ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 1, Winter 1980

# NONLINEAR BOUNDARY VALUE PROBLEMS FOR SOME CLASSES OF ORDINARY DIFFERENTIAL EQUATIONS*

A. GRANAS, R.B. GUENTHER AND J.W. LEE

1. **Introduction.** In this paper we study existence questions for nonlinear boundary value problems of the form,

$$(\mathscr{P}) \qquad \begin{cases} Ly = f(t, y, y', \ldots, y^{(n-1)}), \\ y \in \mathscr{B}, \end{cases}$$

where $L$ is a certain $n$th order linear differential operator, $\mathscr{B}$ is a suitable set of boundary conditions, and $f$ is continuous and subject to various additional requirements. (Most of our results pertain to the special cases $n = 2$, 3, and 4.) In particular, when $n = 2$ we investigate problems in which $f$ exhibits greater than quadratic growth in $y'$. Other parts of this paper treat problems with nonlinear boundary conditions, certain 3rd and 4th order equations, and some nonlinear systems.

In 1912, S. Bernstein in the first part of his celebrated memoir [**2**] devoted to the boundary value problems arising in the calculus of variations, studied the case $n = 2$ and $Ly = y''$. In this case, Bernstein showed that a priori bounds for derivatives of solutions to $(\mathscr{P})$ can be obtained once such bounds are found for the solutions themselves, *provided* that the nonlinearity in $f$ is at most quadratic in $y'$. Furthermore, he showed that a corresponding result is not valid in general if $f$ displays greater than quadratic growth in $y'$. In the former case when a priori bounds are available, existence results follow readily from various fixed point theorems. In the latter case, the negative character of Bernstein's examples seems to have impeded the search for a priori bounds. Also, the true role played by the quadratic growth assumption seems to have been overemphasized and mis-interpreted in certain respects.

For example, the two Dirichlet problems,

$$\begin{cases} y'' = y'^2 + 1 \\ y(a) = 0 = y(b) \end{cases} \quad \text{and} \quad \begin{cases} y'' = y'^2 - 1 \\ y(a) = 0 = y(b) \end{cases}$$

for $a \leqq t \leqq b$ have virtually identical growth as $|y'| \to \infty$; however, the first problem has a solution only for $b - a < \pi$ while the second problem has a solution for any choice of $b > a$. A similar assertion holds for the pair of Dirichlet problems

*This work was partially supported by a grant from the National Research Council of Canada.

Received by the editors on April 13, 1978.

$$\begin{cases} y'' = (y'^2 + 1)^n \\ y(a) = 0 = y(b) \end{cases} \text{and} \quad \begin{cases} y'' = (y'^2 - 1)^n \\ y(a) = 0 = y(b) \end{cases},$$

where $n > 1$. What is essential here is not the presence or absence of quadratic growth, but rather the location of the zeros of the polynomials $y'^2 + 1$ and $y'^2 - 1$, as explained in §4.

This paper is organized as follows. §2 contains a general existence theorem for nonlinear boundary value problems for which a priori bounds on solutions and their derivatives can be established. In §§3, 4 and 5 appropriate a priori bounds are established for specific classes of second order nonlinear boundary value problems. Existence of solutions to these problems follows by application of the results in §2. §6 treats problems in which the boundary conditions are nonlinear. In §7 solutions are established for certain 3rd and 4th order problems. Here the derivation of a priori bounds involves certain $L_2$-estimates. §8 contains existence results for second order systems. Uniqueness theorems are treated in the monograph [**6**]. For recent extensions of Bernstein's original work on nonlinear Dirichlet problems see [**4**] and for another approach [**3**].

2. **An Existence Theorem.** Let $(C, |\cdot|_0)$ be the Banach space of continuous functions on [0, 1] with the sup norm, $|\cdot|_0$. Let $(C^n, |\cdot|_n)$ be the Banach space of $n$-times continuously differentiable functions $u$ in $C$ with

$$|u|_n = \max\{|u|_0, \ldots, |u^{(n)}|_0\}.$$

Let $\mathscr{B}$ denote a set of linear, homogeneous boundary conditions

$$U_i(u) \equiv \sum_{j=0}^{n-1} [a_{ij}u^{(j)}(0) + b_{ij}\, u^{(j)}(1)] = 0, \ i = 1, \ldots, n,$$

$$(\mathscr{B}) \qquad C^n_{\mathscr{B}} = \{u \in C^n \colon u \in \mathscr{B}\}, \text{ and}$$

$$(Lu)(t) = \sum_{j=0}^{n} a_j(t)u^{(j)}(t),$$

where $a_j \in C$ and $a_n(t) \neq 0$ for $t$ in [0, 1].

Consider the boundary value problem,

$$(1) \qquad \begin{cases} Lu = f(t, u, \ldots, u^{(n-1)}), \\ u \in \mathscr{B}, \end{cases}$$

where $f(t, p_1, \ldots, p_n)$ is continuous on $[0, 1] \times R^n$, and the family of problems

$$(1)_\lambda \qquad \begin{cases} Mu = g(t, u, \ldots, u^{(n-1)}, \lambda), \\ u \in \mathscr{B}, \end{cases}$$

where $0 \leqq \lambda \leqq 1$,

$$(Mu)(t) = \sum_{j=0}^{n} b_j(t)u^{(j)}(t),$$

$g$, $b_j$ are continuous, $b_n(t) \neq 0$ for $t$ in [0, 1], and $g(t, y_1, \ldots, y_n, 0) \equiv 0$.

REMARK. In the applications of the theorem below, we usually take $M = L$ and $g = \lambda f$.

THEOREM 2.1. *Let $L$, $M$, $f$ and $g$ be as above. Assume*:

(i) *The problems* $(1)_\lambda$ *and* (1) *are equivalent when* $\lambda = 1$; *that is*, $(1)_\lambda$ *and* (1) *have the same set of solutions.*

(ii) *The differential operator* $(M, \mathscr{B})$ *is invertible* (*one-to-one*) *as a map from* $C^n_{\mathscr{B}}$ *to* $C$.

(iii) *There is a constant* $R$ *independent of* $\lambda$ *such that* $|u|_n < R$ *for each solution* $u$ *to* $(1)_\lambda$, $0 \leqq \lambda \leqq 1$.

*Then the boundary value problem* (1) *has at least one solution.*

PROOF. The proof utilizes the topological transversality theorem in [5]. For the definitions of compact homotopy, essential map, and for full statements of the topological results used here, see [5].

Let $K_R = \{ u \in C^n_{\mathscr{B}} : |u|_n \leqq R\}$ and define

$$T_\lambda : C^{n-1} \to C, \; 0 \leqq \lambda \leqq 1,$$

by

$$(T_\lambda v)(t) = g(t, v(t), \ldots, n^{(n-1)}(t), \lambda).$$

Let $j: C^n_{\mathscr{B}} \to C^{n-1}$ be the completely continuous embedding of $C^n_{\mathscr{B}}$ into $C^{n-1}$. Then

$$H_\lambda = M^{-1}T_\lambda j$$

defines a homotopy $H_\lambda : K_R \to C^n_{\mathscr{B}}$.

It is easily seen that the fixed points of $H_\lambda$ are precisely the solutions to problem $(1)_\lambda$. By (iii) the homotopy is fixed point free on the boundary of $K_R$. Moreover, the complete continuity of $j$ and (ii) imply that the homotopy $H_\lambda$ is compact. Hence $H_0$ is homotopic to $H_1$ ($H_0 \sim H_1$). Since $H_0$ is a constant map ($H_0$ is the zero map) it is essential and, because $H_0 \sim H_1$, $H_1$ is also essential (c.f. [5, Theorem 3]). In particular, $H_1$ has a fixed point, $(1)_\lambda$ has a solution, and, by (i), (1) has a solution.

REMARK 2.2. If zero is not an eigenvalue of $(L, \mathscr{B})$, Theorem 2.1 can often be applied with $M = L$ and $g = \lambda f$. If zero is an eigenvalue of $(L, \mathscr{B})$, then the choices $M = L - cI$ and

$$g(t, p_1, \ldots, p_n, \lambda) = \lambda[f(t, p_1, \ldots, p_n) - p_1]$$

where $c$ is not in the spectrum of $L$ are useful.

REMARK 2.3. In problem $(1)_\lambda$, $M$ can be replaced by a family of operators $\{L_\lambda\}$ if we replace (ii) by (ii)′ The differential operator $(L_\lambda, \mathscr{B})$ is invertible as a map from $C^n_{\mathscr{B}} \to C$ for each $0 \leqq \lambda \leqq 1$, and $\{L_\lambda\}$ is collectively compact; that is, $\{L_\lambda z : 0 \leqq \lambda \leqq 1, |z|_n \leqq 1\}$ is compact.

Collectively compact operators are studied extensively in **[1]**.

The preceding discussion extends to include nonlinear problems in which nonlinear boundary conditions occur. Thus, consider the problem

$$(2) \qquad \begin{cases} Lu = f(t, u, \ldots, u^{(n-1)}), \\ U_i(u) = V_i(u), \; i = 1, 2, \ldots, n, \end{cases}$$

where $L$, $f$ and $U_i$ are as above, and

$$V_i(u) = \phi_i(u(0), \ldots, u^{(n-1)}(0), u(1), \ldots, u^{(n-1)}(1))$$

with $\phi_i : R^{2n} \to R$ continuous. We also consider,

$$(2)_\lambda \qquad \begin{cases} Mu = g(t, u, \ldots, u^{(n-1)}, \lambda), \\ U_i(u) = \lambda V_i(u), \end{cases}$$

where $0 \leqq \lambda \leqq 1$, $M$ and $g$ are as above.

THEOREM 2.4. *Let $L$, $M$, $f$, $g$, $U_i$ and $V_i$, $i = 1, 2, \ldots, n$, be as above. Assume*

(i) *The problems* $(2)_\lambda$ *and* (2) *are equivalent when* $\lambda = 1$.

(ii) *The differential operator* $(M, \mathscr{B})$ *is invertible* (*one-to-one*) *as a map from* $C^n_{\mathscr{B}} \to C$.

(iii) *There is a constant $R$ independent of $\lambda$ such that $|u|_n < R$ for each solution to* $(2)_\lambda$, $0 \leqq \lambda \leqq 1$.

*Then the boundary value problem* (2) *has at least one solution.*

PROOF. The proof is essentially the same as for Theorem 2.1. Modifications are needed: Let $K_R = \{u \in C^n : |u|_n \leqq R\}$ and define

$$S_\lambda : C^{n-1} \to C \times R^n, \; 0 \leqq \lambda \leqq 1$$

by

$$S_\lambda v = (g(t, v(t), \ldots, v^{(n-1)}(t), \lambda), \; V_1(v), \ldots, V_n(v)).$$

Also, define

$$M_1 : C^n \to C \times R^n,$$

by

$$M_1 u = (Mu, U_1(u), \ldots, U_n(u)).$$

As before let $j : C^n \to C^{n-1}$ be the completely continuous embedding of $C^n$ into $C^{n-1}$.

By (ii) of Theorem 2.4, $M_1$ is a continuous, linear, one-to-one map of $C^n$ onto $C \times R^n$ and hence has a continuous inverse $M_1^{-1}$. Now putting

$$H_\lambda = M_1^{-1} S_\lambda j,$$

we obtain a compact homotopy from $K_R \to C^n$ which is fixed point free on $\partial K_R$. We use the topological transversality theorem as above and the proof is completed.

REMARK. Theorems analogous to (2.1) and (2.4) can be formulated for systems of differential equations. The formulations are omitted.

3. **Existence without Quadratic Growth Restrictions.** In this section, we establish the existence of solutions to the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

subject to either the Dirichlet, Neumann, periodic, or Sturm-Liouville boundary conditions, and where the growth of $f$ in $y'$ may be substantially greater than quadratic.

The boundary conditions referred to above are, respectively,

$$
\begin{array}{ll}
\text{(I)} & y(0) = 0,\ y(1) = 0, \\
\text{(II)} & y'(0) = 0,\ y'(1) = 0, \\
\text{(III)} & y(0) = y(1),\ y'(0) = y'(1), \\
\text{(IV)} & -\alpha y(0) + \beta y'(0) = 0, \quad a y(1) + b y'(1) = 0,
\end{array}
$$

where $\alpha, \beta, a, b \geqq 0$, $\alpha^2 + \beta^2 > 0$, $a^2 + b^2 > 0$, and $\alpha^2 + a^2 > 0$. We say that a function $y \in C^2$ satisfies the Dirichlet problem (I) if it satisfies the differential equation (1) and the boundary conditions (I). Similar notation is used for the other problems.

The following assumptions are made on $f$:

(i) There is a constant $M_0 \geqq 0$ such that
$yf(t, y, 0) \geqq 0$ for $|y| > M_0$;

(ii) For $(t, y)$ in $[0, 1] \times [-M_0, M_0]$,
$|f(t, y, p)| \leqq \phi(|p|)$,
where $\phi > 0$ and $1/\phi$ is integrable on each bounded interval in $[0, \infty)$.

THEOREM 3.1. *Let $f(t, y, p)$ be continuous and satisfy* (i) *and* (ii). *Then each of the boundary value problems* (I), (II), (III), *and* (IV) *has a solution, provided*

$$\int_0^\infty \frac{x\,dx}{\phi(x)} > 2M_0. \tag{2}$$

PROOF. Fix a constant $c$, $0 < c < 1$ such that

$$\int_0^\infty \frac{x\,dx}{\phi(x) + cM_0} > 2M_0, \tag{3}$$

and such that zero is not an eigenvalue for $(M, \mathscr{B})$ where

$$My = y'' - cy,$$

and $\mathscr{B}$ denotes either Dirichlet, Neumann, periodic, or Sturm-Liouville boundary conditions.

Consider the family of boundary value problems

$$\begin{cases} My = \lambda[f(t, y, y') - cy], \\ y \in \mathscr{B}, \end{cases} \tag{4}$$

for $0 \leqq \lambda \leqq 1$. Theorem 2.1 implies that $y'' = f(t, y, y')$, $y \in \mathscr{B}$ has a solution provided a priori bounds (independent of $\lambda$) are available for $|y|_0$, $|y'|_0$, and $|y''|_0$ for all solutions $y$ to (4). We proceed to establish such bounds.

We assume first that $yf(t, y, 0) > 0$ if $|y| \geqq M_0$. Then the argument in §2 of [4] shows that $|y(t)| \leqq M_0$ for $t$ in [0, 1] and all solutions to (4).

Next,

$$|\lambda\{f(t, y, y') - cy\}| \leqq \phi(|y'|) + cM_0. \tag{5}$$

Since the derivative $y'$ of any solution to (4) must vanish at least once in [0, 1], each point $t$ in [0, 1] for which $y'(t) \neq 0$ belongs to an interval $[\mu, \nu]$ such that $y'$ maintains a fixed sign on $[\mu, \nu]$ and $y'(\mu)$ and/or $y'(\nu)$ is zero. To be definite (the other cases lead to the same result) assume $y'(\nu) = 0$ and $y' \geqq 0$ on $[\mu, \nu]$. Then from (ii) and (5),

$$\begin{aligned} \frac{y'y''}{\phi(y') + cM_0} &\geqq y', \\ \int_t^\nu \frac{y'dy'}{\phi(y') + cM_0} &\geqq y(\nu) - y(t) \geqq -2M_0, \\ \int_{y'(t)}^0 \frac{x\,dx}{\phi(x) + cM_0} &\geqq -2M_0, \\ \int_0^{y'(t)} \frac{x\,dx}{\phi(x) + cM_0} &\leqq 2M_0. \end{aligned} \tag{6}$$

Since (3) holds, this inequality implies there is a constant $M_1$ such that

$$|y'(t)| \leqq M_1 \text{ for } t \text{ in } [0, 1].$$

The a priori bounds on $y$ and $y'$ together with (4) imply $|y''(t)| \leqq M_2$ for $t$ in [0, 1] and some constant $M_2$. This completes the proof when (i) is strengthened to $yf(t, y, 0) > 0$ for $|y| \geqq M_0$.

Assume (i) holds and consider

$$\begin{cases} y'' = f(t, y, y') + \dfrac{1}{m} y, \\ y \in \mathscr{B} \end{cases}$$

for large $m$. For this problem, $y[f(t, y, 0) + (1/m)\, y] > 0$ if $|y| \geqq M_0$, so by what was just proven, this problem has a solution $y_m$ provided

$$\int_0^\infty \frac{x\,dx}{\psi(x) + \frac{1}{m}M_0} > 2M_0,$$

which holds for all large $m$ by (2). Furthermore, arguing as above, we can show that $|y_m|_2 \leqq R$ for some $R < \infty$. A simple compactness argument implies that a subsequence of $\{y_m\}$ converges to a solution to $y'' = f(t, y, y')$, $y \in \mathscr{B}$, and the proof is complete.

REMARK. In (6) we have $y(\nu) - y(t) \geqq -M_0$ if it is known a priori that the solutions to (4) maintain a fixed sign on [0, 1]. In this case, the condition (2) in Theorem 3.1 may be replaced by

$$\int_0^\infty \frac{x\,dx}{\psi(x)} > M_0. \tag{7}$$

An important special case of (ii) is

(ii)′ $|f(t, y, p)| \leqq A(t, y)p^{2l} + B(t, y)$, where $A$ and $B$ are non-negative functions bounded for $(t, y)$ in $[0, 1] \times [-M_0, M_0]$ and $l \geqq 0$ (but $l$ need not be an integer).
In this case, we may choose

$$\psi(p) = A_0 p^{2l} + B_0,$$

where

$$A_0 = \sup A(t, y) \text{ and } B_0 = \sup B(t, y),$$

for $(t, y)$ in $[0, 1] \times [-M_0, M_0]$. Use of this $\psi$ in (2) leads to

THEOREM 3.2. *Let $f(t, y, p)$ be continuous and satisfy* (i) *and* (ii)′. *Then each of the boundary value problems* (I), (II), (III), *and* (IV) *has a solution provided*

$$\int_0^\infty \frac{dx}{1 + x^l} > 4M_0 A_0^{1/l} B_0^{1-(1/l)}. \tag{2$'$}$$

If it is known a priori that each solution to (4) of one of the preceding problems maintains a fixed sign on [0, 1], then (2)′ can be replaced by

$$\int_0^\infty \frac{dx}{1 + x^l} > 2M_0 A_0^{1/l} B_0^{1-(1/l)}, \tag{2$''$}$$

where $A_0$ and $B_0$ are the maxima computed over $[0, 1] \times [-M_0, 0]$, respectively $[0, 1] \times [0, M_0]$, when the fixed sign of the solutions to (4) is negative, respectively positive.

REMARK. In the classical Bernstein case $l = 1$ and condition (2) always holds.

As a corollary of the preceding result, we have, THEOREM 3.3. *Let* $f(t, y, y') = a(t)y + b(t) + (y')^{2l}A(t, y)$ *where* $a$, $b$, $A$ *are continuous,* $a(t) > 0$, *and* $b(t) \geqq 0$ *for* $t$ *in* $[0, 1]$. *Then the Dirichlet problem* (I) *has a solution if*

$$\int_0^\infty \frac{dx}{1 + x^l} > 2M_0 A_1^{1/l} B_1^{-(1/l)},$$

*where* $M_0 = \max|b(t)/a(t)|$ for $t$ in $[0, 1]$, $A_1 = \max|A(t, y)|$, *and* $B_1 = \max|B(t, y)|$ *for* $(t, y)$ *in* $[0, 1] \times [-M_0, 0]$. (*If* $b(t) \leqq 0$ *on* $[0, 1]$ *and* $A_1$, $B_1$ *are defined as maxima over* $[0, 1] \times [0, M_0]$, *then the same conclusion holds.*)

PROOF. Since Dirichlet conditions are used here the homotopy (4) can be used with $c = 0$. In this case, one easily confirms that any solution to (4) for this $f$ cannot have a positive maximum. If a negative minimum occurs at $t_0$ in $(0, 1)$, then

$$0 \leqq y''(t_0) = a(t_0)y(t_0) + b(t_0),$$
$$y(t_0) \geqq -b(t_0)/a(t_0).$$

Thus, each solution to (4) satisfies $-M_0 \leqq y(t) \leqq 0$ with $M_0 = \max|b(t)/a(t)|$. Also, $yf(t, y, 0) > 0$ for $|y| > M_0$. The theorem follows from (3.2).

EXAMPLES. Consider

$$\begin{cases} y'' = ay'^4 + by + 1, \\ y \in \mathscr{B}, \end{cases} \tag{8}$$

where $\mathscr{B}$ specifies one of the boundary conditions (I)–(IV) and $a$, $b$ are nonzero constants. Here $l = 2$ and Theorem 3.2 applies with $M_0 = 1/|b|$, $A_0 = |a|$, $B_0 = 2$. Thus, (7) has a solution if

$$\frac{\pi}{2} > \frac{4}{|b|}(2|a|)^{1/2}$$

or

$$\frac{\pi}{8(2)^{1/2}} > \frac{(|a|)^{1/2}}{|b|}.$$

We obtain a better result if $a > 0$ and $\mathscr{B}$ specifies the Dirichlet conditions. In this case Theorem 3.3 applies with $M_0 = 1/|b|$, $A_1 = a$, $B_1 = 1$.

Thus (8) has a solution if

$$a/b^2 < \pi^2/16.$$

In particular, the Dirichlet problem

$$\begin{cases} y'' = y'^4 + 2y + 1, \\ y(0) = 0, y(1) = 0 \end{cases}$$

has a solution, and the problem

$$\begin{cases} y'' = y'^4 + 4y + 1, \\ y \in \mathscr{B}, \end{cases}$$

has a solution for any choice $\mathscr{B}$ of boundary conditions (I) – (IV).

REMARK. In connection with problem (8), with say $a > 0$ and $b = 1$, our result only guarantees a solution for $a < \pi^2/16$. It is natural to ask if a solution also exists for larger values of $a$ or indeed for all $a > 0$. Computer experiments indicate that solutions exist up to about $a = 30$. On the other hand, L. Nirenberg (private communication) has proven: If $a > 0$ and $b = 1$ in (8), then if (8) has a solution, $a < 3^8/2^5$.

4. **The Dirichlet Problem for $y'' = f(y')$.** Consider the Dirichlet problem

$$\begin{cases} y'' = f(y'), \\ y(0) = 0, y(1) = 0 \end{cases} \tag{1}$$

and the related family of problems

$$\begin{cases} y'' = \lambda f(y'), \\ y(0) = 0, y(1) = 0 \end{cases} \tag{2}$$

for $0 \leqq \lambda \leqq 1$. If $f(0) = 0$, then (1) has the solution $y \equiv 0$. Thus, in what follows we assume $f(0) \neq 0$.

THEOREM 4.1. *Assume $f$ is continuous on $R$, $f(0) \neq 0$, and that $f$ has two zeros of opposite sign. Then the Dirichlet problem* (1) *has a solution. Moreover, if $r^- < 0 < r^+$ are, respectively, the greatest negative and smallest positive roots of $f$, then for each solution $y$ to* (1),

$$|y|_0,\ |y'|_0 < \max(-r^-, r^+) = M$$
$$|y''|_0 \leqq \max|f(z)|$$

*for $z$ in $[-M, M]$.*

PROOF. The existence of a solution to (1) follows from Theorem 2.1 once the bounds in the conclusion of the present theorem are established for all solutions $y$ to (2) for $0 \leqq \lambda \leqq 1$.

If $\lambda = 0$, (2) has the unique solution $y \equiv 0$. So for the purpose of establishing the a priori bounds we can assume $0 < \lambda \leqq 1$. Suppose $y$ is a solution of (2) and $y'(t_0) \geqq r^+$ for $t_0$ in $[0, 1]$. Since $y'(\tau) = 0$ for some $\tau$ in $(0, 1)$, it follows that $y'(t_1) = r^+$ for some $t_1$ in $[0, 1]$.

Assume temporarily that $f \in C^2(-\infty, \infty)$. Then

$$y''(t_1) = \lambda f(r^+) = 0,$$
$$y'''(t_1) = \lambda f'(r^+) y''(t_1) = 0,$$

and

$$y^{(iv)}(t) = \lambda f''(y'(t))y''(t)^2 + \lambda f'(y'(t))y'''(t).$$

Thus, $u(t) = y''(t)$ satisfies the initial value problem

$$u'' = a(t)u^2 + b(t)u',$$
$$u(\tau) = u'(\tau) = 0,$$

for $t$ in $[0, 1]$ where $a(t) = \lambda f''(y'(t))$ and $b(t) = \lambda f'(y'(t))$ are continuous on $[0, 1]$. By the uniqueness theory for such initial value problems, $u \equiv 0$ on $[0, 1]$. Then $y$ is linear, hence $y \equiv 0$, and this contradicts $y'(t_0) \geqq r^+ > 0$. We have established that

$$y'(t) < r^+, \ t \text{ in } [0, 1],$$

and analogous reasoning implies that

$$r^- < y'(t), \ t \text{ in } [0, 1].$$

Thus

$$|y'|_0 < \max(-r^-, t^+) = M.$$

Consequently

$$|y|_0 < M, |y''|_0 \leqq \max |f(z)|$$

for $z$ in $[-M, M]$. The theorem is proven for $f \in C^2(-\infty, \infty)$.

If $f \in C(-\infty, \infty)$ and $r^- < 0 < r^+$ are the zeros of $f$ as above, then there exists a sequence of functions $f_n \in C^2(-\infty, \infty)$ such that $f_n$ converges uniformly to $f$ on $J = [-M - 1, M + 1]$ with $M$ as above, and $f_n$ has zeros $r_n^- < 0 < r_n^+$ in $J$ such that $r_n^- \to r^-$ and $r_n^+ \to r^+$ as $n \to \infty$.

By what was just proven, there are solutions $y_n$ to

$$\text{(3)} \qquad \begin{cases} y'' = f_n(y'), \\ y(0) = 0, \ y(1) = 0 \end{cases}$$

such that

$$|y_n|_0, |y_n'|_0 < \max(-r_n^-, r_n^+) < M + 1 = B,$$
$$|y_n''|_0 \leqq \max_n \max_{[-B,B]} |f_n(z)| < \infty.$$

Thus there is a subsequence $N$ of positive integers and $y \in C^1$ such that

$$|y_n - y|_1 \to 0, n \to \infty \text{ in } N.$$

Since $|y_n'|_0, |y'|_0 < M + 1$, $\{f_n(y_n')\}_{n \in N}$ converges uniformly to $f(y')$ on $[0, 1]$. From (3) there is a $\tau_n$ in $[0, 1]$ such that $y_n'(\tau_n) = 0$. Then there is a subsequence $N_1 \subset N$ and a $\tau$ in $[0, 1]$ such that $\tau_n \to \tau$ as $n \to \infty$ in $N_1$ and

$$y_n'(t) = \int_{\tau_n}^{t} f_n(y_n'(x))\,dx \longrightarrow \int_{\tau}^{t} f(y'(x))\,dx$$

as $n \to \infty$ in $N_1$. Consequently

$$y'(t) = \int_{\tau}^{t} f(y'(x))\,dx,\ t \text{ in } [0, 1],$$

and $y$ satisfies (1). This completes the proof.

THEOREM 4.2. *Let $f$ be continuous on $R$, $f(0) \neq 0$, and assume that $f$ has a positive zero but no negative zero. Then* (1) *has a solution provided*

$$\int_{-\infty}^{0} \frac{dz}{|f(z)|} > 1.$$

*If instead $f$ has a negative zero and no positive zero, then* (1) *has a solution provided*

$$\int_{0}^{\infty} \frac{dz}{|f(z)|} > 1.$$

PROOF. The result follows as for Theorem 4.1 once a priori bounds are established for $|y|_2$ where $y$ is a solution to (2) for some $0 < \lambda \leqq 1$.

Let $r^+ > 0$ be the smallest positive root of $f$. Just as above we find that

$$y'(t) < r^+,\ t \text{ in } [0, 1].$$

There is a unique number $\alpha = \alpha(\lambda)$ in $(0, 1)$ such that $y'(\alpha) = 0$. Such an $\alpha$ exists since $y(0) = y(1) = 0$ and if there were two, then $0 = y''(t_0) = f(y'(t_0))$ for some $t_0$ in $(0, 1)$, which contradicts $y'(t_0) < r^+$.

Assume $f(0) > 0$. Then $y''(\alpha) = \lambda f(0) > 0$ and, since $y'(t) < r^+$, $y''(t) > 0$ for $t$ in $[0, 1]$. Consequently $y'$ is strictly increasing and $y'(0) < 0$ is the minimum value of $y'$ on $[0, 1]$. Since $y'' > 0$ the boundary conditions imply $y \leqq 0$. Also,

$$y(1) - y(\alpha) = \int_{\alpha}^{1} y'(x)\,dx < (1 - \alpha)r^+,$$

$$y(\alpha) > -r^+,$$

and so

$$-r^+ < y(t) \leqq 0,\ t \text{ in } [0, 1].$$

Thus we have a priori bound for $|y|_0$. Next,

$$\int_{0}^{\alpha} \frac{y''(t)\,dt}{f(y'(t))} = \lambda\alpha,$$

$$\int_{y'(0)}^{0} \frac{dz}{f(z)} = \lambda\alpha.$$

Since $\lambda\alpha < 1$ and $\int_{-\infty}^{0} dz/f(z) > 1$ it follows that $-M < y'(0)$ for some

$M > 0$ (independent of $\lambda$). Thus $-M < y'(t) < r^+$ for $t$ in $[0, 1]$. Finally, a priori bounds for $y''$ follow from the differential equation. Analogous reasoning applies if $f(0) < 0$, and if $f$ has a negative zero but no positive zero. The proof is complete.

REMARK 4.3. The method of proof above establishes several features of the solution to (1). We state these features for the case when $f$ has a positive zero but no negative zero. Let $r^+$ be the smallest positive zero of $f$, and $y$ be a solution to (1). Then

$$y'(t) < r^+, \ t \text{ in } [0, 1].$$

If $f(0) > 0$, then $y$ is concave upward, $y'$ is strictly increasing, and

$$-r^+ < y(t) \leqq 0, \ t \text{ in } [0, 1],$$

while if $f(0) < 0$, $y$ is concave downwards, $y'$ is strictly decreasing, and

$$0 \leqq y(t) < r^+, \ t \text{ in } [0, 1].$$

Similar properties hold for the case when $f$ has a negative zero but no positive zero.

REMARK 4.4. Examination of the proof of Theorem 4.2 reveals that the first integral inequality can be replaced by

$$\int_{-\infty}^{0} \frac{dz}{|f(z)|} > A$$

and the second inequality can be replaced by

$$\int_{0}^{\infty} \frac{dz}{|f(z)|} > B,$$

where

$$A \geqq \sup_{0 \leqq \lambda \leqq 1} \lambda \alpha(\lambda),$$

$$B \geqq \sup_{0 \leqq \lambda \leqq 1} \lambda(1 - \alpha(\lambda)),$$

and $\alpha(\lambda)$ is the unique zero of $y'(t; \lambda)$.

THEOREM 4.5. *Let $f$ be continuous and have no real zeros. Then* (1) *has a solution if*

$$\int_{-\infty}^{0} \frac{dz}{|f(z)|} > 1 \text{ and } \int_{0}^{\infty} \frac{dz}{|f(z)|} > 1.$$

PROOF. Under these conditions $y'$ is strictly monotone for any solution to (2), $0 < \lambda \leqq 1$. From and differential equation,

$$\int_{y'(0)}^{0} \frac{dz}{f(z)} = \lambda\alpha \text{ and } \int_{0}^{y'(1)} \frac{dz}{f(z)} = (1 - \alpha)\lambda,$$

where $y'(\alpha) = 0$ as before. These equations imply a priori bounds for $|y'|_0$ and a priori bounds for $|y|_0$ and $|y''|_0$ follow as above.

The preceding theorem should be considered with the following nonexistence result.

THEOREM 4.6. *Let $f$ be continuous and have no real zeros. Then* (1) *does not have a solution if*

$$\int_{-\infty}^{\infty} \frac{dz}{|f(z)|} \leqq 1.$$

PROOF. If $y$ satisfies (1), then

$$\int_{y'(0)}^{y'(1)} \frac{dz}{f(z)} = 1.$$

EXAMPLES.

(a)
$$\begin{cases} y'' = (y'^2 - 1)^n, & n \geqq 0, \\ y(0) = 0, y(1) = 0, \end{cases}$$

has a solution (in fact a unique solution) by Theorem 4.1.

(b) Consider

$$\begin{cases} y'' = (y'^2 + 1)^n, & n \geqq 1, \\ y(0) = 0, \ y(1) = 0. \end{cases}$$

Since

$$\int_{-\infty}^{\infty} \frac{dz}{(z^2 + 1)^n} = \frac{1 \cdot 3 \cdots (2n - 3)}{2 \cdot 4 \cdots (2n - 2)} \pi,$$

this problem has no solution if $n \geqq 4$ by Theorem 4.6.

(c) Consider

$$y'' = y'^3 + y'^2 + y' + 1,$$
$$y(0) = 0, \ y(1) = 0.$$

Here

$$f(z) = (z + 1)(z^2 + 1)$$

has only one real zero, $z = -1$. Thus

$$-1 < y'(t), \ t \text{ in } [0, 1]$$

for any solution to (2). Since

$$\int_0^\infty \frac{dz}{(z+1)(z^2+1)} = \frac{\pi}{4},$$

Theorem 4.2 does not apply directly; however, the ideas of this section do apply. Consider,

$$\begin{cases} y'' = \lambda(y'^3 + y'^2 + y' + 1) = \lambda f(y'), \\ y(0) = 0, y(1) = 0 \end{cases}$$

for $0 \leqq \lambda \leqq 1$. We have

$$y'(t) > -1 \text{ for } t \text{ in } [0, 1].$$

Also the differential equation shows that $y$ cannot have a positive maximum so $y \leqq 0$ in $[0, 1]$. Since $y(0) = 0$, we have $y'(0) \leqq 0$ and

$$\frac{y''}{1+y'^2} = \lambda(1 + y'),$$

$$\arctan y'(t) = \lambda(t + y(t) + \arctan y'(0)),$$

$$\arctan y'(t) \leqq 1$$

for $t$ in $[0, 1]$. Thus

$$-1 < y'(t) \leqq \frac{\pi}{4}, \ t \text{ in } [0. 1].$$

A priori bounds for $y$ and $y''$ follow easily and the existence theorem applies.

5. **A Neumann Problem for** $y'' = f(t, y, y')$**.** The ideas developed in §4 for the Dirichlet problem are also useful for the Neumann problem. In §4 the zero set of $f(p) = 0$ played a decisive role in establishing a priori bound for the Dirichlet problem. For the Neumann problem, the zero set of $f(t, y, p) - y$ plays a similar role.

The following notation will be used. Let $f(t, y, p)$ be continuous on $[0, 1] \times R \times R$ and let $g(t, y, p) = f(t, y, p) - y$. If for each fixed $(t, y)$, $g(t, y, p) = 0$ has both positive and negative solutions for $p$, define

$$r(t, y) = \sup\{p : g(t, y, p) = 0\}$$

and

$$s(t, y) = \inf\{p : g(t, y, p) = 0\}.$$

Consider the Neumann problem

$$(1) \qquad \begin{cases} y'' = f(t, y, y'), \\ y'(0) = 0, \ y'(1) = 0, \end{cases}$$

where $f(t, y, p)$ is continuous.

THEOREM 5.1. *Let $f(t, y, p)$ be continuous on* $[0, 1] \times R \times R$, *and let* $g(t, y, p) = f(t, y, p) - y$.

*Assume*:

(i) *There is a constant* $M \geqq 0$ *such that*

$$yf(t, y, 0) \geqq 0 \quad \textit{for } |y| > M;$$

(ii) *The equation* $g(t, y, p) = 0$ *has both positive and negative solutions for each fixed* $(t, y)$ *in* $[0, 1] \times [-M, M]$ *and the functions* $r(t, y)$, $s(t, y)$ *defined above are continuous on* $[0, 1] \times [-M, M]$.

*Then the Neumann problem* (1) *has at least one solution.*

REMARK. In the proof below we assume that $yf(t, y, 0) > 0$ for $|y| > M$. Once the theorem is established under this assumption, it follows for the weaker assumption in (i) by the same type of compactness argument used in the proof of Theorem 3.1.

PROOF. Let $0 \leqq \lambda \leqq 1$ and consider the family of problems

$$\begin{cases} y'' - y = \lambda g(t, y, y'), \\ y'(0) = 0, \; y'(1) = 0. \end{cases} \tag{2}$$

If $\lambda = 0$, $y \equiv 0$ and so we assume $0 < \lambda \leqq 1$ for the purpose of establishing a priori bounds.

Let $y(t)$ be a solution to (2) for some $\lambda$, $0 < \lambda \leqq 1$. First, (i) implies

$$|y(t)| \leqq M, \quad t \text{ in } [0, 1]$$

(see **[4]**). By continuity there is a constant $B$ such that

$$-B < s(t, y) < 0 < r(t, y) < B,$$

for $(t, y)$ in $[0, 1] \times [-M, M]$. Then

$$|y'(t)| \leqq B, \quad t \text{ in } [0, 1].$$

To see this first assume $y'(t_1) > B$ for some $t_1$ in $(0, 1)$. Note that $g(t, y, p)$ is nonzero for $(t, y)$ in $[0, 1] \times [-M, M]$ and $p > B$. To be definite, assume

$$g(t, y, p) > 0, \; t \text{ in } [0, 1] \times [-M, M] \times (B, \infty). \tag{3}$$

Since $y'(0) = 0 = y'(1)$, the continuous function

$$h(t) = y'(t) - r(t, y(t))$$

changes sign on $[0, t_1]$ and on $[t_1, 1]$. Thus we can find a largest value $t_0$ such that

$$0 > t_0 > t_1 \text{ with } h(t_0) = 0$$

and a smallest value $t_2$ such that

$$t_1 < t_2 < 1 \text{ with } h(t_2) = 0.$$

For $t$ in $(t_0, t_2)$, $h(t) > 0$; i.e., $y'(t) > r(t, y(t))$ and so

$$y''(t) = g(t, y, (t), y'(t)) > 0 \tag{4}$$

by (3). Then

$$0 < \int_{t_1}^{t_2} y''(t)\, dt = y'(t_2) - y'(t_1),$$

$$B < y'(t_1) < y'(t_2) = r(t_2, y(t_2)) < B,$$

a contradiction. If the inequality in (3), hence also in (4), is reversed, then

$$0 > \int_{t_0}^{t_1} y''(t)\, dt = y'(t_1) - y'(t_0),$$

$$B > r(t_0, y(t_0)) = y'(t_0) > y'(t_1) > B,$$

a contradiction. This proves that $y'(t) \leqq B$ for $t$ in $[0, 1]$, and the bound $y'(t) \geqq -B$ follows analogously.

As usual the a priori bounds $|y|_0 \leqq M$, $|y'|_0 \leqq B$ imply an a priori bound for $|y''|_0$, and existence follows from Theorem 2.1.

Theorem 5.1 can be applied to assert that the Neumann problem for

$$\begin{cases} y'' = \sum_{k=0}^{m} a_k(t, y) y'^k, \\ y'(0) = 0, \; y'(1) = 0 \end{cases}.$$

has a solution provided: The $a_k(t, y)$ are continuous functions on $[0, 1] \times R$ such that

(a) for some $M \geqq 0$,

$$y a_0(t, y) \geqq 0 \quad \text{for } |y| > M;$$

(b) the polynomial

$$\sum_{k=0}^{m} a_k(t, y) p^k = 0,$$

has both positive and negative roots for each $(t, y)$ in $[0, 1] \times [-M, M]$ and $a_m(t, y) \neq 0$ there.

It is easy to check that (a), (b) hold for

$$\begin{cases} y'' = (\cos t) y'^6 + 2e^t y y'^5 + y'^3 + y'^2 - (1 + t^2) y' + (y - 1), \\ y'(0) = 0, \; y'(1) = 0, \end{cases}$$

with $M = 1$. The fact that (b) holds with $(t, y)$ in $[0, 1] \times [-1, 1]$ is clear if $y < 1$. If $y = 1$, consider

$$h(p) = (\cos t) p^6 + 2e^t p^5 + p^3 + p^2 - (1 + t^2) p.$$

Then $h'(0) < 0$, $h(-1) \leqq 0$, and it follows easily that $h(p)$ has both positive and negative zeros. Thus the preceding Neumann problem has a solution.

6. **Problems with Nonlinear Boundary Conditions.** Consider the nonlinear problem

$$\text{(1)} \qquad \begin{cases} y'' = f(t, y, y') \\ y'(0) = \phi(y(0)),\ y'(1) = \psi(y(1)) \end{cases}$$

where $\phi, \psi: R \to R$ are continuous and $f: [0, 1] \times R \times R$ is continuous. Further assume there is a constant $M \geqq 0$ such that

(i) $yf(t, y, 0) > 0$ for $|y| > M$;

(ii) $|f(t, y, p)| \leqq A(t, y)p^2 + B(t, y)$,

where $A(t, y)$, $B(t, y) > 0$ are functions bounded for $(t, y)$ in $[0, 1] \times [-M, M]$;

(iii) $r\phi(r) > 0$, $r\psi(r) < 0$ for $r \neq 0$.

Note that (iii) and continuity imply that $\phi(0) = \psi(0) = 0$.

THEOREM 6.1. *Assume $f$, $\phi$, and $\psi$ satisfy conditions* (i), (ii) and (iii). *Then the nonlinear boundary value problem* (1) *has a solution.*

PROOF. Let $y(t) \neq 0$ be a solution to (1). The equation

$$y(0)y'(0) = y(0)\phi(y(0))$$

and (iii) imply that $y$ cannot achieve its extreme values at $t = 0$. Likewise $y(1)$ cannot be the maximum or minimum of $y$ on [0, 1]. Thus $|y|$ achieves its positive maximum at $t_0$ in (0, 1) and by (i), $|y(t_0)| \leqq M$ (see [**4**, Lemma 2.1). Consequently $|y|_0 \leqq M$, and the same reasoning applies if $y$ is a solution to

$$\text{(2)} \qquad \begin{cases} y'' - y = \lambda[f(t, y, y') - y], \\ y'(0) = \lambda\phi(y(0)),\ y'(1) = \lambda\psi(y(1)) \end{cases}$$

for any $0 \leqq \lambda \leqq 1$.

Furthermore, as we have just seen, any nonzero solution of (2) must achieve its extreme values in (0, 1). Hence $y'$ vanishes at least once in [0, 1]. Since (ii) holds, Lemma 3.1 in [**4**] implies $|y'|_0 \leqq M_1$ for some $M_1 < \infty$. Then $|y''|_0 \leqq M_2 < \infty$ follows from (2).

Comparing equations (1) and (2) above with (2) and $(2)_\lambda$ of §2 and making obvious identifications in the case $n = 2$, we see that Theorem 2.4 is applicable, and so (1) has a solution.

REMARK. The nonlinear boundary conditions in (1) can be regarded as the nonlinear analogues of the Sturm-Liouville conditions in IV, §3. Indeed, if $\alpha, \beta, a, b > 0$, $\phi(r) = \alpha r/\beta$, and $\psi(r) = -ar/b$ in (1), we obtain boundary conditions of type IV.

Problems of the form (1) which satisfy (i), (ii), and (iii) occur, for example, in stationary heat conduction for an insulated rod. (Note here that $t$ refers to displacement, not temperature as is customary.) In this case,

$$y'' = f(t, y, y'),$$

where

$$f(t, y, y') = \frac{1}{k(t, y)} [k_t y' + k_y y'^2 + q(t, y)],$$

$k(t, y) > 0$ is the thermal conductivity at position $t$ and temperature $y$, and $q(t, y)$ describes internal heat sources. (Here subscripts denote partial derivatives, and all functions appearing in the definition of $f$ are assumed to be continuous.) Appropriate boundary conditions for this problem are of the form

$$y'(0) = h_0(y(0))y(0), \; y'(1) = -h_1(y(1))y(1),$$

where $h_i(r) > 0$ and continuous in $r$ for $i = 0, 1$. (In the fully linearized case, one assumes $h_0$, $h_1$ are positive constants.) Here

$$\phi(r) = rh_0(r), \; \psi(r) = -rh_1(r)$$

and (iii) holds.

The proof of Theorem 6.1 actually works for certain mixed boundary conditions. Indeed, consider

$$\text{(3)} \qquad \begin{cases} y'' = f(t, y, y'), \\ y'(0) = \phi(y(0), y'(0), y(1), y'(1)), \\ y'(1) = \psi(y(1), y'(1), y(0), y'(0)), \end{cases}$$

where $\phi, \psi: R^4 \to R$ are continuous and

(iii)′ $r\phi(r, r_1, r_2, r_3) > 0$, $r\psi(r, r_1, r_2, r_3) < 0$ for $r \neq 0$ and all $r_1$, $r_2$, and $r_3$.

THEOREM 6.2. *Assume $f$, $\phi$, and $\psi$ satisfy* (i), (ii), *and* (iii)′. *Then problem* (3) *has a solution.*

As a consequence of Theorem 6.2 we see that

$$\begin{cases} y'' = y^3 + (\sin t)y'^2 + \cos t, \\ y' = a_0 y(0)[1 + a_1 y'(0)^2 + a_2 y(1)^2 + a_3 y'(1)^2], \\ y'(1) = -b_0 y(1)[1 + b_1 \,|y(0)y'(0)y'(1)|] \end{cases}$$

has a solution for any choice of the constants $a_0$, $b_0 > 0$, $a_1$, $a_2$, $a_3$, and $b_1 \geqq 0$.

Finally, the method of proof used for Theorem 6.1 also applies to

$$\text{(4)} \qquad \begin{cases} y'' = f(t, y, y'), \\ y(0) = \phi(y'(0)), y(1) = \psi(y'(1)), \end{cases}$$

where $\phi$, $\psi$ satisfy (iii).

THEOREM 6.3. *Assume $f$, $\phi$, and $\psi$ satisfy* (i), (ii), *and* (iii). *Then* (4) *has a solution.*

REMARK. A rather general theorem which includes (6.1)—(6.3) as special cases can be formulated; however, we omit this. The point is that if $f$ satisfies (i) and (ii) above, and if the boundary conditions $U_i(y) = \lambda V_i(y)$, $i = 1, 2$, imply that $y(0)y'(0) > 0$ and $y(1)y'(1) < 0$, then a priori bounds can be established for solutions of $(2)_\lambda$ in §2 where we use either $g = \lambda f$ or $g = \lambda[f - cy]$, for some constant $c$. Then Theorem 2.4 can be used.

7. **Third and Fourth Order Problems.** In this section we consider a class of nonlinear boundary value problem of fourth order which models the deflections of a beam under various end conditions. Related third order problems are also discussed.

THEOREM 7.1. *The boundary value problem*

$$\begin{cases} y^{(\text{iv})} = f(t, y\ y', y'', y'''), \\ y(0) = y(1) = y'(0) = y'(1) = 0 \end{cases}$$

*has at least one solution provided: the function $f$ is continuous and has a decomposition*

$$f(t, y, y', y'', y''') = \phi(t, y, y', y'', y''') + \psi(t, y, y', y'', y'''),$$

*such that*

(a) $y\phi \leqq 0$ *on* $[0, 1] \times R^4$;

(b) $|\psi(t, y, y', y'', y''')| \leqq B(1 + |y|^\alpha + |y'|^\beta + |y''|^\gamma)$, *where* $B < \infty$, $0 \leq \alpha, \beta, \gamma < 1$;

(c) *For $(t, y, y')$ varying in a bounded set in* $[0, 1] \times R^2$, $\phi(t, y, y', y'', y''')$ *is bounded* (*the bound for $\phi$ depending on the bounded set in* $[0, 1] \times R^2$).

The proof of Theorem 7.1 involves several technical estimates which lead to a priori bounds for $|y|_0$, $|y'|_0$, $|y''|_0$, $|y'''|_0$, and $|y^{(\text{iv})}|_0$. These bounds are inferred from certain $L_p$-estimates based on Hölder's inequality. Thus, in this section let

$$\|y\|_p = \left[\int_0^1 |y(t)|^p\, dt\right]^{1/p}$$

be the norm of $y \in L_p$, $1 \leqq p < \infty$. We recall that the function $p \to \|y\|_p$ is increasing on $[1, \infty)$.

In what follows let $y$ be a solution to

$$y'' = \lambda f(t, y, y', y'', y'''),$$
$$y(0) = y(1) = y'(0) = y'(1) = 0,$$

and notice that $\lambda f$ satisfies (a), (b), and (c) above, with the bounds in (b) and (c) independent of $\lambda$ in [0, 1]. In fact, the bounds for $\lambda = 1$, i.e., the bounds for $f$, also hold for $\lambda f$ for all $\lambda$ in [0, 1]. Finally, $C_1$, $C_2$, ... below denote constants, *independent of* $\lambda$. We also assume for the proofs that $2\alpha, 2\beta, 2\gamma \geq 1$, which entails no loss in generality.

LEMMA 7.2. $\|y\|_2 \leqq \|y'\|_2 \leqq \|y''\|_2$.

PROOF. Since $y(t) = \int_0^t y'(\theta)d\theta$, the Schwarz inequality yields $|y(t)| \leqq \|y'\|_2$ which implies the first part of the lemma. The second inequality follows similarly.

LEMMA 7.3. *There is a constant* $C_1$ *such that* $|y(t)|, |y''(t)|, \|y''\|_2 \leqq C_1$ *for* $t$ *in* [0, 1].

PROOF. Since $\int_0^1 yy^{(\mathrm{iv})}\, dt = \int_0^1 (y'')^2\, dt$, (a) and (b) imply that

$$\begin{aligned}
\|y''\|_2^2 &\leqq B\int_0^1 [|y| + |y|\,|y|^\alpha + |y|\,|y'|^\beta + |y|\,|y''|^\gamma]\, dt \\
&\leqq B\{\|y\|_2 + \|y\|_2\|y\|_{2\alpha}^\alpha + \|y\|_2\|y'\|_{2\beta}^\beta + \|y\|_2\|y''\|_{2\gamma}^\gamma\} \\
&\leqq B\|y\|_2\{1 + \|y\|_2^\alpha + \|y'\|_2^\beta + \|y''\|_2^\gamma\} \\
&\leqq B\|y''\|_2\{1 + \|y''\|_2^\alpha + \|y''\|_2^\beta + \|y''\|_2^\gamma\}
\end{aligned}$$

by Lemma 7.2. Since $\alpha, \beta, \gamma, < 1$, this inequality implies $\|y''\|_2 \leqq C_1$ for some constant $C_1$. In view of the initial conditions imposed on $y$ and the proof of Lemma 7.2, the inequalities

$$|y'(t)| \leqq C_1 \text{ and } |y(t)| \leqq C_1$$

follow.

LEMMA 7.4. $\|y^{(\mathrm{iv})}\|_2 \leqq C_2$ *for some constant* $C_2$.

PROOF. From the differential equation, (b), (c) and Lemma 7.3. we obtain

$$|y^{(\mathrm{iv})}(t)| \leqq C_3 + |\phi| \leqq C_4(1 + |y''(t)|^\gamma)$$

for certain constants $C_3$ and $C_4$. Thus

$$|y^{(\mathrm{iv})}(t)|^2 \leqq C_5 + (1 + |y''(t)|^{2\gamma}),$$
$$\|y^{(\mathrm{iv})}\|_2^2 \leqq C_5(1 + \|y''\|_{2\gamma}^{2\gamma}),$$

$$\|y^{(\text{iv})}\|_2^2 \leqq C_5(1 + \|y''\|_2^{2r}) \leqq C_5(1 + C_1^2)$$

from Lemma 7.3.

LEMMA 7.5. *There is a constant $C_6$ such that $|y''(t)|$, $|y'''(t)|$, $|y^{(\text{iv})}(t)| \leqq C_6$.*

PROOF. The boundary conditions imply that $y'$ has three distinct zeros in $[0, 1]$. Hence $y''(\mu) = y'''(\tau) = 0$ for some $\mu$, $\tau$ in $(0, 1)$ and now the rest of the proof is standard.

Now Theorem 7.1 follows by the familiar application of Theorem 2.1.

REMARK. The proof of Theorem 7.1 requires that the boundary conditions imply:

(1)
$$(yy''' - y'y'')\,]_0^1 = 0, \text{ and that}$$
$$y, y', y'', \text{ and } y''' \text{ vanish at least once in } [0, 1].$$

The simplest boundary conditions implying (i) are those imposed in Theorem 7.1, namely
*clamped ends*: $y(0) = y'(0) = 0$, $y(1) = y'(1) = 0$. The other common boundary conditions for the beam problem also imply (1), namely
*flexibly supported ends*: $y(0) = y''(0) = 0$, $y(1) = y''(1) = 0$; or one end clamped and the other either flexibly supported or *free* (which means that $y''$ and $y'''$ vanish at one end). Thus, the conclusion of Theorem 7.1 holds for all these boundary conditions.

The foregoing analysis also applies to certain third order problems.

THEOREM 7.6. *The boundary value problem,*

$$\begin{cases} y''' = \phi(t, y, y'y'') + \psi(t, y, y', y''), \\ y(0) = 0,\ y'(0) = y'(1) = 0, \end{cases}$$

*has at least one solution provided the functions $\phi$ and $\psi$ are continuous and*
(a) $y'\phi \geqq 0$ *on* $[0, 1] \times R^3$;
(b) $|\psi| \leqq B(1 + |y|^\alpha + |y'|^\beta + |y''|^\gamma)$,
*where* $B < \infty$, $0 \leqq \alpha, \beta, \gamma < 1$;
(c) $\phi(t, y, y', y'')$ *is bounded when* $(t, y, y')$ *varies in a bounded set in* $[0, 1] \times R^2$.

SKETCH OF PROOF. The proof uses the same types of argument used for Theorem 7.1. First, the boundary conditions imply that $y$, $y'$, and $y''$ vanish at least once in $[0, 1]$. Thus we obtain

$$\|y\|_2 \leqq \|y'\|_2 \leqq \|y''\|_2 \leqq \|y'''\|_2,$$
$$|y(t)|,\ |y'(t)|,\ |y''(t)| \leqq \|y'''\|_2$$

as for Lemmas 7.2 and 7.3. Use of integration-by-parts, (a), (b), and the preceding facts yield

$$\int_0^1 y''^2\,dt = -\int_0^1 y'y'''\,dt \leqq \int_0^1 y'\psi,$$

$$\|y''\|_2^2 \leqq B\int_0^1 [|y'| + |y'|\,|y|^\alpha + |y'|\,|y'|^\beta + |y'|\,|y''|^\gamma]\,dt,$$

$$\|y''\|_2^2 \leqq B\,\|y''\|_2\,[1 + \|y''\|_2^\alpha + \|y''\|_2^\beta + \|y''\|_2^\gamma],$$

which implies $\|y''\|_2 \leqq C_7$ for some constant $C_7$ independent of $\lambda$. (Here $y$ represents a solution to the boundary value problem in (7.6) with $\phi$ and $\psi$ replaced by $\lambda\phi$ and $\lambda\psi$, $0 \leqq \lambda \leqq 1$.)

Thus, $|y(t)|$, $|y'(t)| \leqq C_7$ for $t$ in [0, 1] and by (c) there is a constant $C_8$ such that

$$|y'''(t)| \leqq C_8 + |\psi|,$$

which yields

$$|y'''(t)| \leqq C_9(1 + |y''(t)|^\gamma),$$

in view of the a priori bounds on $y$ and $y'$. Hence

$$\|y'''\|_2 \leqq C_{10}.$$

This implies $|y(t)|$, $|y'(t)|$, $|y''(t)| \leqq C_{10}$ for $t$ in [0, 1] and so the differential equation yields $|y'''(t)| \leqq C_{11}$. Now Theorem 2.1 applies.

REMARK. In Theorem 7.6 the stated boundary condition may be replaced by

$$y(1) = 0,\ y'(0) = y'(1) = 0$$

by symmetry. Also, the proof works provided $y'y'']_0^1 = 0$ and $y$, $y'$, $y''$ vanish at least once in [0, 1]. Thus, for example, the boundary conditions in Theorem 7.6 may also be replaced by

$$y(0) = y'(0) = y''(1) = 0.$$

8. **Second Order Systems.** The methods of §7 can be applied to certain second order systems. Consider, for example, the problem

$$\text{(1)} \qquad \begin{cases} y'' = \phi(t, y, y') + \psi(t, y, y'), \\ y(0) = 0,\ y(1) = 0, \end{cases}$$

where $\phi$, $\psi$: $[0, 1] \times R^{2n} \to R^n$ are continuous, and $y = (y_1, \ldots, y_n)$. As usual if $u$ and $v$ are $n$-vectors, then

$$u\cdot v = u_1v_1 + \cdots + u_nv_n,$$

$$|u| = (u\cdot u)^{1/2}.$$

Also, let

$$\|y\|_2 = \left(\int_0^1 |y(t)|^2\,dt\right)^{1/2} = \left(\int_0^1 [y_1(t)^2 + \cdots + y_n(t)^2]\,dt\right)^{1/2}.$$

THEOREM 8.1. *The Dirichlet problem* (1) *has at least one solution provided*:

(i) $|\psi(t, y, p)| \leqq B[1 + |y|^\alpha + |p|^\beta]$ *for some* $B < \infty$ *and* $0 \leqq \alpha, \beta < 1$;

(ii) $y \cdot \phi(t, y, y') \geqq 0$;

(iii) *For* $(t, y)$ *in a bounded subset of* $[0, 1] \times R^n$, *the function* $\phi(t, y, y')$ *is bounded.*

REMARK. Condition (ii) is satisfied if $\phi(t, y, y') = A(t, y, y')y$ where $A(t, y, y')$ is a positive semi-definite $n \times n$ matrix for each $(t, y, y')$.

PROOF. Let $y$ be a solution to $y'' = \lambda(\phi + \psi)$, $y(0) = y(1) = 0$ where $0 \leqq \lambda \leqq 1$. Since $y(0) = 0$, the usual argument yields

$$\|y\|_2 \leqq \|y'\|_2. \tag{2}$$

Next we show

$$|y(t)|,\ \|y'\|_2 \leqq C_1,\ t \text{ in } [0, 1] \tag{3}$$

for some constant $C_1$. First,

$$\int_0^1 y \cdot y''dt = -\int_0^1 |y'|^2\,dt$$

and so

$$\int_0^1 |y'|^2\,dt \leqq \int_0^1 y \cdot \psi\,dt \leqq \int_0^1 |y|\,|\psi|\,dt,$$

$$\begin{aligned}\|y'\|_2^2 &\leqq B\int_0^1 [|y| + |y|\,|y|^\alpha + |y|\,|y'|^\beta]\,dt\\ &\leqq B[\|y\|_2 + \|y\|_2\|y\|_{2\alpha}^\alpha + \|y\|_2\|y'\|_{2\beta}^\beta]\\ &\leqq B[\|y'\|_2 + \|y'\|_2^{1+\alpha} + \|y'\|_2^{1+\beta}]\end{aligned}$$

by (2). This inequality implies the existence of $C_1$ such that $\|y'\|_2 \leqq C_1$ and the pointwise bound for $|y(t)|$ follows as usual.

There is a constant $C_2 < \infty$ such that

$$\|y''\|_2 \leqq C_2. \tag{4}$$

Indeed, since $(t, y)$ is confined to the bounded set $[0, 1] \times [-C_1, C_1]^n$, there is a constant $B_1 < \infty$ such that $|\phi(t, y, y')| \leqq B_1$ by (iii). Then

$$|y''(t)| \leqq B_1 + B[1 + C_1 + |y'(t)|].$$

Squaring and integrating leads to the assertion (4).

There is a constant $C_3 < \infty$ such that

$$|y'(0)| \leqq C_3 \tag{5}$$

We have

$$y'(t) - y'(0) = \int_0^t y''(\tau)\, d\tau,$$

$$|y'(0)| \leqq |y'(t)| + \int_0^1 |y''(\tau)|\, d\tau \leqq |y'(t)| + C_2,$$

$$|y'(0)| - C_2 \leqq |y'(t)|,$$

$$[|y'(0)| - C_2]^2 \leqq \|y'\|_2,$$

which implies (5).

The bounds in (4) and (5) imply that

$$|y'(t)| \leqq C_4, \ t \text{ in } [0, 1] \tag{6}$$

for some $C_4 < \infty$, and then

$$|y''(t)| \leqq C_5, t \text{ in } [0, 1]$$

follows from the differential equation. The analogue of Theorem 2.1 for systems shows the existence of a solution to (1).

REMARK. The proof above requires boundary conditions for (1) which imply that $y$ vanishes at least once in [0, 1] and that $y \cdot y']_0^1 = 0$. Thus (1) also has a solution for the boundary conditions

$$y(0) = 0, y'(1) = 0,$$

for example.

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DEPARTMENT OF MATHEMATICS, UNIVERSITÉ DE MONTRÉAL, Montreal, Quebec, Canada
OREGON STATE UNIVERSITY DEPARTMENT OF MATHEMATICS, CORVALLIS, OR 97331