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A REMARK ON UNIFORM CLASSIFICATION OF BOUNDEDLY COMPACT LINEAR TOPOLOGICAL SPACES

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ABSTRACT. Only linear spaces over reals are considered. It is proved that (i) if a locally convex space is uniformly homeomorphic to a Montel-Fréchet space then it is isomorphic to it and (ii) if two separable conjugate real Banach spaces are uniformly homeomorphic with respect to their w^*-topologies then they are isomorphic.

Introduction. It seems that we are quite far from understanding under what circumstances the existence of a uniform homeomorphism between linear topological spaces over the reals implies that the spaces are isomorphic. Very recently, Aharoni and Lindenstraus [1] have shown that in general this is not the case. they Namely, exhibited an example of two nonisomorphic but Lipschitz homeomorphic nonseparable, nonreflexive Banach spaces. On the other hand, it was known, for example, that if one of the spaces involved is either a Hilbert space [7] or the Fréchet space of all real valued sequences [8] then the existence of a uniform homeomorphism implies that the spaces are isomorphic. Also, recently Ribe [10] has proved that if two Banach spaces are uniformly homeomorphic then they have the same local structure.

The aim of our note is to prove that (i) if a locally convex linear topological space is uniformly homeomorphic to a Montel-Fréchet space then it is isomorphic to it, and (ii) if two separable conjugate Banach spaces are uniformly homeomorphic with respect to their w^*-topologies then they are isomorphic.

We shall consider vector spaces over the field of reals only. In what follows E, F will stand for locally convex topological vector spaces with $\mathscr{P}$ and $\mathscr{Q}$ the corresponding systems of all continuous pseudonorms. Spaces E, F are supposed to be uniform spaces with their natural translation-invariant uniformity. We shall call the mapping $T: E \to F$ Lipschitz if for each $q \in \mathscr{Q}$ there is $p \in \mathscr{P}$ such that $q(Tx - Ty) \leqq p(x - y)$ for all x, y in E.

We shall often use the following lemma, stating that a uniformly continuous map between localy convex spaces has the Lipschitz property for large distances. It is a simple special case of a result of Corson and Klee [5].

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LEMMA 1. *Let T: $E \to F$ be a mapping $q \in \mathcal{Q}$, $p \in \mathcal{P}$, $\varepsilon > 0$, $n \in N$. Assume that for all $x, y \in E$ we have*

$$q(Tx - Ty) < \frac{\varepsilon}{2}, \text{ if } p(x - y) < \frac{1}{n}.$$

Then for all $v, w \in E$ such that $p(v - w) \geqq 1/n$,

$$q(Tv - Tw) \leqq n\varepsilon p(v - w).$$

Recall that a space E is called boundedly compact **[6]** if every bounded subset of E is relatively compact.

We begin the description of one method to obtain in some cases a Lipschitz mapping from a uniformly continuous one.

LEMMA 2. *Let f: $E \to F$ be uniformly continuous. The system $\mathcal{F} = \{f_r: r \geqq 1\}$, where $f_r(x) = (1/r)f(rx)$, is uniformly equicontinuous and maps each point in E onto a bounded subset of F.*

PROOF. Take $\varepsilon > 0$, $q \in \mathcal{Q}$. In view of the uniform continuity of f we can find $n \in \mathbf{N}$, $p \in \mathcal{P}$ such that $q(fx - fy) < \varepsilon/2$ if $p(x - y) < 1/n$. Taking $r \geqq 1$, we obtain:

(a) if $p(rx - ry) < 1/n$, then

$$q(f_r x - f_r y) = \frac{1}{r} q(f(rx) - f(ry)) < \frac{\varepsilon}{2r} < \varepsilon.$$

(b) if $p(rx - ry) \geqq 1/n$, Lemma 1 gives

$$q(f_r x - f_r y) \leqq \frac{1}{r} \varepsilon n r\, p(x - y),$$

the latter being less than ε for $p(x - y) < 1/n$.
(a) and (b) together give the uniform equicontinuity of $\mathcal{F}$. To prove the second assertion we may and shall assume that $f(0) = 0$. Taking $q \in \mathcal{Q}$, there is $p \in \mathcal{P}$, $n \in \mathbf{N}$ such that $q(fx) < 1/2$ if $p(x) < 1/n$. If $p(rx) < 1/n$, we have $q(f_r x) < 1/2r < 1$ for $r \geqq 1$; if $p(rx) \geqq 1/n$, we have $q(f_r x) \leqq np(x)$, using Lemma 1. This proves that the set $\mathcal{F}(x)$ is bounded in F.

Let us recall now some basic facts about convergence along a filter. Having a set J, a family $\mathcal{U}$ of subsets of J is called an ultrafilter, if (i) $\varnothing \notin \mathcal{U}$, (ii) $\mathcal{U}$ is closed under finite intersections, (iii) if $A \in \mathcal{U}$, $A \subset B$, then $B \in \mathcal{U}$, (iv) if $A \notin \mathcal{U}$ then $J \backslash A \in \mathcal{U}$. If P is a topological space and $\{x_a: a \in J\}$ is a family of points in P, we shall say that $\{x_a\}$ converges to a point x along $\mathcal{U}$ if for any neighborhood U of x, there is a set $A \in \mathcal{U}$ such that $x_a \in U$ for $a \in A$. Finally recall that if P is compact, then each family has a limit along each ultrafilter. (For details and proofs see for instance **[3]**.)

PROPOSITION 1. *Let f: $E \to F$ be uniformly continuous, F a boundedly* *compact space. Choose any ultrafilter $\mathcal{U}$ in the set* $[1, \infty]$ *containing all residual intervals* $[a, \infty]$ *for* $a \geqq 1$. *Then the mapping* $h: E \to F$ *assigning to each point* x *the limit of* $\{f_r x: r \geqq 1\}$ *along* $\mathcal{U}$ *is Lipschitz.*

PROOF. First we observe that Lemma 2 gives the boundedness of each $\mathcal{F}(x)$ in F, hence their closures are compact in F and the mapping h is well defined. (The limit along $\mathcal{U}$ exists.) Take $q \in \mathcal{Q}$; we can find $p \in \mathcal{P}$, $n \in \mathbf{N}$ such that $q(fx - fy) < 1/2$ whenever $p(x - y) < 1/n$. Take arbitrary $x, y \in E$, $\eta < 0$.

(a) If $p(x - y) = 0$, take $r_\eta \geqq 1$ such that

$$q(hx - f_{r_\eta}x) < \eta,$$

$$q(hy - f_{r_\alpha}y) < \eta,$$

$$\frac{1}{2r_\eta} < \eta.$$

Then $q(hx - hy) \leqq q(hx - f_{r_\eta}x) + (1/r_\eta)q(f(r_\eta x) - f(r_\eta y)) + q(f_{r_\eta}y - hy) < 3\eta$, hence $q(hx - hy) = 0$.

(b) If $p(x - y) > 0$, take $r_\eta \geqq 1$ such that

$$p(r_\eta x - r_\eta y) \geqq \frac{1}{n},$$

$$q(hx - f_r x) < \eta,$$

$$q(hy - f_r y) < \eta.$$

Using Lemma 1 we obtain that $q(hx - hy) < 2\eta + np(x - y)$, hence $q(hx - hy) \leqq np(x - y)$.

(a) and (b) prove the Lipschitz condition for h.

THEOREM 1. *Let E, F be uniformly homeomorphic and let one of them be boundedly compact. Then E, F are Lipschitz isomorphic.*

PROOF. First observe that if one of the spaces is boundedly compact, then the second one is boundedly compact as well, because both boundedness and compactness are preserved under uniformly continuous mappings. Suppose $f: E \to F$ is a uniform homeomorphism. We define $h: E \to F$ and $h': F \to E$ the respective mappings from Proposition 1 for f and f^{-1}, respectively. Both h, h' are Lipschitz and we shall prove that h' is the inverse of h. A simple computation gives for each $r \geqq 1$:

$$(f^{-1})_r = (f_r)^{-1}.$$

Take any $x \in E$, $p \in \mathcal{P}$, $\eta > 0$. Using Lemma 2 we can find $q \in \mathcal{Q}$, $\varepsilon > 0$ such that for $q(v - w) < \varepsilon$ the inequality $p(f_r^{-1} v - f_r^{-1} w) < \eta$ holds for all $r \geqq 1$. Now because hx, $h'(hx)$ are limits along the same ultrafilter, we can find $r_0 \geqq 1$ such that

$$q(hx - f_{r_0}x) < \varepsilon,$$
$$p(h'hx - f_{r_0}^{-1}hx) < \eta.$$

Then $p(h'hx - x) < 2\eta$, hence $h'hx = x$. In a similar way we can prove that $hh'y = y$ for all $y \in F$, hence h is a Lipschitz isomorphism.

Let f be a mapping from E into F. We say that f is differentiable at a point $x \in E$ iff for every $y \in E$ the limit

$$\lim_{\lambda \to 0} \frac{f(x + \lambda y) - f(x)}{\lambda} = f'_y(x)$$

exists (in the topology of F) and the mapping $(Df)_x: E \to F$ defined by $(Df)_x(y) = f'_y(x)$ is linear.

LEMMA 3. *Let $f: E \to F$ be a Lipschitz homeomorphism between boundedly compact spaces E and F and let f be differentiable at a point $x_0 \in E$. Then $(Df)_{x_0}: E \to F$ is an isomorphism between E and F.*

PROOF. Without any loss of generality we may assume that $f(0) = 0$ and $x_0 = 0$. It can easily be seen (see, e.g., [9]) that $(Df)_0$ is an isomorphic embedding of E into F. It remains to prove that $(Df)_0$ is surjective. Assume the contrary. Then there is $y_0 \in F \backslash F_0$, where $F_0 = (Df)_0(E)$ is a closed subspace of F. Let ϕ be a continuous linear functional on F with $\phi(y) = 0$ for $y \in F_0$ and $\phi(y_0) = 1$. Define $q(y) = |\phi(y)|$ for $y \in G$. Since f is a Lipschitz mapping, there is $p \in \mathscr{P}$ such that

$$q(f(x) - f(z)) \leqq p(x - z) \text{ for } x, z \in E.$$

Set $x_n = nf(n^{-1}y_0)$ for $n \in \mathbf{N}$. Note that the sequence (x_n) is bounded and, therefore, admits an accumulation point, say z. We have

$$\lim_{\lambda \to 0} q\left(\frac{f(\lambda z)}{\lambda}\right) = 0.$$

Thus there is $\delta > 0$ such that

$$q\left(\frac{f(\lambda z)}{\lambda}\right) < \frac{1}{2} \text{ for } |\lambda| < \delta.$$

On the other hand, there is $n \in \mathbf{N}$ such that $p(z - x_n) < 1/2$ and $n^{-1} < \delta$. Hence we have

$$q\left(\frac{f\left(\frac{1}{n}z\right)}{\frac{1}{n}}\right) \geqq -q\left(\frac{f\left(\frac{z}{n}\right) - f\left(\frac{x_n}{n}\right)}{\frac{1}{n}}\right) + q\left(\frac{f\left(\frac{x_n}{n}\right)}{\frac{1}{n}}\right)$$
$$\geqq -np\left(\frac{z}{n} - \frac{x_n}{n}\right) + n\left|\phi\left(f\left(\frac{x_n}{n}\right)\right)\right| \geqq -\frac{1}{2} + \phi(y_0) = \frac{1}{2};$$

a contradiction, which completes the proof.

THEOREM 2. *Let a Montel-Fréchet space X be uniformly homeomorphic with a locally convex vector space Y. Then X is isomorphic to Y.*

PROOF. It is easy to see that Y is a Montel-Fréchet space as well. By Theorem 1, there is a Lipschitz homeomorphism f from X onto Y. By Theorem 4.5 in [**9**], f is differentiable at some point and finally, by the previous lemma, every such differential is an isomorphism between X and Y.

THEOREM 3. *Let X and Y be separable conjugate Banach spaces endowed with their weak* topology and let X be uniformly homeomorphic with Y. Then X and Y are isomorphic with respect to their norm topologies.*

PROOF. Let Y be the conjugate space to a Banach space Z (i.e., $Y = Z^*$). Obviously Z is separable. Let (ϕ_n) be a sequence of points in Z dense in its unit sphere. In the sequel, we shall consider the ϕ_n as functions on Y. Let s be the space of all real-valued sequences with its standard product topology. By Theorem 1, there is a Lipschitz homeomorphism f from X onto Y. Set $f_n = \phi_n \circ f$ for $n \in \mathbf{N}$ and consider the mapping $h\colon X \to s$ given by the formula

$$h(x) = (f_1(x), f_2(x), f_3(x), \ldots) \text{ for } x \in X.$$

It is easy to check that h is a Lipschitz mapping from X with weak* topology into s, but this implies that h is a Lipschitz mapping from X with norm topology into s. Thus by Theorem 4.5 of [**9**] (see also [**2**], [**4**]). there is a point $x_0 \in X$ at which h is differentiable. Hence we deduce that f_n is differentiable at x_0 for each $n \in \mathbf{N}$. This implies that, for each $x \in X$,

$$\left\{\frac{f(x_0 + \lambda x) - f(x_0)}{\lambda}\right\}$$

is a weak* Cauchy system when λ tends to 0. Thus for each $x \in X$, the limit

$$\lim_{\lambda \to 0} \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} = f'_x(x_0)$$

exists. We shall show that the mapping $(Df)_{x_0}$ defined by the formula $(Df)_{x_0}(x) = f'_x(x_0)$ for $x \in X$ is additive. Indeed, assume the contrary and let $a, b \in X$ be such that $(Df)_{x_0}(a) + (Df)_{x_0}(b) \neq (Df)_{x_0}(a + b)$. This implies that there is $n \in \mathbf{N}$ such that

$$\phi_n((Df)_{x_0}(a)) + \phi_n((Df)_{x_0}(b)) \neq \phi_n((Df)_{x_0}(a + b)).$$

On the other hand, it can easily be seen that

$$\phi_n((Df)_{x_0}(x)) = (Df_n)_{x_0}(x) \text{ for every } x \in X.$$

Since $(Df_n)_{x0}(x)$ is linear in x we infer that

$$\phi_n((Df)_{x0}(a)) + \phi_n((Df)_{x0}(b)) = \phi_n((Df)_{x0}(a + b)),$$

a contradiction. Thus $(Df)_{x_0}$ is additive. Also, one can easily verify ([**9**]) that $(Df)_{x_0}$ is continuous. Thus it is linear. By Theorem 2, the mapping $(Df)_{x_0}$ is an isomorphism between X and Y with respect to their weak* topologies. Hence it is also an isomorphism with respect to the norm topologies of X and Y.

COROLLARY. *Separable reflexive Banach spaces are weakly uniformly homeomorphic if and only if they are isomorphic.*

REMARK. Note that in Theorem 3 the assumption that Y is a separable conjugate Banach space can be omitted.

REMARK. By a small modification of the proof of the previous theorem one can obtain that if two conjugate Banach spaces are weak* uniformly homeomorphic then they have the same separable linear dimension; i.e., each separable linear subspace of one of them is isomorphically embeddable in the second and vice versa.

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