofs of the following known results.

THEOREM 4 (GOAD [10]). *Let $p : E^{n+1} \to B^n$ be an approximate fibration of manifolds, $n \geqq 6$, whose fibers are shape equivalent to S^1 and which is already a fiber bundle projection from ∂E to ∂B. Then for any open cover α of B, p is α-homotopic* rel ∂E *to a fiber bundle projection.*

THEOREM 5 (KIRBY-SIEBENMANN [13]). *Let M^n and N^{n+2} be manifolds without boundary, $n \geqq 5$, and assume that M is a locally flat submanifold of N. Then M has a normal microbundle in N.*

As was pointed out in [13], the results of Kister-Mazur [14] and Kneser [15] immediately imply that M has a closed normal 2-disc bundle in N.

Here is how the material of this paper is organized.

§ 2. **A Splitting Lemma.** In this section a result is established which is needed for the proof of Theorem 1. It relies heavily on results from [2].

§ 3. **Proof of Theorem 1.**

§ 4. **Proof of Theorem 2.**

§ 5. **Some Lemmas for Theorem 3.** Here the main technical work in the proof of Theorem 3 is carried out. It relies heavily on a similar program from [5].

§ 6. **Proof of Theorem 3.**

§ 7. **Proof of Theorem 4.**

§ 8. **Proof of Theorem 5.**

For the proofs of Theorems 1 and 2 we refer the reader to [3] for background on Q-manifold theory. The reader who is interested only in our finite-dimensional results can read §§ 5–8 more or less independent of §§ 2–4.

2. **A Splitting Lemma.** The purpose of this section is to establish a splitting result (Lemma 2.2) which will be needed in the proof of Theorem 1. Our main tool is Lemma 2.1, which follows from a splitting result of [2].

There is one new definition which we will need. Let $f: X \to Y$ be a proper map of spaces, let α be an open cover of Y, and let $A \subset Y$ be closed. We say that f is an *α-equivalence over A* if there is a map $g: A \to X$ so that there is an α-homotopy $fg \simeq id$ and an $f^{-1}(\alpha)$-homotopy $gf \mid f^{-1}(A) \simeq id$. This is just a local version of the notion of an α-equivalence which was defined in § 1.

Here is some notation which will be used throughout this section. Let Y be a polyhedron which is written as the union of closed subpolyhedra Y_1 and Y_2, where Y_1 is compact. Choose compacta C and D in Y so that $Y_1 \subset \mathring{C} \subset C \subset \mathring{D}$ (where "$\circ$" denotes topological interior). K will be a compact polyhedron such that π_1 of each component of K is free abelian, and $p = \text{proj}: Y \times K \to Y$.

LEMMA 2.1 ([2, THEOREM 7.2]). *For each open cover α of Y there exists an open cover β of Y so that if X is a polyhedron and $f: X \to Y \times K$ is a $p^{-1}(\beta)$-equivalence over $D \times K$, then there is an $m \geqq 0$, a subdivision of $X \times I^m$ into closed subpolyhedra, $X \times I^m = X_1 \cup X_2$, and a proper map $f': X \times I^m \to Y \times K$ such that*

(1) *$f' \mid X_1: X_1 \to Y_1 \times K$ is a $p^{-1}(\alpha)$-equivalence,*
(2) *$f' \mid X_1 \cap X_2: X_1 \cap X_2 \to (Y_1 \cap Y_2) \times K$ is a $p^{-1}(\alpha)$-equivalence,*
(3) *$f' \mid X_2: X_2 \to Y_2 \times K$ is a $p^{-1}(\alpha)$-equivalence over $(Y_2 \cap C) \times K$,*
(4) *f' is $p^{-1}(\alpha)$-homotopic to $f \circ$* proj: $X \times I^m \to Y \times K$.

REMARKS. Here I^m is the m-cell $[0, 1]^m$. If we represent Q as $[0, 1]^\infty$, then for each m we have a canonical factorization $Q = I^m \times Q_{m+1}$. We will agree to identify I^m with $I^m \times \{0\}$ in Q, where $0 = (0, 0, \cdots) \in Q_{m+1}$.

LEMMA 2.2. *For each open cover α of Y there exists an open cover β of Y so that if M is a Q-manifold and $f: M \to Y \times K$ is a $p^{-1}(\beta)$-equivalence over $D \times K$, then there exists a subdivision of M into closed Q-manifolds, $M = M_1 \cup M_2$, and a proper map $g: M \to Y \times K$ such that*

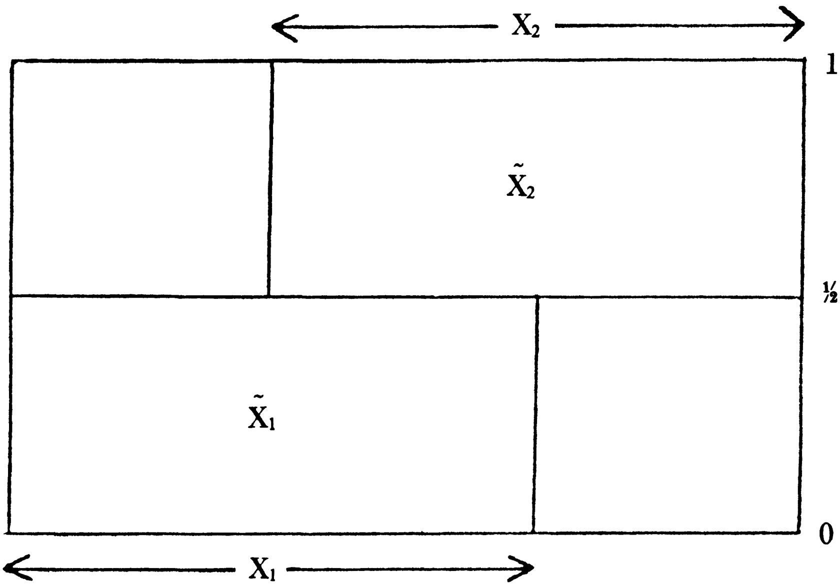
(1) *$M_1 \cap M_2$ is a Q-manifold which is a Z-set in M_1 and in M_2,*
(2) *$g \mid M_1: M_1 \to Y_1 \times K$ is a $p^{-1}(\alpha)$-equivalence,*
(3) *$g \mid M_1 \cap M_2 \cdot M_1 \cap M_2 \to (Y_1 \cap Y_2) \times K$ is a p^{-1} (α)-equivalence,*
(4) *$g \mid M_2: M_2 \to Y_2 \times K$ is a p^{-1} (α)-equivalence over $(Y_2 \cap C) \times K$,*
(5) *g is p^{-1} (α)-homotopic to f.*

PROOF. We may write $M = X \times Q$, for some polyhedron X. Now let β be the open cover of Lemma 2.1 and for any integer m consider the composition,

$$f: X \times I^m \hookrightarrow X \times Q \xrightarrow{f} Y \times K.$$

If m is large enough, then f is a $p^{-1}(\beta)$-equivalence over $D \times K$. Thus by Lemma 2.1 we may choose m large enough so that there is a subdivision $X \times I^m = X_1 \cup X_2$ and a proper map $f' : X \times I^m \to Y \times K$ satisfying properties (1)–(4) stated there.

Consider the following subset of $X \times I^{m+1}$, $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$, where $\tilde{X}_1 = X_1 \times [0, \tfrac{1}{2}]$ and $\tilde{X}_2 = X_2 \times [\tfrac{1}{2}, 1]$. Here is a picture of $X \times 1^{m+1}$.



It is not hard to construct a PL retraction $r : X \times 1^{m+1} \to \tilde{X}$ with contractible point inverses such that the $X \times I^m$-coordinates are moved as little as we please. Recalling the canonical factorization $X \times Q = X \times I^{m+1} \times Q_{m+2}$, we get a CE map $r \times id : X \times Q \to \tilde{X} \times Q_{m+2}$. By [3, p. 103] we can find a homeomorphism $h : X \times Q \to \tilde{X} \times Q_{m+2}$ as close to $r \times id$ as we want. Our required $g : M \to Y \times K$ is defined by the composition

$$X \times Q \xrightarrow{h} \tilde{X} \times Q_{m+2} \xrightarrow{\text{proj}} \tilde{X} \xrightarrow{\text{proj}} X \times I^m \xrightarrow{f} Y \times K.$$

M_1 and M_2 are defined by $M_i = h^{-1}(\tilde{X}_i \times Q_{m+2})$. By [3, p. 54] we conclude that M_1, M_2 and $M_1 \cap M_2$ are Q-manifolds. Clearly $M_1 \cap M_2$ is a Z-set in M_1 and in M_2.

Because h is a homemorphism, our required properties (2)–(4) are obviously satisfied. For (5) we know that g is close to

$$X \times Q \xrightarrow{\text{proj}} X \times I^m \xrightarrow{f'} Y \times K,$$

and we know that f' is $p^{-1}(\alpha)$-homotopic to $\bar{f} \mid X \times I^m$.

3. **Proof of Theorem 1.** In what follows B will be any Q-manifold and $p: E \to B$ will be a fiber bundle with fiber a compact ANR F for which π_1 of each component is free abelian. It will simplify matters to write $B = B_1 \times Q$, where B_1 is a polyhedron, and then consider the fiber bundle $p_1: E \xrightarrow{p} B \xrightarrow{\text{proj}} B_1$. This has fiber $F \times Q$, which is a compact Q-manifold by Edwards' ANR Theorem. Since the factorization $B = B_1 \times Q$ can be chosen so that each $\{b\} \times Q$ has small diameter, it will suffice to prove Theorem 1 with the fiber bundle $p: E \to B$ replaced by $p_1: E \to B_1$. This means that in the statement of Theorem 1 we may assume that B is a polyhedron and the fiber is a compact Q-manifold, F. We first treat the compact case.

Proof of Theorem 1 (B compact). The procedure is to induct on $\dim B$. If $\dim B = 0$, we just use the fact that any homotopy equivalence between compact Q-manifolds whose fundamental group is free abelian must be homotopic to a homeomorphism [3, p. 56]. Passing to the inductive step let $\dim B = n$ and assume that the result is true for all $(n-1)$-dimensional base spaces. Choose a fine subdivision of B and let B_1 be the $(n-1)$-skeleton of B. Without loss of generality, assume that $B - B_1$ is a single n-cell. Let $B = B_1 \cup \Delta$, where Δ is an n-simplex whose combinatorial interior, $B - B_1$, is denoted by R^n (euclidean n-space). We let $rB^n = [-r, r]^n \subset R^n$, with $1B^n = B^n$, and ∂B^n, $\mathring{B}^n$ denote the boundary and interior, respectively, of B^n.

Consider the restriction,

$$\bar{f} = f \mid (pf)^{-1}(R^n) : (pf)^{-1}(R^n) \to p^{-1}(R^n),$$

which is a proper map that is a p-small equivalence over $p^{-1}((r+2)B^n)$, for some large r. (By a *p-small* equivalence we mean a $p^{-1}(\gamma)$-equivalence, for some fine γ. This will eliminate the need to add up a number of estimates.) If we factor $F = K \times Q$, for some compact polyhedron K, then $p^{-1}(R^n) = R^n \times K \times Q$. By Lemma 2.2 we have a subdivision, $(pf)^{-1}(R^n) = M_1 \cup M_2$, and a proper map $g: (pf)^{-1}(R^n) \to p^{-1}(R^n)$ such that

(1) $M_1 \cap M_2$ is a Q-manifold which is a Z-set in M_1 and in M_2,

(2) $g \mid M_1 : M_1 \to p^{-1}(rB^n)$ is a p-small equivalence,

(3) $g \mid M_1 \cap M_2 : M_1 \cap M_2 \to p^{-1}(\partial rB^n)$ is a p-small equivalence,

(4) $g \mid M_2 : M_2 \to p^{-1}(R^n - r\mathring{B}^n)$ is a p-small equivalence over $p^{-1}((r+1)B^n - r\mathring{B}^n)$,

(5) g is p-small homotopic to $\bar{f}$.

By using the (estimated) Homotopy Extension Theorem [5, Proposition 2.1] we may assume that $g = f$ over $p^{-1}(R^n - (r+2)\mathring{B}^n)$, and g is p-small homotopic to $\tilde{f}$ rel $(pf)^{-1}(R^n - (r+2)\mathring{B}^n)$. Then g extends to $\tilde{g} : M \to E$ so that $\tilde{g} = f$ over $p^{-1}(B - R^n)$. We conclude that $\tilde{g}$ is p-small homotopic to f. Let $\tilde{M}_2 = M_2 \cup (M - (pf)^{-1}(\mathring{R}^n))$ and use [5, Proposition 3.2] to conclude that $\tilde{g} \mid \tilde{M}_2 : \tilde{M}_2 \to p^{-1}(B - r\mathring{B}^n)$ is a p-small equivalence. (Proposition 3.2 of [5] is a result which enables one to sew together α-equivalences.)

Let $s : B - r\mathring{B}^n \to B_1$ be a small CE retraction and let $\tilde{s} : p^{-1}(B - r\mathring{B}^n) \to p^{-1}(B_1)$ be a CE retraction which covers s. Consider the composition

$$\tilde{s}\tilde{g} : \tilde{M}_2 \to p^{-1}(B - r\mathring{B}^n) \to p^{-1}(B_1),$$

which must be a p-small equivalence for r large. By our inductive assumption we have $\tilde{s}\tilde{g}$ p-small homotopic to a homeomorphism $h_2 : \tilde{M}_2 \to p^{-1}(B_1)$. Let $h_0 : M_1 \cap \tilde{M}_2 \to p^{-1}(\partial rB^n)$ be a homeomorphism which is p-small homotopic to $g \mid M_1 \cap M_2 : M_1 \cap M_2 \to p^{-1}(\partial rB^n)$. By [3, p. 30] we may correct h_2 to get a homeomorphism $\tilde{h}_2 : \tilde{M}_2 \to p^{-1}(B_1)$ which is p-small homotopic to h_2 and which agrees with h_0 on $M_1 \cap M_2$.

We have a p-small homotopy $\tilde{h}_2 \simeq h_2 \simeq \tilde{s}\tilde{g} \mid \tilde{M}_2 \simeq \tilde{g} \mid \tilde{M}_2 \simeq f \mid \tilde{M}_2$, because there is a p-small homotopy $\tilde{s} \simeq id$. By the (estimated) homotopy Extension Theorem we can construct a p-small homotopy of f to $f' : M \to E$, where $f' = \tilde{h}_2$ on $\tilde{M}_2$ and $f'(M_1) \subset p^{-1}(rB^n)$. We know that there is a p-small homotopy of $f' \mid M_1$ to $g \mid M_1 : M_1 \to p^{-1}(rB^n)$. This implies that $f' \mid M_1 : M_1 \to p^{-1}(rB^n)$ is a homotopy equivalence. This means that $f' \mid M_1 : M_1 \to p^{-1}(rB^n)$ is homotopic to a homeomorphism $h_1 : M_1 \to p^{-1}(rB^n)$ rel $M_1 \cap \tilde{M}_2$. Then $\tilde{h}_2$ and h_1 piece together to define a homeomorphism $h : M \to E$ which is p-small homotopic to f.

Proof of Theorem 1 (B arbitrary). Write $B = B_1 \cup B_2 \cup \cdots$, where the B_i are compact subpolyhedra such that $B_i \cap B_j = \Phi$ for $|i - j| \geqq 2$. By repeatedly applying Lemma 2.2 to the interiors of the cells in $\mathring{B}_{2i}$, in order of decreasing dimension, we can carve out of M a closed Q-manifold $N_{2i} \subset M$ and a proper map $f_{2i} : N_{2i} \to p^{-1}(B - \mathring{B}_{2i})$ such that f_{2i} is a p-small equivalence and f_{2i} is p-small homotopic to $f \mid N_{2i}$. In fact, N_{2i} can be constructed to contain $(pf)^{-1}(B - \tilde{B}_{2i})$ in its interior, for any compact neighborhood $\tilde{B}_{2i}$ of B_{2i}. Arrange the B_i so that each $B_i \cap B_{i+1}$ is collared in B_i and B_{i+1}. Let us write $N_{2i} = R_{2i} \cup S_{2i}$, where $f_{2i}(R_{2i}) \subset B_1 \cup \cdots \cup B_{2i-1}$, $f_{2i}(S_{2i}) \subset B_{2i+1} \cup B_{2i+2} \cup \cdots$. Then R_{2i} and S_{2i} are disjoint Q-manifolds such that R_{2i} is compact. Moreover, there is a p-small equivalence $g_{2i} : R_{2i} \to p^{-1}(B_1 \cup \cdots \cup$

B_{2i-1}) which is p-small homotopic to $f \mid R_{2i}$.

Using the compact case already established, let $h_{2i} : R_{2i} \to p^{-1}(B_1 \cup \cdots \cup B_{2i-1})$ be a homeomorphism which is p-small homotopic to g_{2i} and therefore p-small homotopic to $f \mid R_{2i}$. For each i let $A_i = h_{2i+2}^{-1} p^{-1}(B_i \cap B_{i+1})$, which is a bicollared compact Q-manifold in M. The A_i naturally subdivide M into compact Q-manifolds M_i so that

(1) $M = M_1 \cup M_2 \cup \cdots$,

(2) $M_i \cap M_j = \Phi$ for $|i - j| \geqq 2$,

(3) $M_i \cap M_{i+1} = A_i$,

(4) A_i is a Z-set in M_i and M_{i+1}.

Moreover, it is easy to see that f is p-small homotopic to a map $f' : M \to E$ such that $f' = h_{2i+2}$ on A_i and $f' \mid M_i : M_i \to p^{-1}(B_i)$ is a p-small equivalence.

Now proceeding as in the proof of the compact case we can find a p-small homotopy of $f' \mid M_i : M_i \to p_1{}^{-1}(B_i)$ rel $A_{i-1} \cup A_i$ to a homeomorphism of M_i onto $p^{-1}(B_i)$. These homeomorphisms then piece together to give our desired homeomorphism of M to E.

4. **Proof of Theorem 2.** We will first need to establish two lemmas. The Hurewicz fibrations which appear in these results have total spaces which are only separable metric spaces. All other spaces are locally compact.

LEMMA 4.1. *Let $p : E \to B$ be an approximate fibration between ANRs and let $q : \mathscr{E} \to B$ be any Hurewicz fibration such that the fibers and the total space are homotopy equivalent to countable complexes. If there is a homotopy equivalence $h : E \to \mathscr{E}$ for which $qh \simeq p$, then for every open cover α of B there are maps $E \underset{g}{\overset{f}{\rightleftarrows}} \mathscr{E}$ such that fg is $q^{-1}(\alpha)$-homotopic to id and gf is p^{-1} (α)-homotopic to id.* (We call f an *approximate* fiber homotopy equivalence.)

REMARKS ON PROOF. In case $E \to B$ is also a Hurewicz fibration, the given homotopy $qh \simeq p$ enables us to homotop h to a fiber preserving map $f : E \to \mathscr{E}$. (This means that $qf = p$.) By [4, Theorem 2.2] we conclude that f is a fiber homotopy equivalence (f.h.e). If $g : \mathscr{E} \to E$ is a fiber homotopy inverse of f, then f and g fulfill our requirements.

In the general case $E \to B$ is only assumed to be an approximate fibration, but we can still homotop h to a fiber preserving map $f : E \to \mathscr{E}$. However f cannot be a f.h.e. in all cases. There is a "delooping trick" which enables one to construct a map $g : \mathscr{E} \to E$ which is approximately fiber preserving, and so that there are homotopies $fg \simeq id$, $gf \simeq id$ which are approximately fiber preserving [9]. It is clear that such maps f and g fulfill our requirements.

LEMMA 4.2. *If $q : \mathscr{E} \to B$ is any Hurewicz fibration over an* ANR *whose fibers are homotopy equivalent to S^1 and whose total space is homotopy equivalent to a countable complex, then $\mathscr{E}$ is f.h.e. to a fiber bundle over B with fibers S^1.*

PROOF. First assume that B is a polyhedron. By inductively working our way through the skeleta of B our problem is quickly reduced to the following: *If $q : \mathscr{E} \to \Delta$ is a Hurewicz fibration over an n-cell and $f : q^{-1}(\partial\Delta) \to E$ is a f.h.e. to an S^1-fiber bundle $E \to \partial\Delta$ then f extends to a f.h.e. $\tilde{f} : \mathscr{E} \to \tilde{E}$, where $\tilde{E} \to \Delta$ is an S^1-fiber bundle extending $E \to \partial\Delta$.* It is well-known that the inclusion of the homeomorphisms of S^1 into the self-homotopy equivalences of S^1 is a homotopy equivalence. From this we conclude that the bundle $E \to \partial\Delta$ is trivial and f extends in the required manner.

For the general case of an ANR base let B_1 be a polyhedron for which there is a homotopy equivalence $h : B_1 \to B$. Form the following diagram:

$$\begin{tikzcd}
E_1 \arrow[r, "h_3"] \arrow[d, "q_3"'] & E \arrow[r, "h_2"] \arrow[d, "q_2"'] & \mathscr{E}_1 \arrow[r, "h_1"] \arrow[d, "q_1"'] & \mathscr{E} \arrow[d, "q"'] \\
B \arrow[r, "h^{-1}"] & B_1 \arrow[r, "id"] & B_1 \arrow[r, "h"] & B
\end{tikzcd}$$

The last rectangle is a pull-back diagram. Thus $q_1 : \mathscr{E}_1 \to B_1$ is a Hurewicz fibration over a polyhedron and h_1 is a homotopy equivalence for which $qh_1 = hq_1$. (To see that h_1 is a homotopy equivalence use the five lemma and the homotopy sequences of q_1 and q.) In the middle rectangle $q_2 : E \to B_1$ is an S^1-fiber bundle and h_2 is a f.h.e, which follows from the special case above. The first rectangle is also a pull-back diagram, where h^{-1} is a homotopy inverse of h. Then $E_1 \to B$ is an S^1-fiber bundle over B and there is a homotopy equivalence $h_4 : E_1 \to \mathscr{E}$ for which $qh_4 \simeq q_3$(Let $h_4 = h_1h_2h_3$.) Just as in Lemma 4.1, h_4 must be homotopic to a f.h.e. ∎

PROOF OF THEOREM 2. 1. If $q : \mathscr{E} \to B$ is the mapping path fibration of the map $p : E \to B$, then there is a homotopy equivalence $h : E \to \mathscr{E}$ such that $qh \simeq p$[17, p. 99]. We can use Lemma 4.1 to construct an approximate f.h.e. $f : E \to \mathscr{E}$. Since the fibers of $p : E \to B$ are shape equivalent to S^1 we conclude that the fibers of $q : \mathscr{E} \to B$ are homotopy equivalent to S^1. Thus by Lemma 4.2 there is an S^1-fiber bundle $q_1 : E_1 \to B$ and a f.h.e. $h_1 : \mathscr{E} \to E_1$. Then $h_1h : E \to E_1$ is a $q_1{}^{-1}(\alpha)$-

equivalence which is homotopic to an approximate fiber preserving homeomorphism $g: E \to E_1$, by Theorem 1. Thus $q_1 g: E \to B$ is a fiber bundle projection which is close to p. 2. With the given conditions it follows from [4, Theorem 4] that the mapping path fibration of p is f.h.e. to a fiber bundle $q_1: E_1 \to B$ with fiber a compact Q-manifold whose fundamental group is free abelian. By Theorem 1 we conclude that there is a homeomorphism $g: E \to E_1$ which is approximately fiber preserving. Then proceed as in 1.

3. If B is a 2-dimensional polyhedron, then it follows from [4, Theorem 3] that the mapping path fibration of p is f.h.e. to a fiber bundle with fiber a simply connected compact Q-manifold. Then proceed as in 1. If B is only assumed to be homotopy equivalent to a 2-dimensional polyhedron, then we can use pull-back diagrams as in the proof of Lemma 4.2 to prove that the mapping path fibration of p is still f.h.e. to a fiber bundle with fiber a simply connected compact Q-manifold. (The use of these pull-back diagrams avoids the unnecessary assumption in [4, Theorem 3] that B be simple homotopy equivalent to a 2-dimensional polyhedron.)

4. We proceed as in 3.

5. **Some Lemmas for Theorem 3.** In this section we establish two results which will be needed for the proof of Theorem 3. These results, the Handle Lemma and Handle Theorem, are generalizations of two similarly-named results from [5]. It turns out that the proofs of these generalizations are quite similar to the proofs given in [5]. Thus we will assume that the reader is familiar with [5], and our duty will be to describe only the necessary changes in proofs.

Here is some notation which we will need for our Handle Lemma. Let V^{n+1} be a topological manifold, $n = m + k \geqq 6$, and let $f: V \to B^k \times R^m \times S^1$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times R^m \times S^1)$ and f is a homeomorphism over $(B^k - {}^1/_2 \mathring{B}^k) \times R^m \times S^1$. p will denote proj: $B^k \times R^m \times S^1 \to B^k \times R^m$.

Handle Lemma. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if f is a $p^{-1}(\delta)$-equivalence over $B^k \times 3B^m \times S^1$ and $m \geqq 1$, then*

(1) *there exists a $p^{-1}(\epsilon)$-equivalence $F: B^k \times R^m \times S^1 \to B^k \times R^m \times S^1$ such that $F = id$ over $[(B^k - {}^5/_6 \mathring{B}^k) \times R^m \cup B^k \times (R^m - 4\mathring{B}^m)] \times S^1$.*

(2) *there exists a homeomorphism $\varphi: f^{-1}(U) \to F^{-1}(U)$ such that $F\varphi = f \mid f^{-1}(U)$, where $U = [(B^k - {}^5/_6 \mathring{B}^k) \times R^m \cup B^k \times 2B^m] \times S^1$.*

Remarks. (1) The reader will notice that the only difference between the above statement and the statement of the Handle Lemma of [5] is the extra S^1-factor.

(2) δ depends only on n and ϵ. It is calculated with respect to the standard metric on $B^k \times R^m$.

Changes in Proof. Here is the Main Diagram which has to be constructed. Note that this is obtained from the corresponding diagram of [5] by multiplying everything by S^1.

$$\begin{tikzcd}
B^k \times R^m \times S^1 \arrow[rr, "F"] & & B^k \times R^m \times S^1 \\
B^k \times R^m \times S^1 \arrow[u, "j \times id"] \arrow[rr, "F'"] \arrow[d, "id \times e^m \times id"] & & B^k \times R^m \times S^1 \arrow[u, "j \times id"'] \arrow[d] \\
B^k \times T^m \times S^1 & W_3 \arrow[l, "h"'] \arrow[r, "f_3"] & B^k \times T^m \times S^1 \\
 & W_2 \arrow[u, hook] \arrow[r, "f_2"] \arrow[d, hook] & (B^k \times T^m - \mathring{D}^n) \times S^1 \arrow[u, hook, "id \times e^m \times id"'] \arrow[d, hook] \\
 & W_1 \arrow[r, "f_1"] & [B^k \times T^m - (\tfrac{2}{3} B^k \times \{x_0\})] \times S^1 \\
 & W_0 \arrow[u, hook] \arrow[r, "f_0"] \arrow[d, "i_0"'] & B^k \times T_0^m \times S^1 \arrow[u, hook] \arrow[d, "id \times i \times id"] \\
 & V \arrow[r, "f"] & B^k \times R^m \times S^1
\end{tikzcd}$$

1. **Construction of W_0.** Just as in [5], W_0 is the fiber product of f and $id \times i \times id$. It easily follows from [5] that for any compactum C in $B^k \times T_0^m$ and any $\delta_0 > 0$, δ can be chosen small enough so that f_0 is a $p^{-1}(\delta_0)$-equivalence over $C \times S^1$. (We have in mind the compactum $C = B^k \times Y_3$ as defined in [5].) Also p will be used to denote projection of $B^k \times T_0^m \times S^1$ to $B^k \times T_0^m$.

II. **Construction of W_1.** Just as in [5], W_1 is formed so that f_1 is a $p^{-1}(\delta_1)$-equivalence over

$$[B^k \times T^m - \frac{3}{4}\mathring{B}^k \times (T^m - Y_2)] \times S^1.$$

III. **Construction of W_2.** This is the first step in which there is a significant variation from the corresponding step in [5]. Consider the open set

$$G = \left[\frac{4}{5}\ \mathring{B}^k \times (T^m - Y_1)\ \right] - \left[\frac{3}{4}\ B^k \times (T^m - \mathring{Y}_2)\ \right],$$

which we identify with $S^{n-1} \times R$. If δ_1 is small enough, then f_1 restricts to a proper map

$$f_1 \mid f_1^{-1}(S^{n-1} \times R \times S^1) : f_1^{-1}(S^{n-1} \times R \times S^1) \to S^{n-1} \times R \times S^1$$

which is a $p^{-1}(\delta_1)$-equivalence over $S^{n-1} \times [-2, 2] \times S^1$. By Theorem 6.10 of [5] there is a codimension 1, bicollared, compact submanifold S of $f_1^{-1}(S^{n-1} \times (-1, 1) \times S^1)$ such that S separates $f_1^{-1}(S^{n-1} \times \{-1\} \times S^1)$ from $f_1^{-1}(S^{n-1} \times \{1\} \times S^1)$ and $f_1 \mid S : S \to S^{n-1} \times R \times S^1$ is a homotopy equivalence.

Assertion. *S is homeomorphic to $S^{n-1} \times S^1$.*

Proof. We have a homotopy equivalence,

$$S \xrightarrow{f_1} S^{n-1} \times R \times S^1 \xrightarrow{\text{proj}} S^{n-1} \times S^1.$$

Since dim S $\geqq 6$ it follows from the Fibering Theorem of [7] that

$$S \xrightarrow{f_1} S^{n-1} \times R \times S^1 \xrightarrow{\text{proj}} S^1$$

is homotopic to a fiber bundle projection $p : S \to S^1$. The fiber of this map is S^{n-1} and the characteristic map of the bundle $p : S \to S^1$ is a homeomorphism $w : S^{n-1} \to S^{n-1}$, which must be homotopic to *id* because S is homotopy equivalent to a trivial bundle over S^1. Therefore w is isotopic to *id* (see [12, p. 34]). This means that the bundle $p : S \to S^1$ is trivial and therefore $S \simeq S^{n-1} \times S^1$.

Define an n-ball by

$$D^n = \tfrac{3}{4} B^k \times (T^m - \mathring{Y}_2)$$

and let W_2 be the closure of the component of $W_1 - S$ containing $f_1^{-1}(Y_0 \times S^1)$. Our map $f_2 : W_2 \to (B^k \times T^m - \mathring{D}^n) \times S^1$ is defined by $f_2 = f_1 \mid W_2$.

IV. **Constuction of W_3.** W_3 is constructed from W_2 by attaching a copy of $B^n \times S^1$ to W_2 along S. W_3 is a compact $(n+1)$-manifold which is homotopy equivalent to $B^k \times T^m \times S^1$. Just as in [5] we can construct a $p^{-1}(\delta_3)$-equivalence $f_3 : W_3 \to B^k \times T^m \times S^1$ which agrees with f_1 over

$$[(B^k - \tfrac{5}{6}\mathring{B}^k) \times T^m \cup B^k \times Y_0] \times S^1.$$

δ_3 can be made as small as we please by making δ_1 small.

V. **Construction of h.** It follows from [16, p. 280] that there is a homeomorphism $h : W_3 \to B^k \times T^m \times S^1$ which agrees with f_3 over $(B^k-(5/6)\mathring{B}^k) \times T^m \times S^1$ and which is homotopic to f_3 rel $f_3^{-1}((B^k - (5/6)\mathring{B}^k) \times T^m \times S^1)$.

VI. **Construction of F'.** In this step we also encounter a significant variation from the corresponding step in [5]. Let $\tilde{F} : B^k \times R^m \times S^1 \to B^k \times R^m \times S^1$ be the covering of f_3h^{-1} which is the *id* on $(B^k - (5/6)\mathring{B}^k) \times R^m \times S^1$. Since $f_3h^{-1} \simeq id$ it follows that $\tilde{F}$ is bounded. If δ_3 is small, then $\tilde{F}$ must be a $p^{-1}(\epsilon_1)$-equivalence for a small ϵ_1. Note that the homotopy $f_3h^{-1} \simeq id$ also implies that there is a bounded homotopy $\tilde{F} \simeq id \operatorname{rel} (B^k - (5/6)\mathring{B}^k) \times R^m \times S^1$. By using this homotopy of $\tilde{F}$ to *id* only in the S^1-factor we can homotop $\tilde{F}$ to a map $F' : B^k \times R^m \times S^1 \to B^k \times R^m \times S^1$ such that

(1) $F' = \tilde{F}$ over $[(B^k - (5/6)\mathring{B}^k) \times R^m \cup B^k \times 4B^m] \times S^1$,

(2) $pF' = p\tilde{F}$,

(3) $qF' = q$ over a neighborhood of ∞, where $q = \text{proj}: B^k \times R^m \times S^1 \to S^1$.

VII. **Construction of $\hat{f}$.** This goes exactly as in [5].

VIII. **Construction of F.** This also goes exactly as in [5]. This is one small catch. The verification that F is a $p^{-1}(\epsilon)$-equivalence is more complicated than what occurs in [5]. For details, the reader should consult Step C in the proof of the Handle Lemma of [2].

IX. **Construction of φ.** This goes exactly as in [5].

For the Handle Theorem we use the same notation as in the Handle Lemma.

HANDLE THEOREM. *For every $\epsilon > 0$ there exists a $\delta > 0$ so that if f is a $p^{-1}(\delta)$-equivalence over $B^k \times 3B^m \times S^1$, then there exists a proper map $\tilde{f} : V \to B^k \times R^m \times S^1$ such that*

(1) $\tilde{f}$ is a $p^{-1}(\epsilon)$-equivalence over $B^k \times 2.5B^m \times S^1$,

(2) $\tilde{f} = f$ over $[(B^k - 2/3\,\mathring{B}^k) \times R^m \cup B^k \times (R^m - 2\mathring{B}^m)] \times S^1$,

(3) $\tilde{f}$ is a homeomorphism over $B^k \times B^m \times S^1$.

PROOF. In [5] this result was established in the case that $S^1 = \{\text{point}\}$. The main ingredients of proof were a Handle Lemma and the following fact: *If W^n is a compact manifold and $g: W^n \to B^n$ is a homotopy equivalence which is a homeomorphism from ∂W to ∂B^n, then $g \mid \partial W$ extends to a homeomorphism $\tilde{g}: W \to B^n$.*

Using the Handle Lemma of this paper we can repeat the proof given in [5] provided that we have the following result: *If W^{n+1} is a compact manifold and $g: W \to B^n \times S^1$ is a homotopy equivalence which is a homeomorphism from ∂W to $\partial B^n \times S^1$, then $g \mid \partial W$ extends to a homeomorphism.* But this is established in [16, p. 280].

6. **Proof of Theorem 3.** As in § 5 we can rely on [5] for most of our details. Let $f: M \to E$ be a $p^{-1}(\beta)$-equivalence which is a homeomorphism from ∂M to ∂E. Just as in the proof of the α-Approximation Theorem of [5] we can use the Handle Theorem of § 5 to construct a homeomorphism $g: M \to E$ which is $p^{-1}(\alpha_1)$-close to f and which agrees with f on ∂M. This requires an induction over small handles in B, and α_1 can be chosen fine by choosing β fine. We have to do a little more work to get a homeomorphism which is $p^{-1}(\alpha)$-homotopic to f.

Consider the map $fg^{-1}: E \to E$, which is $p^{-1}(\alpha_1)$-close to id. Then fg^{-1} is easily seen to be $p^{-1}(\alpha_2)$-homotopic to a fiber-preserving map $k: E \to E$, where α_2 can be chosen fine by choosing β fine. Now k is a homotopy equivalence which is fiber preserving. By [4, Theorem 2.2] we conclude that k is a f.h.e. Just as in the proof of Lemma 4.2 we see that k is fiber homotopic to a fiber preserving homeomorphism $k_1: E \to E$. Then $h = k_1 g: M \to E$ is a homeomorphism which is $p^{-1}(\beta)$-homotopic to f.

7. **Proof of Theorem 4.** We proceed as in the proof of Theorem 2. By Lemmas 4.1 and 4.2 there is a fiber bundle $q: E_1 \to B$ which fiber S^1 and an approximate f.h.e. $f: E \to E_1$. It is easy to adjust f so that $f \mid \partial E: \partial E \to \partial E_1$ is a f.h.e. Since $f \mid \partial E$ is a f.h.e. between bundles with fiber S^1, then $f \mid \partial E$ is fiber homotopic to a fiber preserving homeomorphism $f_0: \partial E \to \partial E_1$. Then f can be homotoped to an approximate f.h.e. $f': E \to E_1$ so that $f' \mid \partial E = f_0$. Using Theorem 3 there is a homeomorphism $h: E \to E_1$ so that qh is close to qf' and $h \mid \partial E$ is fiber preserving. Then $qh: E \to B$ is a fiber bundle projection which equals p on ∂E and which is $p^{-1}(\alpha)$-homotopic to p rel ∂E.

8. **Proof of Theorem 5.** The following handle lemma is the main step in the proof of Theorem 5. This should be compared with the Handle Lemma 4.1 of [13].

LEMMA 8.1. *Let $h: B^k \times R^m \times R^2 \to B^k \times R^m \times R^2$ be an open embedding, $k + m \geqq 5$, such that $h = id$ on $B^k \times R^m \times \{0\}$ and h is fiber preserving on $(B^k - 1/2\, \mathring{B}^k) \times R^m \times R^2$ (i.e.,*

$h(\{x\} \times R^2) \subset [x] \times R^2$, for all $x \in (B^k - 1/2\,\mathring{B}^k) \times R^m$). Then there exists a homeomorphism $f: \mathrm{B}^k \times R^m \times R^2 \to B^k \times R^m \times R^2$ such that

(1) *$f = id$ on $(B^k \times R^m \times \{0\}) \cup (\partial B^k \times R^m \times R^2)$,*
(2) *f has compact support,*
(3) *fh is fiber preserving on some neighborhood of $B^k \times \{0\} \times \{0\}$.*

PROOF. Without loss of generality we may assume that $h(\{x\} \times \partial(tB^2)) = \{x\} \times \partial(tB^2)$ for all $x \in (B^k - 1/2\,\mathring{B}^k) \times R^m$ and all t. To see this we first use the Kister-Mazur result to construct a new open embedding $h_1 : B^k \times R^m \times R^2 \to B^k \times R^m \times R^2$ for which
(1) $h_1 = id$ on $B^k \times R^m \times \{0\}$,
(2) h_1 is fiber preserving on $(B^k - 1/2\,\mathring{B}^k) \times R^m \times R^2$,
(3) $h_1(\{x\} \times R^2) = \{x\} \times R^2$, for all $x \in (3/4\,B^k - 2/3\,\mathring{B}^k) \times R^m$,
(4) $h_1 = h$ on $1/2\,B^k \times R^m \times R^2$.
By the Kneser result we may assume that each restriction,

$$h_1 \mid \{x\} \times R^2 : \{x\} \times R^2 \to \{x\} \times R^2,$$

lies in the orthogonal group $O(2)$, for all $x \in (3/4\,B^k - 2/3\,\mathring{B}^k) \times R^m$. This implies that $h_1(\{x\} \times \partial(tB^2)) = \{x\} \times \partial(tB^2)$, for all $x \in (3/4\,B^k - 2/3\,\mathring{B}^k) \times R^m$. Then all we have to do is establish our result for the restriction,

$$h_1 \mid 3/4\,B^k \times R^m \times R^2 : 3/4\,B^k \times R^m \times R^2 \to 3/4\,B^k \times R^m \times R^2.$$

By a *squeeze* we will mean an open embedding s: $B^k \times 2\mathring{B}^m \times R^2 \to B^k \times 2\mathring{B}^m \times R^2$ which is of the form

$$s(x, y) = (x, \theta(x) \cdot y),$$

where $\theta : B^k \times 2\mathring{B}^m \to (0, 1]$ is a map which sends $B^k \times B^m$ to 1 and for which $\lim \{\theta(x) \mid x \to B^k \times \partial(2B^m)\} = 0$. Note that squeezes s can be chosen so that

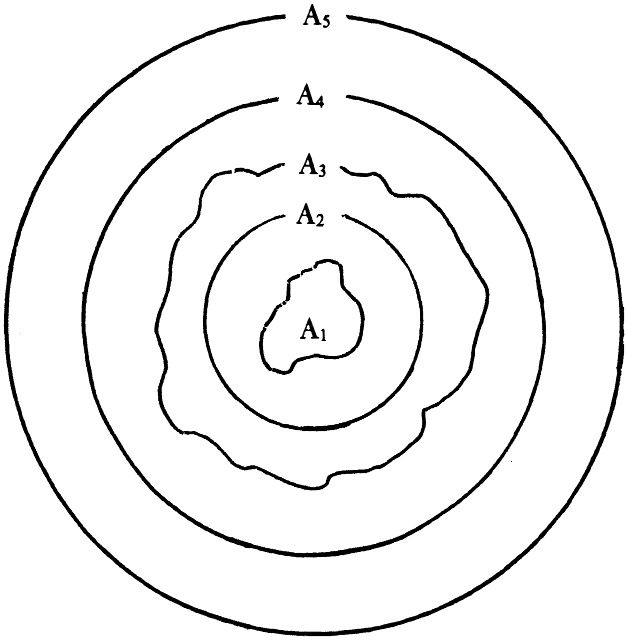
$$hs(B^k \times 2\mathring{B}^m \times 2B^2) \subset B^k \times 2\mathring{B}^m \times R^2.$$

Now choose squeezes s_i along with associated maps θ_i, $1 \leqq i \leqq 5$, so that
(1) $\theta_4 = 1/2\,\theta_5$,
(2) $\theta_3 = \theta_4$ on $\partial B^k \times 2\mathring{B}^m$,
(3) $hs_i(B^k \times 2\mathring{B}^m \times B^2) \subset s_{i+1}(B^k \times 2\mathring{B}^m \times B^2)$, for $i = 1$ and 3,
(4) $s_2(B^k \times 2\mathring{B}^m \times B^2) \subset hs_3(B^k \times 2\mathring{B}^m \times B^2)$.

Let

$$A_i = s_i(B^k \times 2\mathring{B}^m \times B^2), \qquad \text{for } i = 2, 4, 5,$$
$$A_i = hs_i(B^k \times 2\mathring{B}^m \times B^2), \qquad \text{for } i = 1, 3.$$

Here is a picture.



Let $r : A_5 - \mathring{A}_2 \to A_5 - \mathring{A}_4$ be the natural fiber preserving retraction. ($\mathring{A}_i$ = topological interior, $Bd(A_i)$ = topological boundary, ∂A_i = combination boundary.) By restriction, this induces a retraction

$$\bar{r} : A_5 - \mathring{A}_3 \to A_5 - \mathring{A}_4.$$

ASSERTION 1. $\bar{r} \simeq \mathrm{id}\ \mathrm{rel}\ A_5 - \mathring{A}_4$.

PROOF. If $u : h^{-1}(A_3 - \mathring{A}_1) \to h^{-1}(Bd(A_3))$ is the natural fiber preserving retraction, then the retraction

$$huh^{-1} : A_3 - \mathring{A}_1 \to Bd(A_3)$$

induces a retraction $v : A_5 - \mathring{A}_1 \to A_5 - \mathring{A}_3$. If $r_t : r \simeq \mathrm{id}\ \mathrm{rel}\ A_5 - \mathring{A}_4$ is the natural fiber preserving homotopy, then $ur_t \mid A_5 - \mathring{A}_3 : \bar{r} \simeq id$ is our desired homotopy.

Note that s_5^{-1} gives a homeomorphism of $A_5 - \mathring{A}_4$ onto $B^k \times 2\mathring{B}^m \times (B^2 - 1/2\mathring{B}^2)$. Then define

$$g = s_5^{-1}\bar{r} : A_5 - \mathring{A}_3 \to B^k \times 2\mathring{B}^m \times (B^2 - 1/2\,\mathring{B}^2).$$

The restriction $g \mid Bd(A_3)$ takes $Bd(A_3)$ to $B^k \times 2\mathring{B}^m \times \partial(1/2B^2)$, and therefore we have a map,

$$g_1 : B^k \times 2\mathring{B}^m \times \partial B^2 \xrightarrow{hs_3} Bd(A_3) \xrightarrow{g} B^k \times 2\mathring{B}^m \times \partial(1/2\, B^2).$$

If the s_i squeeze everything sufficiently close to $B^k \times 2\mathring{B}^m \times \{0\}$, then g_1 is nearly fiber preserving. Also $g_1 \mid \partial B^k \times 2\mathring{B}^m \times \partial B^2$ *is* fiber preserving. By performing a nearly fiber preserving homotopy we can adjust g_1 slightly rel $\partial B^k \times 2\mathring{B}^m \times \partial B^2$ so that it is fiber preserving. Since the inclusion of the homeomorphisms of ∂B^2 into the homotopy equivalences of ∂B^2 is a homotopy equivalence, we conclude that there is a nearly fiber preserving homotopy $g_1 \simeq g_2$ rel $\partial B^k \times 2\mathring{B}^m \times \partial B^2$, where g_2 is a fiber preserving homeomorphism. Using the (estimated) Homotopy Extension Theorem, there is a nearly fiber preserving homotopy $g \simeq \bar{g}$ rel $\partial(A_5 - \mathring{A}_3)$, where $\bar{g}$ is a map of $A_5 - \mathring{A}_3$ to $B^k \times 2\mathring{B}^m \times (B^2 - 1/2\mathring{B}^2)$ which agrees with g on $\partial(A_5 - \mathring{A}_3)$ and which agrees with $g_2(hs_3)^{-1}$ on $Bd(A_3)$. Thus $\bar{g} \mid \partial(A_5 - \mathring{A}_3)$ is a homeomorphism of $\partial(A_5 - \mathring{A}_3)$ onto $\partial(B^k \times 2\mathring{B}^m \times (B^k - 1/2\, \mathring{B}^k))$.

If all the s_i squeeze everything sufficiently close to $B^k \times 2\mathring{B}^m \times \{0\}$, the map $\bar{g}$ is easily seen to be a $p^{-1}(\alpha)$-equivalence, for a fine open cover α of $B^k \times 2\mathring{B}^m \times [0, 1]$. (Here p is the projection of $B^k \times 2\mathring{B}^m \times (B^2 - 1/2\, \mathring{B}^2)$ to $B^k \times 2\, \mathring{B}^m \times [0, 1]$, where $B^2 - 1/2\, \mathring{B}^2$ is naturally homeomorphic to $[0, 1] \times S^1$.) Using Theorem 3 there is a homeomorphism $k : A_5 - \mathring{A}_3 \to B^k \times 2\, \mathring{B}^m \times (B^2 - 1/2\, \mathring{B}^2)$ which is p-close to $\bar{g}$ and such that $k = \bar{g}$ on $\partial(A_5 - \mathring{A}_3)$. Then $w = s_5 k : A_5 - \mathring{A}_3 \to A_5 - \mathring{A}_4$ is a homeomorphism which is *id* on $\partial(A_5 - A_4)$.

Assertion 2. *$w \mid Bd(A_3) : Bd(A_3) \to Bd(A_4)$ extends to a homeomorphism $\tilde{w} : A_3 \to A_4$ which is the identity on $(B^k \times 2\mathring{B}^m \times \{0\}) \cup hs_3(\partial B^k \times 2\, \mathring{B}^m \times B^2)$.*

Proof. w is the composition

$$Bd(A_3) \xrightarrow{(hs_3)^{-1}} B^k \times 2\, \mathring{B}^m \times \partial B^2 \xrightarrow{g_2} B^k \times 2\, \mathring{B}^m \times \partial(1/2\, B^2) \xrightarrow{s_5} Bd(A_4).$$

The first and third maps have obvious extensions, and the middle map g_2 has an extension by coning.

Now w and $\tilde{w}$ piece together to give a homeomorphism $\alpha : A_5 \to A_5$ such that

(1) $\alpha \mid \partial A_5 = id$,

(2) $\alpha \mid B^k \times 2\, \mathring{B}^m \times \{0\} = id$,

(3) $\alpha h s_3 \mid B^k \times 2\, \mathring{B}^m \times B^2$ is fiber preserving,

(4) α extends to a homeomorphism $\tilde{\alpha} : \overline{A}_5 \to \overline{A}_5$ via the identity, where $\overline{A}_5$ = topological closure.

(Condition (4) is true because the homeomorphism k is p-close to $\bar{g}$.) Then α extends via the identity to our required map f.

PROOF OF THEOREM 5. Theorem 5 follows from Lemma 8.1 in a standard way. Cover M by $\{R_i^n\}_{i=1}^{\infty}$, a star-finite collection of coordinate-charts, $R_i^n \equiv R^n$. Then each R_i^n has a normal microbundle in N, and the idea is to use Lemma 8.1 to mesh them together. We can choose $\{R_i^n\}$ so that it has a refinement $\{C_i\}$, where $C_i \subset R_i^n$ is compact and $M = \bigcup_{i=1}^{\infty} C_i$. So we only need to mesh the normal microbundles together over the C_i.

Looking at R_1^n and R_2^n, we have normal microbundles $\nu_i : E_i \to R_i^n$, $i = 1, 2$, where the E_i are open subsets of N. By inductively applying Lemma 8.1 over handles in R_1^n we can construct a normal microbundle $\nu : E \to U$, where $U \subset R_1^n \cup R_2^n$ is an open set containing $C_1 \cup C_2$. This is essentially the inductive step in the construction of a global normal microbundle. Details are left to the reader.

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