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# THE DIVERGENT BEAM X-RAY TRANSFORM

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1. **Introduction.** Until recently there was little need for mathematics in radiology. Films were examined individually, and by eye, and mathematics had little to offer to the procedure. The picture changed radically in the late 1960's with a breakthrough in radiology called Computed Tomography [10] in which the attenuation in the x-ray beam is measured in an extremely sensitive quantitative way, and the information from many x-rays from different sources is assembled and analysed on a computer. In this new situation mathematics can make significant practical contributions concerning the nature of the total information conveyed by x-rays from many sources, the extent to which this information determines the object x-rayed, suitable methods for using the information to build a detailed reconstruction of the object, etc.

Mathematically, the divergent beam x-ray transform, or radiograph, of a function $f$ on $R^n$ from a source point $a$ is the function $\mathscr{D}_a f$ defined by

$$(1.1) \qquad \mathscr{D}_a f(\theta) = \int_0^\infty f(a + t\theta)\, dt \text{ for } \theta \in S^{n-1}.$$

Physically, $f$ is the density function of the object x-rayed, so that $\mathscr{D}_a f(\theta)$ is the total mass of the object along the half line with origin $a$ and direction $\theta$. In practice this number is determined by measuring the attenuation in the x-ray beam along the half line. The basic problem from either the practical or the mathematical point of view, is the extraction of information about the unknown function $f$ from knowledge about certain of the radiographs $\mathscr{D}_a f$, $a \in A$. Throughout the article it is assumed that $f$ is integrable and that $f$ vanishes outside a bounded open set $\Omega$ with closure $\bar{\Omega}$ and closed convex hull $\hat{\Omega}$.

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In practice, the dimension $n$ is of course 3. In Computed Tomography, however, as the name implies, attention is confined to plane cross sections so that $n = 2$. Thus, in practice the relevant dimensions are $n = 2$ and $n = 3$; and there is a radical difference between the two. From a given source point $a$ it is usually feasible to x-ray full 2 dimensional cross sections of the 3 dimensional object, but out of the question to x-ray the full 3 dimensional object itself. In the 3 dimensional case the beam is "coned down" to the region of interest. For each source point $a \in A$, there is chosen a cone $C_a$ with vertex $a$, and the attenuation is measured only along half lines in $C_a$. If

$$S_a = (C_a - a) \cap S^{n-1} \tag{1.2}$$

is the corresponding set of directions, then the measured x-ray data consist of the numbers

$$\mathscr{D}_a f(\theta) \text{ for } a \in A \text{ and } \theta \in S_a. \tag{1.3}$$

The set

$$\Omega_m = \bigcup_{a \in A} (C_a \cap \Omega), \tag{1.4}$$

consisting of the points in $\Omega$ on at least one half line along which the attenuation is measured, is called the *measured region*. Outside the measured region alterations in the density function produce no change in the x-ray data. Even within the measured region, the effects of having only partial information are rather peculiar.

The practical necessity for dealing with 3 dimensional objects directly, rather than a succession of 2 dimensional cross sections, and hence with the 3 dimensionally divergent x-ray beam, comes from the need for extremely fast x-ray scan times in the reconstruction of moving objects. A problem of major current interest for example, is the 3 dimensional reconstruction of the beating heart. With a human patient the x-ray data must be collected during the fraction of a second in which the heart movement is insignificant, and this is impracticable with a succession of 2 dimensionally divergent beams. At the present time a very sophisticated machine called the Dynamic Spatial Reconstructor is being built at the Mayo Clinic for the study of the functional dynamics of the heart, lungs, and circulation by means of 3 dimensional reconstructions from 3 dimensionally divergent x-ray beams **[17]**, **[22]**.

The general subject of this article is the extraction of information about the unknown function $f$ from the line integrals $\mathscr{D}_a f(\theta)$ for $a \in A$ and $\theta \in S_a$.

Section 2 summarizes the definitions and properties of some standard transforms that are needed: the Fourier transform, the Riesz potential, the Riesz singular integral operator, the operator $\Lambda$ of Calderón, the Radon transform, and the parallel beam x-ray transform.

Section 3 contains elementary basic properties of the divergent beam x-ray transform.

Section 4 establishes relations between the divergent beam x-ray transform and the Radon transform, which in some situations can be used to reduce $n$ dimensional questions to 1 dimensional questions.

Section 5 contains the two main uniqueness theorems, corresponding to the cases where full information is available from each source $a \in A$, i.e., $S_a = S^{n-1}$; and where only partial information is available, i.e., $S_a \subset S^{n-1}$. In the first case uniqueness holds when $A$ is any infinite set outside $\hat{\Omega}$. In the second uniqueness holds when $A$ is an infinite set satisfying additional conditions which seem rather curious, but which are shown to be more or less necessary, and which are entirely feasible in practice. A high degree of non-uniqueness is shown to obtain whenever $A$ is a finite set.

Section 6 contains a short discussion of the practical role of the uniqueness theorems.

Section 7 gives a constructive procedure for approximating the unknown function $f$ by the use of its measured x-ray data (1.3).

One of the principles of Computed Tomography has been that each point of the measured region $\Omega_m$ be seen from numerous sources. Section 8 gives a description of the measured regions for which this is possible, showing that they are very restricted.

Section 9 gives examples of x-ray setups that may be of use in practice.

2. **Some standard transforms.** In the course of the discussion various standard transforms will be used. The definitions and some properties are summarized below. The references are **[2, 7, 14, 18, 19, 20, 21]**.

a) The *Fourier transform* is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-i\langle x, \xi\rangle} f(x)\, dx. \tag{2.1}$$

b) The *Riesz potential* $\mathscr{R}^\alpha$ is defined by

$$\mathscr{R}^\alpha f(x) = R_\alpha * f(x) = \int f(y) R_\alpha(x-y)\, dy, \text{ where} \tag{2.2}$$

$$R_\alpha(x) = \frac{1}{C(n, \alpha)} |x|^{\alpha - n},\ 0 < \alpha < n,\ C(n, \alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}.$$

With the indicated choice of the constant $C(n, \alpha)$ the Fourier transform of $R_\alpha$ is

$$\hat{R}_\alpha(\xi) = (2\pi)^{-n/2} |\xi|^{-\alpha},\ \textit{hence}\ (\mathscr{R}_\theta f)^\wedge(\xi) = |\xi|^{-\alpha} \hat{f}(\xi). \tag{2.3}$$

Note that therefore $\mathscr{R}^\alpha = (\mathscr{R}^1)^\alpha$.

c) The *Riesz transform* $\mathscr{H}_j$ is defined by

(2.4) $$\mathscr{H}_j f(x) = H_j * f(x) = \int f(y) H_j(x - y)\, dy, \text{ where}$$

$$H_j(x) = \frac{n-1}{C(n,1)} x_j / |x|^{n+1} = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} x_j / |x|^{n+1}.$$

In this case the convolution integral is singular, and is taken as a Cauchy principal value. If $n > 1$, then

(2.5) $$H_j = -(\partial/\partial x_j) R_1$$

so that by (2.3)

(2.6) $$\hat{H}_j(\xi) = -(2\pi)^{-n/2}\, i\xi_j |\xi|^{-1}, \text{ hence}$$
$$(\mathscr{H}_j f)\hat{}(\xi) = -i\,\xi_j |\xi|^{-1} \hat{f}(\xi),$$

which also holds in dimension 1.

d) The operator $\Lambda$ is defined by

(2.7) $$\Lambda = \sum_{j=1}^{n} (\partial/\partial x_j)\, \mathscr{H}_j.$$

According to (2.6)

(2.8) $$(\Lambda f)\hat{}(\xi) = |\xi| \hat{f}(\xi).$$

It follows from (2.3) that

(2.9) $$\Lambda = (\mathscr{R}^1)^{-1} = \Lambda^2 \mathscr{R}^1 = -\Delta \mathscr{R}^1,$$

where

(2.10) $$\Delta = \sum_{j=1}^{n} (\partial/\partial x_j)^2$$

is the usual Laplace operator.

e) The *Radon transform* $\mathscr{R}_\theta$ is defined by

(2.11) $$\mathscr{R}_\theta f(t) = \int_{\langle x, \theta\rangle = t} f(x) dx \text{ for } \theta \in S^{n-1},\ t \in R^1.$$

If $\rho$ is a function of 1 variable, then

(2.12) $$\int_{-\infty}^{\infty} \mathscr{R}_\theta f(t) \rho(t)\, dt = \int_{R^n} f(x) \rho(\langle x, \theta\rangle)\, dx,$$

as can be seen simply by writing out $\mathscr{R}_\theta f$. With $\rho(t) = t^m$, it follows that the function

(2.13) $$p_m(\theta) = \int_{-\infty}^{\infty} \mathscr{R}_\theta f(t) t^m\, dt$$

is a homogeneous polynomial of degree $m$. (This is the consistency condition of Helgason [7] and Ludwig [14] which characterizes the range of the Radon transform.) With $\rho(t) = e^{-i\tau t}$, (2.12) becomes the formula for the Fourier transform of $\mathscr{R}_\theta f$:

$$(\mathscr{R}_\theta f)\hat{}(\tau) = (2\pi)^{(n-1)/2}\hat{f}(\tau\theta). \tag{2.14}$$

Apart from constants the polynomials $p_m(\theta)$ in (2.13) are the Taylor coefficients in the expansion of $\hat{f}(\tau\theta)$ in powers of $\tau$. For the validity of these formulas it is assumed that $f \in L_0^1$, i.e., that $f$ is integrable with bounded support.

f) The *parallel beam x-ray transform* $\mathscr{P}_\theta$ is defined by

$$\mathscr{P}_\theta f(x) = \int_{-\infty}^{\infty} f(x + t\theta)\, dt \textit{ for } \theta \in S^{n-1},\ x \in \theta^\perp. \tag{2.15}$$

In this case $\theta$ is the direction of the parallel x-ray beam and the point $x$ in the orthogonal subspace $\theta^\perp$ can be thought of as a point on the film. Thus, while the formula is almost identical to that defining the divergent beam x-ray transform, the variables play quite different roles. The relation between the two is:

$$\mathscr{P}_\theta f(E_\theta a) = \mathscr{D}_a f(\theta) + \mathscr{D}_a f(-\theta), \tag{2.16}$$

where $E_\theta$ is the orthogonal projection in $R^n$ on $\theta^\perp$.

If $\rho$ is a function of 1 variable, then

$$\int_{\theta^\perp} \mathscr{P}_\theta f(x)\rho(\langle x, \xi\rangle)\, dx = \int_{R^n} f(x)\rho(\langle x, \xi\rangle)\, dx \textit{ for } \xi \in \theta^\perp, \tag{2.17}$$

as can be seen simply by writing out $\mathscr{P}_\theta f$. With $\rho(t) = t^m$, it follows that the polynomials

$$p_{m,\theta}(\xi) = \int_{\theta^\perp} \mathscr{P}_\theta f(x)\, \langle x, \xi\rangle^m\, dx,\ \xi \in \theta^\perp,$$

must fit together to determine a polynomial on $R^n$. (This is the consistency condition of [20] which effectively characterizes the range of the parallel beam x-ray transform.) With $\rho(t) = e^{-it}$, (2.17) becomes the formula for the Fourier transform of $\mathscr{P}_\theta f$:

$$(\mathscr{P}_\theta f)\hat{}(\xi) = (2\pi)^{1/2}\hat{f}(\xi) \textit{ for } \xi \in \theta^\perp. \tag{2.18}$$

Again, for the validity of these formulas it is assumed that $f \in L_0^1$. Both the parallel beam x-ray transform and the Radon transform are discussed in considerable detail in [19, 20].

3. **Basic properties.** The divergent beam x-ray transform, or radiograph, of the function $f$ from the source $a$ is defined by

$$\mathscr{D}_a f(\theta) = \int_0^{\infty} f(a + t\theta)\, dt \textit{ for } \theta \in S^{n-1}. \tag{3.1}$$

Although the transform can be defined more generally, it is assumed throughout the article that $f \in L_0^1(\Omega)$, i.e., that $f$ is integrable and vanishes outside $\Omega$, where $\Omega$ is a bounded open subset of $R^n$ with diameter $\delta$, closure $\bar{\Omega}$, and closed convex hull $\hat{\Omega}$.

Most of the basic properties of the divergent beam x-ray transform result from a simple formula expressing the scalar product of $\mathscr{D}_a f$ with a function $h$ on $S^{n-1}$ as the convolution of $f$ with the positively homogeneous extension of $h(-\theta)$ of degree $1 - n$.

THEOREM 3.2. *If $h$ is a measurable function on $S^{n-1}$ and $H(x) = |x|^{1-n}h(-x/|x|)$, then*

$$\int_{S^{n-1}} \mathscr{D}_a f(\theta) h(\theta) d\theta = \int_{R^n} f(x) H(a - x)\, dx = f * H(a). \tag{3.3}$$

*The formula holds if both $f$ and $h$ are non-negative, or if either side is finite when both $f$ and $h$ are replaced by their absolute values.*

PROOF. We have

$$\int_{S^{n-1}} \mathscr{D}_a f(\theta) h(\theta) d\theta = \int_{S^{n-1}} \int_0^\infty f(a + t\theta) H(-t\theta) t^{n-1} dt,$$

which is the right side of (3.3) in polar coordinates with origin at $a$. The calculation is justified automatically by Fubini if both $f$ and $h$ are non-negative, then a posteriori if either side is finite when $f$ and $h$ are replaced by absolute values.

THEOREM 3.4. *If $f \in L_0^1(\Omega)$, then for almost every source $a$, $\mathscr{D}_a f(\theta)$ is defined by an absolutely convergent integral for almost all $\theta$, and $\mathscr{D}_a f \in L^1(S^{n-1})$. If $f \in L_0^2(\Omega)$, then for almost every source $a$, $\mathscr{D}_a f \in L^2(S^{n-1})$.*

PROOF. Taking $h = 1/C(n, 1)$ in (3.3), and assuming initially that $f$ is non-negative, we obtain from (2.2)

$$\frac{1}{C(n, 1)} \int_{S^{n-1}} \mathscr{D}_a f(\theta) d\theta = R_1 * f(a) = \mathscr{R}^1 f(a). \tag{3.5}$$

If $|a| \leqq r$, the kernel $R_1$ can be replaced by the kernel $K$ which is equal to $R_1$ for $|y| \leqq r + \delta$ and is 0 otherwise without affecting the convolution on the right of (3.5). Since $K$ is integrable, $K * f$ is finite almost everywhere. Consequently, $R_1 * f$ is finite almost everywhere on $|a| \leqq r$, and since this holds for every $r$, $R_1 * f$ is finite almost everywhere. This proves the first assertion in the theorem.

If $f \in L_0^2(\Omega)$, we apply that has been proved to $|f|^2$, noting first that

$$|\mathscr{D}_a f(\theta)|^2 \leqq \delta \mathscr{D}_a |f|^2(\theta). \tag{3.6}$$

This and (3.5) give

$$\int_{S^{n-1}} |\mathscr{D}_a f(\theta)|^2 d\theta \leqq \delta C(n, 1)\, R_1{}^*|f|^2(a), \tag{3.7}$$

so the proof is complete by what has been done for $L^1$ functions.

Formulas (3.5) and (2.9) contain the inversion formula for the divergent beam $x$-ray transform:

$$f = \frac{1}{C(n, 1)} \Lambda \int_{S^{n-1}} \mathscr{D}_a f(\theta)\, d\theta = \frac{-1}{C(n, 1)} \Delta \mathscr{R}^1 \int_{S^{n-1}} \mathscr{D}_a f(\theta)\, d\theta, \tag{3.8}$$

the second of which has also been observed by L. Shepp (oral communication).

When account is taken of the relationship (2.16) between the divergent and parallel beam transforms, the inversion formula for the latter results:

$$f = \frac{1}{2C(n, 1)} \Lambda \int_{S^{n-1}} \mathscr{P}_\theta f(E_\theta a)\, d\theta. \tag{3.9}$$

REMARK 3.10. In formula (3.8) the operators $\Lambda$, $\Delta$, and $\mathscr{R}^1$ act on functions of the source $a$. Thus the formula requires the knowledge of $\mathscr{D}_a f$ for every point $a \in R^n$. However, it is obvious that $\mathscr{D}_a f(\theta) = 0$ if the half line with initial point $a$ and direction $\theta$ misses $\Omega$, and also that $\mathscr{D}_a f(\theta) + \mathscr{D}_a f(-\theta) = \mathscr{D}_b f(\theta) + \mathscr{D}_b f(-\theta)$ if this half line contains $b$. Consequently, it is enough to know $\mathscr{D}_a f$ for all points $a$ on a sphere which surrounds $\bar{\Omega}$, and with these all other $\mathscr{D}_a f$ can be expressed simply.

REMARK 3.11. Account being taken of Remark 3.10, formulas (3.8) and (3.9) contain in principle (i.e., apart from specific numerical implementations) the divergent and parallel beam "convolution inversion formulas" in current use in the Computed Tomography scanners for the reconstruction of 2 dimensional cross sections of the body **[13]**, **[16]**.

For source points $a$ outside $\bar{\Omega}$, Theorem 3.4 becomes considerably stronger.

THEOREM 3.12. *For fixed $a$ outside $\bar{\Omega}$, $\mathscr{D}_a$ is a bounded linear map from $L_0^1(\Omega)$ to $L^1(S^{n-1})$ satisfying*

$$\|\mathscr{D}_a f\|_{L^1(S^{n-1})} \leqq d(a, \Omega)^{1-n} \|f\|_{L_0^1(\Omega)} \tag{3.13}$$

*where $d(a, \Omega)$ is the distance from $a$ to $\Omega$. Moreover, for fixed $f \in L_0^1(\Omega)$, the map $a \to \mathscr{D}_a f$ is continuous from $R^n - \bar{\Omega}$ to $L^1(S^{n-1})$.*

PROOF. Since $C(n, 1)R_1 = |x|^{1-n}$, the inequality (3.13) follows immediately from (3.5). The continuity of the map $a \to \mathscr{D}_a f$ is obvious for continuous $f$, and from this it follows for general $f \in L_0^1(\Omega)$ because of the uniformity of the bound in (3.13).

THEOREM 3.14. *For fixed $a$ outside $\bar{\Omega}$, $\mathscr{D}_a$ is a bounded linear map from $L_0^2(\Omega)$ to $L^2(S^{n-1})$ satisfying*

$$(3.15)\qquad \|\mathscr{D}_a f\|^2_{L^2(S^{n-1})} \leqq \delta\, d(a, \Omega)^{1-n}\, \|f\|^2_{L^2_0(\Omega)}.$$

*Moreover, for fixed* $f \in L^2_0(\Omega)$, *the map* $a \to \mathscr{D}_a f$ *is continuous from* $R^n - \bar{\Omega}$ *to* $L^2(S^{n-1})$.

PROOF. Since $C(n, 1)R_1 = |x|^{1-n}$, the inequality (3.15) follows immediately from (3.7), and the rest of the proof is the same as the last one.

COROLLARY 3.16. *Under the conditions of the theorem the adjoint of* $\mathscr{D}_a$ *is given by*

$$(3.17)\qquad \mathscr{D}_a^* h(x) = |x - a|^{1-n}\, h((x - a)/|x - a|) \text{ for } x \in \Omega.$$

PROOF. This is obvious from (3.3).

For later use we record a minor extension of Theorem 3.2. For this purpose we define

$$(3.18)\qquad \mathscr{D}_a^k f(\theta) = \int_0^\infty t^k f(a + t\theta)\, dt \text{ for } \theta \in S^{n-1}.$$

THEOREM 3.19. *If* $h$ *is a measurable function on* $S^{n-1}$ *and* $H(x) = |x|^{k+1-n} h(-x/|x|)$, *then*

$$(3.20)\qquad \int_{S^{n-1}} \mathscr{D}_a^k f(\theta) h(\theta)\, d\theta = \int_{R^n} f(x) H(a - x)\, dx = f * H(a).$$

*The formula holds if both* $f$ *and* $h$ *are non-negative, or if either side is finite when both* $f$ *and* $h$ *are replaced by their absolute values.*

PROOF. The proof is the same as the proof of Theorem 3.2.

4. **Relations with the Radon transform.** The first useful relationship between the divergent beam x-ray transform and the Radon transform arises from formulas (3.3) and (2.12) upon taking $h(\theta) = |\langle \theta, \phi \rangle|^{1-n}$ and $\rho(t) = |\langle a, \phi \rangle - t|^{1-n}$ with $\phi \in S^{n-1}$ fixed, as will be explained presently, so that the integrals converge. Assuming convergence of the integrals, the cited formulas give

$$\int_{S^{n-1}} \mathscr{D}_a f(\theta) |\langle \theta, \phi \rangle|^{1-n}\, d\theta = \int_{R^n} f(x) |\langle a - x, \phi \rangle|^{1-n}\, dx$$
$$= \int_{-\infty}^{\infty} \mathscr{R}_\phi f(t) |\langle a, \phi \rangle - t|^{1-n}\, dt.$$

The functions $h$ and $\rho$ have rather bad singularities, and the point $\phi$ must be restricted so that these singularities do not produce divergent integrals.

The support function of the convex set $\hat{\Omega}$ is defined by

$$(4.1)\qquad \|\xi\|_{\hat{\Omega}} = \sup_{x \in \Omega} \langle x, \xi \rangle.$$

If $t > \|\phi\|_{\hat{\Omega}}$, then $\mathscr{R}_\phi f(t) = 0$, for in this case the plane $\langle x, \phi \rangle = t$ misses

$\hat{\Omega}$. Thus, if $\langle a, \phi \rangle > \|\phi\|_{\hat{\Omega}}$, then the integral on the right above is absolutely convergent. Under the same condition, $\mathscr{D}_a f(\theta)$ vanishes on a neighborhood of the set where $\langle \theta, \phi \rangle \geqq 0$, so that the integral on the left is also absolutely convergent.

THEOREM 4.2. *If* $f \in L_0^1(\Omega)$ *and* $\langle a, \phi \rangle > \|\phi\|_{\hat{\Omega}}$, *then*

$$\text{(4.3)} \quad \begin{aligned} \int_{S^{n-1}} \mathscr{D}_a f(\theta)|\langle \theta, \phi \rangle|^{1-n} \, d\theta &= \int_{-\infty}^{\infty} \mathscr{R}_\phi f(t) \, (\langle a, \phi \rangle - t)^{1-n} \, dt \\ &= \int_{-\|-\phi\|_{\hat{\Omega}}}^{\|\phi\|_{\hat{\Omega}}} \mathscr{R}_\phi f(t) \, (\langle a, \phi \rangle - t)^{1-n} \, dt. \end{aligned}$$

COROLLARY 4.4. *If* $\langle a, \phi \rangle > \max(\|\phi\|_{\hat{\Omega}}, \|-\phi\|_{\hat{\Omega}})$, *then*

$$\text{(4.5)} \quad \int_{S^{n-1}} \mathscr{D}_a f(\theta)|\langle \theta, \phi \rangle|^{1-n} \, d\theta = \sum_{m=0}^{\infty} \binom{m+n-2}{m} \langle a, \phi \rangle^{1-n-m} \, p_m(\phi),$$

*where* $p_m$ *is the polynomial in* (2.13).

PROOF. In the right hand integral in (4.3) we expand $(\langle a, \phi \rangle - t)^{1-n}$ in a power series and integrate term by term.

A second useful relationship with the Radon transform arises on applying Parseval's equality to the right side of (3.3). The result is

$$\text{(4.6)} \qquad \int_{S^{n-1}} \mathscr{D}_a f(\theta) h(\theta) \, d\theta = \int_{R^n} e^{i\langle a, \xi \rangle} \hat{f}(\xi) \hat{H}(\xi) \, d\xi.$$

Applying the same thing to $h(-\theta)$, adding the two together, and expressing the result in polar coordinates, we get

$$\begin{aligned} &\int_{S^{n-1}} (\mathscr{D}_a f(\theta) + \mathscr{D}_a f(-\theta)) h(\theta) \, d\theta \\ &\qquad = \int_{S^{n-1}} \int_{-\infty}^{\infty} e^{i\tau\langle a, \theta \rangle} \, |\tau|^{n-2} \hat{f}(\tau\theta) \hat{H}(\theta) \, d\tau \, d\theta. \end{aligned}$$

THEOREM 4.7. *If* $f$ *is of class* $C_0^s(\Omega)$, $s > (n-2)/2$, $h \in L^2(S^{n-1})$, $H(x) = |x|^{1-n} h(-x/|x|)$, *and* $a \notin \bar{\Omega}$, *then*

$$\text{(4.8)} \qquad \begin{aligned} &\int_{S^{n-1}} (\mathscr{D}_a f(\theta) + \mathscr{D}_a f(-\theta)) h(\theta) \, d\theta \\ &\qquad = (2\pi)^{(2-n)/2} \int_{S^{n-1}} \Lambda^{n-2} \mathscr{R}_\theta f(\langle a, \theta \rangle) \hat{H}(\theta) \, d\theta. \end{aligned}$$

PROOF. By (2.8) and (2.14), the right hand side in the last formula above is equal to the right hand side of (4.8), provided the separation of the integrals in the polar coordinate representation is justified. It is easily seen that this separation is justified if $f$ belongs to the Sobolev space $\mathscr{H}_0^s(\Omega)$ with $s > (n-2)/2$, therefore, in particular, if $f \in C_0^s(\Omega)$ with $s > (n-2)/2$ **[21]**.

5. **Uniqueness theorems.** The first of the two main uniqueness theorems is as follows.

THEOREM 5.1. *Let $f \in L_0^1(\Omega)$, and let $A$ be any infinite set of sources bounded away from $\hat{\Omega}$. If $\mathscr{D}_a f = 0$ for each $a \in A$, then $f = 0$.*

NOTE. According to Theorem 3.12, $\mathscr{D}_a f$ is defined almost everywhere on $S^{n-1}$. The hypothesis that $\mathscr{D}_a f = 0$ means that $\mathscr{D}_a f = 0$ almost everywhere on $S^{n-1}$. Similarly, the conclusion that $f = 0$ means that $f = 0$ almost everywhere on $R^n$.

PROOF. If $A$ is bounded away from $\hat{\Omega}$, then clearly there is a ball $B \supset \hat{\Omega}$ such that an infinite subset of $A$ is bounded away from $B$. Replacing $A$ by such a subset, we can assume that $A$ itself is bounded away from $B$. Since the theorem is invariant under translation and dilation, we can assume that $B$ is the unit ball in $R^n$, i.e., the ball with center 0 and radius 1. Thus we can assume that $\hat{\Omega} = B(0, 1)$ and that $|a| > 1 + 2\varepsilon$ for each $a \in A$. Note, for the purpose of applying Corollary 4.4, that now $\|\phi\|_{\hat{\Omega}} = |\phi| = 1$.

If $\phi_0$ is an accumulation point of the set $\{a/|a| : a \in A\}$, then the condition, $\langle a, \phi\rangle > 1 + \varepsilon$ holds for $\phi$ in some fixed neighborhood $U$ of $\phi_0$ on $S^{n-1}$ and an infinite number of points $a \in A$. Discarding the remaining points of $A$ we have

$$\langle a, \phi\rangle > 1 + \varepsilon \quad \text{for } a \in A \text{ and } \phi \in U, \tag{5.2}$$

where $U$ is an open subset of $S^{n-1}$. Thus formula (4.5) holds for every $a \in A$ and every $\phi \in U$. It is clear that for almost every $\phi \in U$ the set of values $s = \langle a, \phi\rangle$ is an infinite set. Consequently, for such a $\phi$, (4.5) shows that the function

$$\sum_{m=0}^{\infty} \binom{m + n - 2}{m} s^{1-n-m} p_m(\phi)$$

has infinitely many zeros $s = \langle a, \phi\rangle > 1 + \varepsilon$, which is possible only if each coefficient is 0, i.e., each $p_m(\phi) = 0$. Thus each $p_m$ vanishes almost everywhere on $U$, and hence vanishes identically, since $p_m$ is a homogeneous polynomial. From this and (2.13) it follows that $\mathscr{R}_\theta f = 0$ for every $\theta$, and then from (2.14) that $f = 0$.

In practice, a full 3 dimensionally divergent x-ray beam is rarely measured, and indeed in medical procedures it is never measured. Instead, the beam is "coned down" to the region of interest. For each source point $a \in A$ there is chosen a cone $C_a$ with vertex at $a$, and the attenuation is measured only along half lines in $C_a$. If

$$S_a = (C_a - a) \cap S^{n-1} \tag{5.3}$$

is the corresponding set of directions, then the *measured x-ray data* consist of the numbers

(5.4) $$\mathscr{D}_a f(\theta) \quad \text{for } a \in A \text{ and } \theta \in S_a.$$

The set

(5.5) $$\Omega_m = \left(\bigcup_{a \in A} C_a\right) \cap \Omega = \bigcup_{a \in A} (C_a \cap \Omega)$$

consisting of those points in $\Omega$ on at least one half line along which the attenuation is measured is called the *measured region*. Outside of $\Omega_m$ alterations in the density function produce no change in the x-ray data.

Theorem 5.1 treats the case in which $A$ is any infinite set outside $\hat{\Omega}$ and the measured region $\Omega_m$ is all of $\Omega$. The next theorem treats the case in which $A$ is a dense subset of a suitable rectifiable arc and $C_a = a + C$, where $C$ is a fixed open cone with vertex 0.

THEOREM 5.6. *Let $f \in L_0^1(\Omega)$. Let $C$ be an open cone with vertex* 0. *Let $\bar{A}$, the closure of $A$, be a rectifiable arc outside $\bar{\Omega}$ such that for each half line $\ell$ in $C$ there is a point $a \in \bar{A}$ for which $a + \ell$ misses $\bar{\Omega}$. If the measured x-ray data $\mathscr{D}_a f(\theta)$, $a \in A$ and $\theta \in S_a = C \cap S^{n-1}$, are* 0, *then $f = 0$ on the measured region $\Omega_m = (A + C) \cap \Omega = (\bar{A} + C) \cap \Omega$.*

Before turning to the proof we give an example to clarify the somewhat peculiar situation which obtains in regard to uniqueness. Figure 1 below simulates a 2 dimensional cross section of the chest $\Omega$ with the heart $\Omega_0$ being the region of medical interest. The large circle is $\Omega$, the smaller one $\Omega_0$. If the source set $A$ is the arc $ab$, then the measured region $\Omega_m$ is the region shaded by the rays coming from points on $ab$. In this case, although the measured region $\Omega_m$ contains the region of interest $\Omega_0$, the density function $f$ on $\Omega_0$ is not uniquely determined by the measured x-ray data. Indeed, if $f$ is $+1$ between the upper pair of horizontal lines, $-1$ between the lower pair (the two pairs being equally spaced), and 0 elsewhere on $\Omega_m$, then clearly its x-ray data are 0. On the other hand, if $A$ is the arc $ac$, then the hypotheses in the theorem are satisfied, and the density function is uniquely determined by the measured x-ray data on the measured region, which is now the entire shaded region.

Other examples are given in section 9.

PROOF OF THE THEOREM. Recall that

(5.7) $$\mathscr{D}_a^k f(\theta) = \int_0^\infty t^k f(a + t\theta)\, dt \text{ for } \theta \in S^{n-1}.$$

According to Theorem 3.19, if $H$ is positively homogeneous of degree $k + 1 - n$ and of class $C^1$ on $R^n - \{0\}$, then

(5.8) $$\int_{S^{n-1}} \mathscr{D}_a^k f(\theta) H(\theta)\, d\theta = \int_{R^n} f(x) H(x - a)\, dx$$

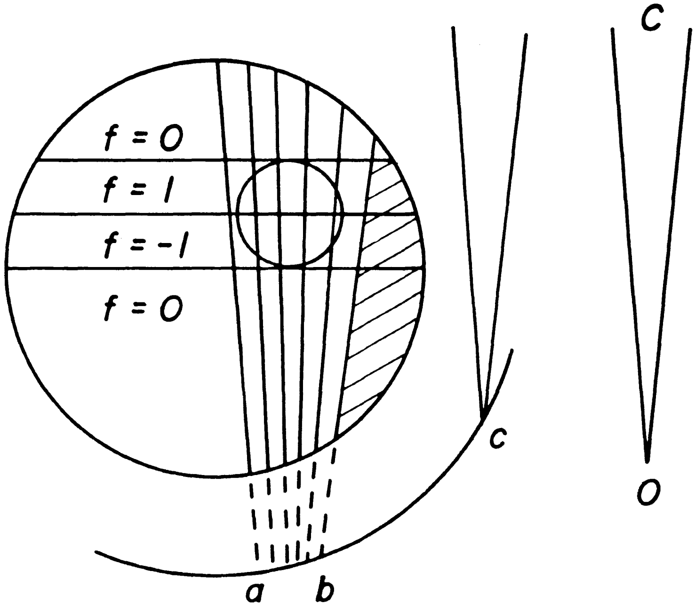


Figure 1

and

$$\int_{S^{n-1}} \mathscr{D}_a^{k-1} f(\theta)\partial H/\partial x_j(\theta)\, d\theta = \int_{R^n} f(x)\partial H/\partial x_j(x - a)\, dx. \tag{5.9}$$

First we note that if $\mathscr{D}_a^k f = 0$ on $S = C \cap S^{n-1}$ for $a \in A$, then the same is true for $a \in \bar{A}$. Indeed, the right hand integral in (5.8) is a continuous function of $a$, while if $H$ vanishes outside $C$, then the left side is 0 for $a \in A$. Therefore, the two are 0 for $a \in \bar{A}$. Since this is true for every $H$ vanishing outside $C$, the assertion follows.

The following statement, which will be proved by induction on $k$, will establish the theorem; the case $k = 0$ being taken care of by the hypothesis in the theorem and the remark above:

$$\mathscr{D}_a^k f = 0 \textit{ on } S = C \cap S^{n-1} \textit{ for } a \in \bar{A}. \tag{5.10}$$

Let $a(s)$, $0 \leqq s \leqq L$, be the parameterization of the rectifiable arc $\bar{A}$ by arc length, so that $a(s)$ is absolutely continuous and $|a'(s)| = 1$ a.e. Let $H$ be positively homogeneous of degree $k + 1 - n$, of class $C^1$ on $R^n - \{0\}$, and set

$$\alpha(s) = \int_{S^{n-1}} \mathscr{D}_{a(s)}^k f(\theta)H(\theta)\, d\theta. \tag{5.11}$$

By formula (5.8)

$$\alpha'(s) = -\sum_{j=1}^{n} a_j'(s) \int_{R^n} f(x)\partial H/\partial x_j(x - a(s))\, dx.$$

Therefore, by (5.9)

$$\alpha'(s) = -\sum_{j=1}^{n} a_j'(s) \int_{S^{n-1}} \mathscr{D}_{a(s)}^{k-1} f(\theta)\partial H/\partial x_j(\theta)\, d\theta. \tag{5.12}$$

Note for future reference that, without any hypothesis of vanishing, if $\alpha(s)$ is defined by (5.11), then $\alpha'(s)$ is given by (5.12).

Now suppose inductively that (5.10) holds for $k - 1$, and let $H$ vanish outside $C$. Then $\partial H/\partial x_j$ vanishes outside $C$ also and we have $\alpha'(s) = 0$ for almost every $s$. Let $\theta_0$ be an arbitrary point of $S = C \cap S^{n-1}$, and choose $s_0$ so that the half line $a(s_0) + t\theta_0$ misses $\bar{\Omega}$. Then choose a neighborhood $U$ of $\theta_0$ in $S$ so that the half line $a(s_0) + t\theta$ misses $\bar{\Omega}$ for every $\theta \in U$. Let $s_1$ be arbitrary, and let $H$ vanish outside the open cone generated by $U$. Since $\alpha'(s) = 0$, we have $\alpha(s_1) = \alpha(s_0) = 0$. Since $H$ is arbitrary, it follows that for the arbitrary $a = a(s_1)$, (5.10) holds with $S$ replaced by $U$. Since this is true for some neighborhood $U$ of an arbitrary $\theta_0$, (5.10) itself holds, and the inductive proof is complete.

With a finite number of sources there is always a high degree of non-uniqueness. In the parallel beam case (where the parallel beam directions correspond to the divergent beam sources) the following has been shown **[19]**: *For any given directions $\theta_1, \ldots, \theta_M$, any given compact set $K \subset \Omega$, and any given function $f_0$ which is $C^\infty$ on $\Omega$, there is a function $f \in C_0^\infty(\Omega)$ such that $f = f_0$ on $K$ and $\mathscr{P}_{\theta_j} f = 0$ for $j = 1, \ldots, M$.* Something similar must be true in the divergent beam case also. At present we have a weaker result, which nevertheless establishes a high degree of non-uniqueness.

THEOREM 5.13. *Let sources $a_1, \ldots, a_M$ outside $\hat{\Omega}$ be given. For any function $u \in C_0^\infty(\hat{\Omega})$, set*

$$g(\theta, t) = \left\{\prod_{j=1}^{M} (t - \langle a_j, \theta\rangle)^{n-1}\right\} (d^m/dt^m)\mathscr{R}_\theta u(t) \text{ with} \tag{5.14}$$
$$m = M(n - 1).$$

*There is a unique function $f \in C_0^\infty(\hat{\Omega})$ with $\mathscr{R}_\theta f(t) = g(\theta, t)$, and for this $f$*

$$\mathscr{D}_{a_j} f = 0 \text{ for } j = 1, \ldots, M. \tag{5.15}$$

If the support of $u$ lies in a given convex set, then the support of $f$ also lies in this set ([7], **[14]**, **[19]**), so the addition of such functions $u$ with supports in small disjoint convex sets provides a large supply of functions with zero radiographs from the given sources.

PROOF. The necessary and sufficient conditions of Helgason [7] and Ludwig **[14]** in order that $g(\theta, t) = \mathscr{R}_\theta f(t)$ for some $f \in C_0^\infty(\hat{\Omega})$ are:

a) *$g \in C^\infty$ and $g(\theta, t) = 0$ for $t > \|\theta\|_\Omega$.*

b) $g(-\theta, -t) = g(\theta, t)$.

c) *The function*

$$p_m(\theta) = \int_{-\infty}^{\infty} t^m g(\theta, t)\, dt$$

*is a homogeneous polynomial of degree m.*

With the definition of $g$ in (5.14) these conditions are easily verified, so that $g(\theta, t) = \mathscr{R}_\theta f(t)$ for some $f \in C_0^\infty(\hat{\Omega})$.

With the above definition of $g$ it is also easily verified that

$$\Lambda^{n-2} \mathscr{R}_\theta f(\langle a_j, \theta \rangle) = \Lambda^{n-2} g(\theta, \langle a_j, \theta \rangle) = 0 \textit{ for all } \theta \in S^{n-1}.$$

Therefore, according to Theorem 4.7, $\mathscr{D}_{a_j} f(\theta) + \mathscr{D}_{a_j} f(-\theta) = 0$ for all $\theta$. However, since $a_j$ lies outside $\hat{\Omega}$, one of the two summands is automatically 0 for any given $\theta$, as the corresponding half line misses $\bar{\Omega}$, so the other is 0 too.

6. **The practical role of uniqueness theorems.** The ultimate practical objective of a study of the x-ray transform is to provide the means for the reconstruction of an unknown density function $f$ on a region of interest from the measured x-ray data taken from a finite number of sources. The Non-uniqueness Theorem 5.13 shows that this objective cannot be achieved exactly, even in theory. However, the Uniqueness Theorems 5.1 and 5.6, since they require only a countably infinite set of sources, show that if the unknown function $f$ cannot be reconstructed exactly, at least arbitrarily good approximations can be. This will now be discussed.

Let $A$ be a set of sources outside $\bar{\Omega}$. As in the last section, for each source $a$, let $C_a$ be the chosen cone of measured half lines, and let

$$S_a = (C_a - a) \cap S^{n-1} \tag{6.1}$$

be the corresponding set of directions. In this section and the next it suffices that $C_a$ be measurable with positive measure, but in sections 5 and 8 it is assumed that $C_a$ is open.

The set

$$N_A = \{u \in L_0^2(\Omega) \colon \mathscr{D}_a u(\theta) = 0 \textit{ for } a \in A \textit{ and } \theta \in S_a\} \tag{6.2}$$

is the null space, in the Hilbert space $L_0^2(\Omega)$, of the measured x-rays, and $f + N_A$ consists of all functions $h$ in this Hilbert space with the same measured x-rays as $f$. According to Theorem 3.14, $N_A$ is a closed subspace of $L_0^2(\Omega)$, and therefore $f + N_A$ is a closed plane. Let $P_A$ be the orthogonal projection operator in $L_0^2(\Omega)$ on the plane $f + N_A$. The following is a translation of a standard theorem in Hilbert space.

THEOREM 6.3. *If $\{A_k\}$ is an increasing sequence of sources with union $A$, then for each $g \in L_0^2(\Omega)$,* (6.4) $$P_{A_k}g \to P_A g \text{ in } L_0^2(\Omega).$$

In the next section a constructive method is given for approximating $P_{A_k}g$ whenever $A_k$ is a finite set outside $\bar{\Omega}$. According to the above theorem, this provides a constructive method for approximating $P_A g$ whenever $A$ is a countable set outside $\bar{\Omega}$.

Consider first the case where $A$ is any countably infinite set outside $\hat{\Omega}$ and $S_a = S^{n-1}$. Theorem 5.1 says that in this case $N_A = \{0\}$, so that $f + N_A = \{f\}$. Consequently,

(6.5) $$P_A g = f \text{ for every } g \in L_0^2(\Omega),$$

and we have a constructive method for approximating the unknown $f$ on $\Omega_m = \Omega$, starting with any $g \in L_0^2(\Omega)$.

Consider next the case where $A$ is a countably infinite set dense in a rectifiable arc satisfying the hypotheses of Theorem 5.6 with respect to the cone $C$, and $S_a = C \cap S^{n-1}$. Theorem 5.6 says that in this case

$$N_A = \{u \in L_0^2(\Omega) : u = 0 \text{ on } \Omega_m = (A + C) \cap \Omega\},$$

hence that

$$f + N_A = \{h \in L_0^2(\Omega) : h = f \text{ on } \Omega_m = (A + C) \cap \Omega\}.$$

Consequently,

(6.6) $$P_A g = f \text{ on } \Omega_m \text{ for every } g \in L_0^2(\Omega),$$

and we have a constructive method for approximating the unknown $f$ on the measured region $\Omega_m$, starting with any $g \in L_0^2(\Omega)$.

On the other hand, the infinite sequence $\{P_{A_k}g\}$ certainly cannot be computed. Some fixed $P_{A_k}$ must be used. In practice, partly on theoretical and partly on empirical grounds, a reasonable choice for $k$ can be determined. Nevertheless, no matter how large $k$ is, the non-uniqueness theorem shows that $P_{A_k}g$ can differ dramatically from the desired $f$. The practical meaning of the non-uniqueness theorem is that a priori information must be brought into play in order that the function in $f + N_{A_k}$ chosen by the method be a good approximation to $f$. This fact has become more widely recognized recently, and some examples of the use of a priori information in practice can be found in **[8]**, **[11]**, **[19]**, but much remains to be done.

7. **The Kacmarz method.** The purpose of this section is to give a constructive method for approximating the orthogonal projection $P_A$ on the plane $f + N_A$ when $A = \{a_1, \ldots, a_M\}$ is a finite set of sources outside $\bar{\Omega}$. There are two steps. The first is a constructive method for approximating $P_A$ by means of the projections $P_{a_j}$ corresponding to the individual sources. The second is an explicit formula for the individual projections $P_{a_j}$. The notations are those of sections 5 and 6.

The approximation of $P_A$ by means of the $P_{a_j}$ is taken care of by a theorem in abstract Hilbert space.

THEOREM 7.1. *Let $N_1, \cdots, N_M$ be closed planes in a Hilbert space $\mathscr{H}$ with nonempty intersection $N_0$. Let $P_j$ be the orthogonal projection on $N_j$, and let $Q$ be the product $P_M \cdots P_1$. Then*

i) *For every $g \in \mathscr{H}$, $Q^k g \to P_0 g$ in $\mathscr{H}$ as $k \to \infty$.*

ii) *If $\alpha_j$ is the angle between the plane $N_j$ and the intersection of the following ones, then*

$$\|Q^k g - P_0 g\|^2 \leqq c^k \|g - P_0 g\|^2, \tag{7.2}$$

$$\text{where } c \leqq 1 - \prod_{j=1}^{M-1} \sin^2 \alpha_j.$$

Part i) is a result of Halperin [5], and an elegant proof can also be found in Amemiya and Ando [1].

Part ii) is proved in [19]. In the 2 dimensional parallel beam case with equally spaced x-ray directions, the angles $\alpha_j$ have been computed explicitly, and it has been shown that when the error predicted by (7.2) is computed in a rather sophisticated way, this error is consistent with that which shows up in practice [6]. In the present divergent beam case nothing is known yet about the angles $\alpha_j$.

To derive the formula for an individual projection $P_a$, we set

$$\Omega_{a,\theta} = \begin{cases} \{t > 0 : a + t\theta \in \Omega\} & \text{if } \theta \in S_a \\ \text{empty set} & \text{if } \theta \notin S_a \end{cases} \tag{7.3}$$

$$\mu(a, \theta) = \int_{\Omega_{a,\theta}} t^{1-n}\, dt. \tag{7.4}$$

THEOREM 7.5. *If $h \in L_0^2(\Omega)$, then the orthogonal projection of $h$ on the subspace $N_a^\perp$ orthogonal to $N_a$ is the function $h_1$ defined by*

$$h_1(a + t\theta) = \begin{cases} c(a, \theta) t^{1-n} & \text{if } t \in \Omega_{a,\theta} \\ 0 & \text{if } t \notin \Omega_{a,\theta} \end{cases} \quad \text{where} \tag{7.6}$$

$$c(a, \theta) = \mathscr{D}_a h(\theta)/\mu(a, \theta). \tag{7.7}$$

PROOF. We show first that $h_1$ is square integrable, in which case $h_1 \in L_0^2(\Omega)$, for clearly $h_1$ vanishes outside $\Omega$. By the Cauchy-Schwarz inequality

$$|\mathscr{D}_a h(\theta)|^2 \leqq \mu(a, \theta) \int_0^\infty |h(a + t\theta)|^2\, t^{n-1}\, dt, \tag{7.8}$$

from which it follows that

$$|h_1(a + t\theta)|^2\, t^{n-1} \leqq (t^{1-n}/\mu(a, \theta)) \int_0^\infty |h(a + t\theta)|^2\, t^{n-1}\, dt,$$

and integration with respect to $t$ and $\theta$ shows that $h_1$ is square integrable.

Now, any function $h_1 \in L_0^2(\Omega)$ of the form (7.6) must lie in $N_a^\perp$, as s plain when the inner product with an arbitrary $u \in N_a$ is computed in polar coordinates centered at $a$. On the other hand, it is also plain that $\mathscr{D}_a h = \mathscr{D}_a h_1$, i.e., that $h - h_1 \in N_a$, if $c(a, \theta)$ is defined by (7.7).

COROLLARY 7.9. *$N_a^\perp$ consists of the functions $h_1 \in L_0^2(\Omega)$ of the form* (7.6).

THEOREM 7.10. *The orthogonal projection $P_a$ on $f + N_a$ is given by*

$$(7.11) \qquad P_a g(a + t\theta) = \begin{cases} g(a + t\theta) + \dfrac{\mathscr{D}_a f(\theta) - \mathscr{D}_a g(\theta)}{t^{n-1}\mu(a, \theta)} & \text{if } t \in \Omega_{a,\theta} \\ g(a + t\theta) & \text{if } t \notin \Omega_{a,\theta} \end{cases}$$

PROOF. If $g'$ is the function on the right in (7.11), it follows from Theorem 7.5 that $g - g'$ is the orthogonal projection of $g - f$ on $N_a^\perp$, and from this it follows that $g'$ is the orthogonal projection of $g$ on $f + N_a$.

REMARK 7.12. For reasons explained below, the combination of the explicit projection formula (7.11) with the iterative procedure of Theorem 7.1 is called the Kacmarz method. As an approximation to the unknown function $f$, it produces the orthogonal projection of an arbitrary initial guess $g \in L_0^2(\Omega)$ on the plane of functions with the same measured x-ray data as $f$. It would seem that a priori information about $f$ might be introduced into the initial guess $g$ so that, despite the non-uniqueness theorem, the projection of $g$ would necessarily be a good approximation to $f$. Normally, however, characteristics of $f$ which can be detected and introduced into the guess $g$ are also detected very quickly by the method, so that at best an iteration or two is saved, while the quality of the reconstruction remains the same. So far no one has found a basis for choosing a better guess than $g = 0$, in which case the Kacmarz method yields the function with minimum $L^2$ norm among all functions with the same measured x-ray data.

In the parallel beam case a fixed x-ray direction $\theta$ plays the role of the fixed divergent beam source. If $N_\theta'$ is the null space of $\mathscr{P}_\theta'$ and $P_\theta'$ is the orthogonal projection on the plane $f + N_\theta'$, then the analog of formula (7.11) is

$$(7.13) \quad P_\theta' g(a + t\theta) = g(a + t\theta) + \frac{\mathscr{P}_\theta f(a) - \mathscr{P}_\theta g(a)}{|\Omega_{a,\theta}| + |\Omega_{a,-\theta}|} \text{ for } a + t\theta \in \Omega.$$

The combination of this formula with the iterative procedure in Theorem 7.1 is called the Kacmarz procedure for the parallel beam transform. Again, it produces the orthogonal projection of an arbitrary initial guess on the plane of functions with the same measured x-ray data as the unknown function $f$.

In numerical implementations of the Kacmarz method, in order to combat the effects of the non-uniqueness theorems and of noise in the data, a priori information is often incorporated into the procedure. Usually this takes the form of periodic projection of the current guess on a closed convex set to which it is known a priori that the unknown function $f$ belongs, e.g. the set $h \in L_0^2(\Omega)$ satisfying $0 \leqq h(x) \leqq b$. The theoretical justification of the validity of periodic projections on closed convex sets is given by a theorem of Brègman [3]: *If, in Theorem* 7.1, *the* $N_j$ *are closed convex sets, then for every* $g \in \mathscr{H}$, $Q^k g$ *converges weakly to a point in* $N_0$. The conclusion here is of course much weaker than that of Theorem 7.1. Not only is the convergence weak instead of strong, but nothing is known about the limit point, except that it is lies in $N_0$ — and even in the Hilbert space $R^2$ simple examples show that the limit point depends upon the order in which the $N_j$ are used. The incorporation of a priori information, both of this kind and of others, into the reconstruction method seems to be a practical necessity, and it remains a problem on which much is yet to be done [**8**], [**11**], [**19**].

REMARK 7.14. Consider a system of $M$ linear equations in $N$ unknowns

$$L_j(u) = \langle u, c_j \rangle = v_j, \; j = 1, \ldots, M. \tag{7.15}$$

If $f$ is a solution, $N_j$ is the null space of $L_j$, and $P_j$ is the orthogonal projection on $f + N_j$, then [**12**]

$$P_j g = g + \left( \frac{L_j(f) - L_j(g)}{|c_j|^2} \right) c_j. \tag{7.16}$$

S. Kacmarz seems to have been the first to use this projection formula in combination with the iterative procedure of Theorem 7.1 (the convergence being trivial in this finite dimensional case) in the solution of such systems of equations. The solution produced is the orthogonal projection of the arbitrary initial guess on the plane of solutions.

The first appearance of the Kacmarz method in x-ray work, or at least its first practical appearance, was in the original EMI scanner of G. N. Hounsfield [**10**]. The scanner used a parallel beam and used the formula (7.13), but probably Hounsfield had in mind a physical or geometrical interpretation of the formula, rather than the interpretation as a projection in Hilbert space. Indeed, with $a$ and $\theta$ fixed, (7.13) simply calls for correction of the guess $g$ by the addition of a constant $c(a, \theta)$ along the line $a + t\theta$, the constant $c(a, \theta)$ being chosen so that the integral along this line gives the measured value. This version of the Kacmarz method for use with parallel beam x-rays was also discovered independently by Gordon, Bender, and Herman [**4**].

The divergent beam Kacmarz method can be given a similar, but much less intuitive, physical or geometric interpretation. With $a$ and $\theta$ fixed

(7.11) calls for correction of the guess $g$ by the addition of the function $c(a, \theta)t^{1-n}$ along the half line $a + t\theta$, the constant $c(a, \theta)$ being chosen so that the integral along this half line gives the measured value. The role of the weight factor $t^{1-n}$ can be seen by by imagining the correction that would be called for by the presence of a small region of high density. The correction is made throughout the cone generated by the region and the source. Even though the region is small, if the source is close, then the cone is large. The factor $t^{1-n}$ tempers the correction in the region where it is not wanted.

REMARK 7.17. Recently there has been interest in the development of a 3 dimensional analog of the 2 dimensional convolution method of reconstruction. The true analog is given by the inversion formula (3.8), but it requires sources distributed uniformly around a sphere. This is impractical in most 3 dimensional problems, and what is desired is a non-iterative method making use of sources on an arc. Such a method is contained implicitly in the proof of the Uniqueness Theorem 5.6, for the formulas (5.11) and (5.12) allow the computation of all the moments of the function $f(a + t\theta)$ from the measured x-ray data. Whether these formulas can be implemented in a practical way, however, we have no idea. Even in dimension 2 there is no known analog of the divergent beam convolution method which allows either a) the sources to be confined to any arc substantially smaller than a full circle surrounding the object; or b) the x-ray directions to be confined to a proper subset of the circle — both of which are allowed in the procedure of Theorem 5.6.

8. **Regions measured from multiple sources.** The purpose of this section is to describe the measured regions in which each point is seen from multiple sources. The discussion is confined to dimension 3, although the 2 dimensional case is also covered by Theorem 8.5. For technical reasons the measured cones $C_a$ are assumed to be open, and the source set $A$ is assumed to lie outside $\bar{\Omega}$.

Since the measured region $\Omega_m$ is the union of the sets $C_a \cap \Omega$, it follows that

$$C_a \cap \Omega \subset \Omega_m \quad \textit{for each } a \in A. \tag{8.1}$$

If all of $\Omega_m$ is seen from the source $a$, i.e. if $C_a \supset \Omega_m$, then (8.1) becomes

$$C_a \cap \Omega = \Omega_m \textit{ if all of } \Omega_m \textit{ is seen from } a. \tag{8.2}$$

Most of our proofs simply involve an examination of the consequences of (8.1) and (8.2) as they bear on the set

$$\Omega' = \partial\Omega_m \cap \Omega, \tag{8.3}$$

which is the part of the boundary of $\Omega_m$ lying in $\Omega$. From (8.2) it follows that

$$\partial C_a \cap \Omega = \Omega' \textit{ if all of } \Omega_m \textit{ is seen from } a, \tag{8.4}$$

simply because $C_a$ and $\Omega$ are open sets.

If $\pi$ is a plane and $S$ is a subset of $\pi$, we write $\partial_\pi S$ for the boundary of $S$ relative to $\pi$, i.e., the closure of $S$ (which automatically is contained in $\pi$) minus the relative interior.

THEOREM 8.5. *If each point of $\Omega_m$ is seen from the same two sources $a$ and $b$, and $\pi$ is any plane containing $a$ and $b$, then the set $\partial_\pi(\Omega_m \cap \pi) \cap \Omega$ lies on the line $ab$.*

PROOF. From (8.2) it follows that

$$(C_a \cap \pi) \cap \Omega = \Omega_m \cap \pi, \tag{8.6}$$

and from this it follows that

$$\partial_\pi(C_a \cap \pi) \cap \Omega = \partial_\pi(\Omega_m \cap \pi) \cap \Omega, \tag{8.7}$$

simply because $C_a \cap \pi$ and $\Omega \cap \pi$ are relatively open subsets of $\pi$. Now, let $p$ be any point in $\partial_\pi(\Omega_m \cap \pi) \cap \Omega$. The half line $ap$ meets $\Omega$ in a union of disjoint open segments. Let $\sigma$ be the one containing $p$. By (8.7), $p \in \partial_\pi(C_a \cap \pi)$, hence $ap \subset \partial_\pi(C_a \cap \pi)$, hence, by (8.7) again, $\sigma \subset \partial_\pi(\Omega_m \cap \pi) \cap \Omega$. The same argument, using the source $b$ instead of $a$, shows that the sector $b\sigma \subset \partial_\pi(C_b \cap \pi)$, and hence, by (8.7) again, that

$$b\sigma \cap \Omega \subset \partial_\pi(\Omega_m \cap \pi). \tag{8.8}$$

If $p$ is not on the line $ab$, then the sector $b\sigma$ is a 2 dimensional sector in the plane $\pi$, and the set $b\sigma \cap \Omega$ is a 2 dimensional neighborhood of $p$ in the plane $\pi$, and formula (8.8) asserts that there are no points of $\Omega_m \cap \pi$ close to the boundary point $p$. This is a contradiction, and the theorem is proved.

THEOREM 8.9. *Suppose that each point of $\Omega_m$ is seen from the same two sources $a$ and $b$, and that the line $ab$ does not meet $\bar{\Omega}$. Any subset of $\Omega'$ which is both connected and locally connected must lie on a plane through $a$ and $b$.*

PROOF. Let $\Gamma$ be a subset of $\Omega'$ which is both connected and locally connected, and let $\pi_0$ be any plane through $a$ and $b$ which intersects $\Gamma$. It will be shown that $\pi_0 \cap \Gamma$ is open in $\Gamma$. Since $\pi_0 \cap \Gamma$ is also closed in $\Gamma$, it will follow that $\pi_0 \cap \Gamma = \Gamma$, and hence that $\Gamma \subset \pi_0$.

Let $p_0$ be any point of $\pi_0 \cap \Gamma$, choose $r_0$ so that the ball $B(p_0, r_0)$ is contained in $\Omega$, and let $U$ be a neighborhood of $p_0$ contained in $B(p_0, r_0/2)$ such that $U \cap \Gamma$ is connected. It will be shown that $U \cap \Gamma \subset \pi_0$, which will show that $\pi_0 \cap \Gamma$ is open in $\Gamma$.

Suppose there is a point $p_1 \in U \cap \Gamma$ which is not on $\pi_0$, and let $\pi_1$ be the plane $abp_1$. If $\pi$ is any plane through $ab$ "between" $\pi_0$ and $\pi_1$ (i.e., separating $p_0$ and $p_1$), then $\pi$ must contain points of $U \cap \Gamma$, for otherwise $\pi$ would separate $U \cap \Gamma$. Let $p_\pi$ be such a point, and let $\sigma_\pi$ be the segment

$$\sigma_\pi = ap_\pi \cap B(p_0, r_0),\ p_\pi \in U \cap \Gamma \cap \pi \subset B(p_0, r_0/2).$$

Since $p_\pi \in \Omega'$, it follows from (8.4) that $p_\pi \in \partial C_a$, hence that $ap_\pi \subset \partial C_a$, and, from (8.4) again, that $\sigma_\pi \subset \Omega'$. Now, in the same way, it follows that the sector $b\sigma_\pi \subset \partial C_b$, and hence that

$$b\sigma_\pi \cap B(p_0, r_0) \subset \Omega'.$$

This is a contradiction, for, as $\pi$ runs through all planes "between" $\pi_0$ and $\pi_1$, the sets $b\sigma_\pi \cap B(p_0, r_0)$ fill an open set in $R^3$, while the boundary $\Omega'$ cannot contain an open set. Thus, $U \cap \Gamma \subset \pi_0$, and the proof is complete.

THEOREM 8.10. *If each point of $\Omega_m$ is seen from the same three non-collinear sources a, b, and c, then*

a) *$\Omega'$ is contained in the plane abc.*

b) *If H is either of the open half spaces determined by the plane abc, then $\Omega_m \cap H$ is a union of connected components of $\Omega \cap H$. In particular, $\Omega_m \cap H = \Omega \cap H$ when the latter is connected.*

COROLLARY 8.11. *If each point of $\Omega_m$ is seen from the same four non-coplanar sources, then $\Omega_m$ is a union of connected components of $\Omega$. In particular, $\Omega_m = \Omega$ when the latter is connected.*

PROOF. Let $p \in \Omega'$, and assume, contrary to a), that $p$ is not in the plane $abc$. As in the proofs of Theorems 8.5 and 8.9, we see that $U = b\sigma \cap \Omega$ is a 2 dimensional neighborhood of $p$ in the plane $abp$ which is contained in $\Omega'$. Hence, by (8.4), the cone $cU$ is contained in $\partial C_c$, and, again by (8.4), the open set $cU \cap \Omega$ is contained in $\Omega'$. This is a contradiction, so part a) of the theorem is proved. Part a) shows that $\Omega_m \cap H$ is both open and closed in $\Omega \cap H$, which proves part b).

In the case of the four non-coplanar sources of the corollary, part a) shows that $\Omega'$ must be contained in each of the four planes determined by the faces of the corresponding tetrahedron. Since these planes have no common point, $\Omega'$ must be empty, so $\Omega_m$ must be both open and closed in $\Omega$, which proves the corollary.

Until now we have been considering the case where each point of $\Omega_m$ is seen from the *same* two, three, or four sources, and have seen that the limitations on $\Omega_m$ are severe. Now we take up the case where each point is seen by two, three, or four sources, but not necessarily the same ones. In practical cases the limitations are equally severe.

DEFINITION 8.12. *We say that $\Omega'$ has a tangent plane at the point $p \in \Omega'$ if there is a neighborhood $U$ of $p$ such that*

$$\Omega_m \cap U = \{x \in U: F(x) > 0\}, \tag{8.13}$$

*where $F$ is continuous on $U$ and differentiable at $p$ and $\nabla F(p) \neq 0$. The tangent plane is then the plane*

$$\pi_p = p + \nabla F(p)^{\perp}. \tag{8.14}$$

*We say that $\Omega'$ is smooth at $p$ if $F$ can be taken of class $C^1$ on $U$. We write $\Omega_t'$ for the set of points where the tangent plane exists and $\Omega_s'$ for the set of smooth points.*

It is clear that $\Omega_s' \subset \Omega_t' \subset \Omega'$ and that $\Omega_s'$ is an open subset of $\Omega'$.

If $p \in \Omega'$, and if the coordinates in $R^n$ are chosen so that $p = 0$ and so that the tangent plane $\pi_p$ is the plane $x_n = 0$, then according to the implicit function theorem, $U$, $\delta$, and $\varepsilon$ can be chosen so that

$$\Omega_m \cap U = \{x: |x'| < \delta \text{ and } f(x') < x_n < \varepsilon\}, \tag{8.15}$$

where $x = (x', x_n)$ and $f$ is continuous for $|x'| < \delta$ and differentiable at 0 with $f(0) = 0$ and $\nabla f(0) = 0$. If $p \in \Omega_s'$, then $f$ is $C^1$ on $|x'| < \delta$.

LEMMA 8.16. *If $p \in \Omega_t' \cap \bar{C}_a$, then $a \in \pi_p$.*

PROOF. As in the earlier proofs, the half line $ap$ meets $\Omega$ in a union of disjoint open segments, and we take $\sigma$ to be the one containing $p$. Since $\sigma$ lies in $\bar{C}_a$, it follows from (8.1) that $\sigma$ lies in $\bar{\Omega}_m$. It is easily seen from (8.15) that this forces $\sigma$ to lie in the tangent plane $\pi_p$, and hence the whole line $ap$ to lie in $\pi_p$.

Henceforth it is assumed that the number of sources is *finite*.

LEMMA 8.17. *Suppose that the set $A$ of sources is finite. If each point of $\Omega_m$ is seen from at least $k$ sources, then for each point $p \in \Omega'$ there are at least $k$ sources $a$ for which $p \in \bar{C}_a$.*

The proof is obvious.

THEOREM 8.18. (COLLINEAR SOURCES). *Suppose that each point of $\Omega_m$ is seen from at least two among a finite number of sources on a line $\ell$ which does not meet $\bar{\Omega}$. Then each connected component of $\Omega_s'$ lies in a plane containing $\ell$.*

PROOF. Notice first that according to Lemmas 8.16 and 8.17 every tangent plane to $\Omega_s'$ must contain the line $\ell$. Let $\Gamma$ be a component of $\Omega_s'$, and let $\pi$ be the tangent plane at any point. The set $\pi \cap \Gamma$ is non-empty and closed in $\Gamma$, and we shall show that it is also open in $\Gamma$.

If $p$ is any point in $\pi \cap \Gamma$, then $\pi_p = \pi$, for both planes contain the line

$\ell$ and the point $p \notin \ell$. For simplicity of notation we write $(x, y, z)$ in place of $(x_1, x_2, x_3)$, and we choose the coordinates as in (8.15) so that $p = 0$ and $\pi_p = \pi$ is the plane $z = 0$. In the $x, y$ plane we choose the coordinates so that $\ell$ is the line $x = c$, $z = 0$. If $q = (x, y, f(x, y))$ is a point near $p$ which is in $\bar{C}_a \cap \bar{C}_b$ (in accordance with Lemma 8.17), then by Lemma 8.16, $(\partial f/\partial x, \partial f/\partial y, -1)$ is orthogonal to both $a - q$ and $b - q$, and hence to $a - b$. By virtue of the choice of coordinates this means that $\partial f/\partial y = 0$, and hence that $f$ is a function of $x$ alone, and we write $f(x)$ instead of $f(x, y)$.

Since the tangent plane at any point must contain the line $\ell$, it follows that $f(x) = (x - c)f'(x)$, and hence that $f(x) = k(x - c)$. Since $f'(0) = 0$, $f$ must be 0, so in a neighborhood of $p$, $\Omega'$ is contained in $\pi$. This shows that $\pi \cap \Gamma$ is both open and closed in $\Gamma$, hence equal to $\Gamma$, and therefore that $\Gamma \subset \pi$.

COROLLARY 8.19. (COLLINEAR SOURCES). *Along with the conditions of the theorem, suppose also that $\Omega'_s$ is dense in $\Omega'$ and that $\Omega'_s$ has only a finite number of components. If $W$ is one of the wedges bounded by two adjacent planes corresponding to components of $\Omega'_s$, then $\Omega_m \cap W$ is a union of components of $\Omega \cap W$. In particular, $\Omega_m \cap W = \Omega \cap W$ when the latter is connected.*

PROOF. With the conditions of the corollary, $\Omega'$ is contained in the same finite union of planes as $\Omega'_s$, so that $\Omega_m \cap W$ is both open and closed in $\Omega \cap W$.

THEOREM 8.20. (COPLANAR SOURCES). *Suppose that each point of $\Omega_m$ is seen from three non-collinear sources among a finite number of sources on a plane $\pi$. Then $\Omega'_t \subset \pi$. If $\Omega'_t$ is dense in $\Omega'$ and $H$ is either of the two open half spaces determined by $\pi$, then $\Omega_m \cap H$ is a union of components of $\Omega \cap H$; in particular, $\Omega_m \cap H = \Omega \cap H$ when the latter is connected.*

PROOF. By a slight extension of Lemma 8.17, if $p \in \Omega'$, then $p \in \bar{C}_a$ for three non-collinear sources $a$. Therefore, by Lemma 8.16, if $p \in \Omega'_t$, then $p \in \pi$. If $\Omega'_t$ is dense in $\Omega'$, it follows that $\Omega' \subset \pi$, and hence that $\Omega_m \cap H$ is both open and closed in $\Omega \cap H$.

THEOREM 8.21. (STRONGLY NON-COPLANAR SOURCES). *Suppose that each point of $\Omega_m$ is seen from four non-coplanar sources among a finite number of sources altogether. Then $\Omega'_t$ is empty. If $\Omega'_t$ is dense in $\Omega'$, then $\Omega_m$ is a union of components of $\Omega$; in particular, $\Omega_m = \Omega$ when the latter is connected.*

PROOF. Again by a slight extension of Lemma 8.17, if $p \in \Omega'$, then $p \in \bar{C}_a$ for four non-coplanar sources $a$. Therefore, by Lemma 8.16, if $p \in \Omega'_t$, then $p$ lies in each of the planes determined by the faces of the corresponding tetrahedron. Since these planes have no common point, $\Omega_t'$ is empty. If $\Omega_t'$ is dense in $\Omega'$, then $\Omega'$ is also empty, and $\Omega_m$ is both open and closed in $\Omega$.

9. **Examples.** This section contains some examples of x-ray setups that may be of use in 3 dimensional practice. Two criteria are considered:

i) *The setup is consistent with the uniqueness theorems in the sense that an infinite number of sources will guarantee uniqueness, so that a sufficiently large finite number, if used correctly, will provide a good approximation.*

ii) *Each point of the measured region $\Omega_m$ is seen from multiple sources.*

In practice the number of sources is finite, the region $\Omega$ is a simple region containing the actual physical object, and the measured cones $C_a$ are simple objects, too, with smooth boundaries except perhaps along a finite number of edges. Thus, the various smoothness and connectedness hypotheses in the last section are satisfied.

a) COLLINEAR SOURCES. If the sources are to lie on a line $\ell$ which does not meet $\bar{\Omega}$, then the only realistic setup meeting ii) is that where each $C_a$ is a fixed connected wedge $W$ with edge $\ell$. According to Corollary 8.19, any setup would have to be effectively a union of these. In this case each point of $\Omega_m$ is seen by all of the sources. If $\pi$ is any plane containing $\ell$, the Uniqueness Theorem 5.1 can be applied to the section $\Omega \cap \pi$ to show that uniqueness does hold with any infinite number of sources on $\ell$. Thus, this setup does satisfy both criteria i) and ii). Reconstruction can be performed either by the 3 dimensional Kacmarz method of section 7 or by the use of any of the known 2 dimensional methods within the sections $\Omega \cap \pi$.

With sources on a line, however, it seems likely that the angles in Theorem 7.1 will be rather small and that the problem will be rather "ill-conditioned" whatever reconstruction method is used.

b) COPLANAR SOURCES. Suppose that the sources lie in a plane $\pi$. Unless it is feasible to measure the entire part of $\Omega$ in one of the two half spaces determined by $\pi$ (which is not often the case), Theorem 8.20 shows that criterion ii) can be met only in ways that are basically collinear. For example, the sources might be taken on a polygonal arc with each segment of the arc treated as in the collinear case. This would give an acceptable setup in terms of the criteria i) and ii). An important practical question would be whether the sources can be ordered so as to produce reasonable angles in Theorem 7.1. If not, the problem may be rather ill-conditioned no matter what reconstruction method is used.

c) STRONGLY NON-COPLANAR SOURCES. Probably the most efficient 3 dimensional reconstruction setups, those producing the largest angles in Theorem 7.1 and the best conditioning in the problem, will require strongly non-coplanar sources. However, Theorem 8.21 shows that in this case criterion ii) can be met only when all of $\Omega$ can be measured.

d) MIXED SOURCES. A practical way to meet the criteria i) and ii), and at the same time produce both a reasonable measured region and a problem that is reasonably conditioned, may be to use sources that are not coplanar, but not strongly non-coplanar either. The following is an example. (See Figure 2.)

In the typical 3 dimensional reconstruction problem $\Omega$ is a cylinder

$$\Omega = D \times (-c, c)$$

where $D$ is the unit disk in the $x$, $y$ plane, and the reconstruction is desired on a part

$$\Omega_0 = D \times (-b, b), b < c.$$

Take $S$ to be the set of directions in the upper half space $R^3_+$ making an angle $< \alpha = \arctan(b/2)$ with the $x$, $y$ plane, and let $C$ be the cone with vertex 0 generated by $S$. To begin with, let the sources set $A_1$ be the circle with center 0 and radius 3 in the plane $z = -b$, and, for each $a_1 \in A_1$, let $C_{a_1} = a_1 + C$. Then, as in Theorem 5.6, the measured region is

$$\Omega_m = (A_1 + C) \cap \Omega.$$

In the picture below, $\Omega_m$ is obtained by revolving the region shaded with solid lines around the $z$ axis.

This is a situation where Theorem 5.6 is applicable, so uniqueness holds on the measured region $\Omega_m$, and criterion i) is satisfied. Note that $\Omega_m$ is contained in the desired region $\Omega_0$ and that it includes all of this region on or below the plane $z = 0$. Indeed, each point in the latter set is seen from every source.

With a finite number of sources criterion ii) is not satisfied. Indeed, with a finite number of sources each point of $\Omega'$, except those where there is no tangent plane, has a neighborhood in which the points are seen from only one source.

Now, however, adjoin a second source set $A_2$, the circle with center 0 and radius 3 in the plane $z = b$, taking, for each $a_2 \in A_2$, $C_{a_2} = a_2 - C$. With $A = A_1 \cup A_2$ as the source set, the measured region becomes precisely the desired region $\Omega_0$, uniqueness holds, since it holds on the two parts individually, and every point in $\Omega_0$ is seen either from all sources in $A_1$ or from all sources in $A_2$. With a finite number of sources, half from $A_1$ and half from $A_2$, each point of the measured region $\Omega_0$ is seen from at least half of the sources.

With this setup both the criteria i) and ii) are well satisfied. (The same effect is achieved by taking $A_1$ and $A_2$ to have any radius $r \geqq 3$ and any $\alpha$ satisfying $b/(r - 1) \leqq \tan \alpha \leqq 2b/(r + 1)$.)

In the interest of economy it may be advantageous, particularly in

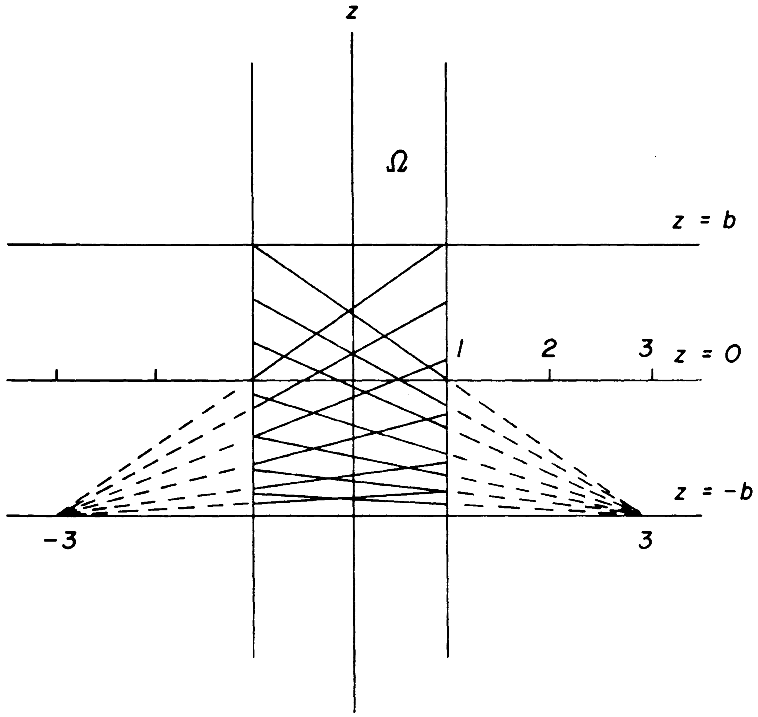


Figure 2

relatively high contrast problems, to try to reconstruct from data which are insufficient to provide uniqueness, even in the limiting case of infinitely many sources. Figure 3 below shows a simulation of a chest cross section with the x-ray setup suggested in [9]. $\Omega$ is a circle of radius $r$ containing the full cross section, and $\Omega_0$ is a concentric circle of radius $r_0$ containing the heart, which is the region of interest. The attenuation is measured only along rays which meet the intermediate concentric circle $\Omega_1$ of radius $r_1$. In this 2 dimensional situation the issue is not that there is difficulty in measuring the full beam, but rather that it is desirable to reduce the reconstruction computations by making use of only a part. In order to isolate the effect of the nonuniqueness inherent in the use of a partial beam of this kind, we assume that the source set $A$ is an infinite set dense in a circle (or any other curve) surrounding $\Omega$.

In this case the measured region is the whole of $\Omega$. Points inside $\Omega_1$ are seen from all sources. A point outside $\Omega_1$ is seen from all sources on the two arcs cut off by the two tangent lines from the point to $\Omega_1$. Thus all

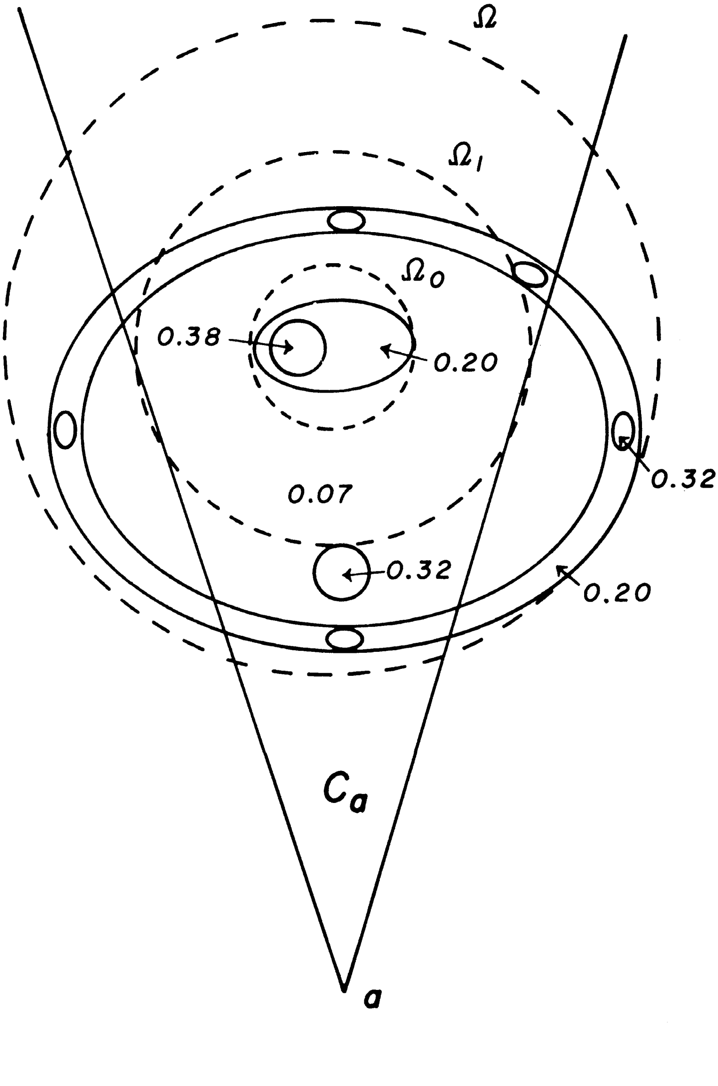


Figure 3. Simulated chest cross section showing the heart with the left chamber filled with 10% contrast dye, the spine, the chest wall, and the ribs. The indicated x-ray attenuation coefficients are taken from [23]. $\Omega$ is a circle containing the entire cross section, and $\Omega_0$ is a concentric circle containing the heart, which is the region of interest. The attenuation is measured only along rays which meet the concentric circle $\Omega_1$ between the two.

points are seen from infinitely many sources (and with finitely many sources, all points are seen from a large fraction).

The null space $N_A$, however, is large. A function lies in the null space if and only if its line integrals vanish for all lines meeting $\Omega_1$, i.e., if and only if its Radon transform vanishes for $|t| < r_1$ and $|t| > r$. If $g(\theta, t)$ is any function which satisfies the consistency conditions of Helgason and Ludwig (they are listed in the proof of Theorem 5.6) and which vanishes for $|t| < r_1$ and $|t| > r$, the Radon inversion formula (see [19]),

$$u(x) = \frac{1}{4\pi} \Lambda \int_{S^1} g(\theta, \langle x, \theta \rangle)\, d\theta, \tag{9.1}$$

provides a function $u \in N_A$. In particular, if $\rho$ is a function of one variable defined for $t > 0$, and

$$g(\theta, t) = g(\theta, -t) = \rho(t)$$

the consistency conditions are automatically satisfied, and the formula (9.1) simplifies. The result is as follows.

*If $\rho(t) = 0$ for $t < r_1$ and $t > r$, then*

$$u(x) = \int_{|x|}^{\infty} \rho'(s)\, (s^2 - |x|^2)^{-1/2}\, ds \tag{9.2}$$

*lies in the null space $N_A$.*

Obviously, therefore, the lack of uniqueness is large.

On the other hand, if the functions $u \in N_A$ do not vary much on the region $\Omega_0$, then the lack of uniqueness does little harm to the reconstruction. If $g(\theta, t) = \mathscr{R}_\theta u(t)$ vanishes for $|t| < r_1$ and $|t| > r$, formula (9.1) gives

$$u(x) = \frac{1}{4\pi^2} \int_{S^1} \int_{r_1 \leqq |s| \leqq r} g(\theta, s)\, (s - \langle x, \theta \rangle)^{-2}\, ds\, d\theta \quad \textit{for } |x| \leqq r_0. \tag{9.3}$$

With this and the Cauchy-Schwarz inequality it is easy to establish that

$$\begin{aligned} &|u(x) - u(0)| \leqq C_1 \|g\|_{L^2(S^1 \times R^1)} \quad \textit{for } |x| \leqq r_0, \\ &C_1^2 = \frac{1}{8\pi^4} \int_0^{\pi/2} \int_{r_1}^{r} \left( \frac{1}{S^2} - \frac{1}{(S + r_0 \cos\varphi)^2} \right)^2 \\ &\qquad + \left( \frac{1}{(S - r_0 \cos\varphi)^2} - \frac{1}{S^2} \right)^2 ds\, d\varphi. \end{aligned} \tag{9.4}$$

It is also easy to establish that

$$\begin{aligned} &|u(x) - u(0)| \leqq C_2 \int_{S^1} \sup_s |g(\theta, s)|\, d\theta \quad \textit{for } |x| \leqq r_0, \\ &C_2 = \frac{r_0}{2\pi^2} \left[ \frac{1}{r_1^2 - r_0^2} - \frac{1}{r^2 - r_0^2} \right]. \end{aligned} \tag{9.5}$$

Since

$$\|g\|_{L^2(S^1\times R^1)} \leqq 2\sqrt{\pi}(r^2 - r_1^2)^{1/4}\,\|u\|_{L^2(R^2)}, \tag{9.6}$$

we also have

$$\begin{gathered}|u(x) - u(0)| \leqq C_3\|u\|_{L^2(R^2)} \textit{ for } |x| \leqq r_0,\\ C_3 = 2\sqrt{\pi}(r^2 - r_1^2)^{1/4}\, C_1.\end{gathered} \tag{9.7}$$

With reasonable choices of the radii the constants $C_1$, $C_2$, and $C_3$ are quite small. For example, with $r_0 = 4.7$ and $r = 19.5$ (see the chest phantom dimensions below) they are as follows.

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $r_1 = 11$ | .000634 | .001742 | .00902 |
| $r_1 = 12$ | .000477 | .001288 | .00663 |
| $r_1 = 13$ | .000367 | .000956 | .00496 |

The constant $C_1$ was obtained by numerical integration in (9.4) by Mr. Al Chu of the Mayo Clinic.

Now, let $f$ be the unknown density function, and let $f_0$ be the solution produced by the reconstruction, so that $u = f - f_0$. There are various possibilities for completing the estimate of $|u(x) - u(0)|$. On the one hand, $\mathscr{R}_\theta f$ can be computed from the x-ray data, provided the full beam is measured, even though only part is used, and of course $\mathscr{R}_\theta f_0$ can be computed. Thus, in any given case the norm and integral on the right of (9.4) and (9.5) can be computed explicitly.

On the other hand, if the Kacmarz reconstruction method is used with the usual initial guess 0, then $f_0$ is the orthogonal projection of 0 on the plane $f + N_A$, so $u$ is the orthogonal projection of $f$ on $N_A$, and hence

$$\|u\|_{L^2(R^2)} \leqq \|f\|_{L^2(R^2)}.$$

Consequently,

$$|u(x) - u(0)| \leqq C_3\|f\|_{L^2(R^2)} \textit{ for } |x| \leqq r_0. \tag{9.8}$$

In the case of the chest phantom of Figure 3 the features are all represented by ellipses with axes (in cm.) as follows:

| | | | |
|---|---|---|---|
| outer ellipse | 25.8 by 34 | chamber | 3 by 3 |
| inner ellipse | 22 by 30 | spine | 3 by 3 |
| heart | 5.2 by 9.4 | ribs | 1.2 by 2.6 |

The $L^2$ norm turns out to be 3.6, and the evaluations of $|u(x) - u(0)|$ given by (9.8) are as follows.

$$
\begin{array}{lcc}
 & & |u(x) - u(0)| \\
 & r_1 = 11 & .032 \\
(9.9) & r_1 = 12 & .024 \\
 & r_1 = 13 & .018
\end{array}
$$

The density differences sought in this problem are those of heart muscle vs. dye filled chamber and heart muscle vs. air, which are .18 and .13 respectively.

The estimate (9.8) is appealing because it is almost independent of the patient. Once the dimensions are roughly fixed the $L^2$ norm will change very little from one patient to the next. Consequently, the estimates in (9.9) are almost absolute.

On the other hand, much tighter estimates probably result from (9.4) and (9.5). It seems likely that a great deal is lost in the inequality (9.6).

REMARK 9.10. In the above discussion it is only assumed that the density function $f$ vanishes outside $\Omega$. In the example of Figure 3 $f$ actually vanishes outside the large ellipse, and this additional constraint is sufficient to provide uniqueness, by virtue of Theorem 5.1. (The measured data contain the full beam data for all sources $a$ in $\Omega_1$ and outside the ellipse.) Some reconstruction methods can make use of such a constraint, while others cannot.

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