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ON THE TOPOLOGICAL TRIVIALITY OF SOLUTION SETS

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To N. Aronszajn, on the occasion of his seventieth birthday

1. **Introduction.** Consider the initial value problem

$$x' = f(x, t),\ x(0) = 0, \tag{1}$$

where $x \in R^n$, $t \in I = [0, T]$, and f is bounded and continuous on $R^n \times I$. Aronszajn [1] has proved that the set S of all solutions of (1) is an R_δ-set in the space $C[I]$ of continuous functions from I into R^n. (Recall that an R_δ-set is defined to be the intersection of a decreasing sequence of compact absolute retracts.) It follows that, although solutions of (1) are not unique, the set of all such solutions is topologically equivalent to a point. Also, the theorem of Kneser that $\{x(T)|x \in S\}$ is connected follows easily from Aronszajn's theorem.

The purpose of this note is to give a new and more elementary proof of Aronszajn's result, and to make some progress on obtaining a similar result for the sets of solutions of the differential inclusion

$$x' \in F(x, t),\ x(0) = 0. \tag{2}$$

In (2), F is a set valued function whose values are compact convex subsets of R^n; it is assumed that all the values of F are contained in some ball in R^n, and that F is a continuous function from $R^n \times I$ to the space of all compact subsets of R^n topologized by the Hausdorff metric. Aronszajn's proof does not work in the latter situation, because it depends on an elegant fixed point theorem which appears to have no suitable counterpart for set valued functions.

Our approach is to approximate (1) by a control problem

$$x'(t) = f_n(x, t) + u(t),\ x(0) = 0, \tag{3}$$

where f_n is Lipschitzian and u belongs to a suitably restricted set U_n of control functions. U_n is chosen so that the set

$$S_n = \{x: I \to R^n \mid x \text{ solves (3) for some } u \in U_n\}$$

is a compact absolute retract containing S and so that for all $\varepsilon > 0$, almost all S_n are contained in the ε-neighborhood $N_\varepsilon(S)$ of S. It then will follow

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from another theorem of Aronszajn that S is an R_δ-set. The details are in §2.

It appears likely that this approach will yield the same result for the solution set of (2), though some formidable technical difficulties remain to be overcome. We have so far been able to show that the solution set of (2) is an R_δ only in case $R^n = R^1$. This is done in §3.

2. **The Solution Set of (1) is an R_δ.** To simplify the exposition we will work only with the autonomous version of (1):

$$x' = f(x), \; x(0) = 0. \tag{1a}$$

We assume f is continuous on R^n and that $|f(x)| \leqq M$ for all $x \in R^n$. With S now denoting the set of all solutions of (1a), we have, for all $t \in I = [0, T]$ and all $x \in S$, that $|x'(t)| = |f(x(t))| \leqq M$. It follows that $|x(t)| \leqq MT$ for all $t \in I$, and so we may restrict the domain of f in (1a) to the closed ball B_{MT} of radius MT. We are going to prove that S is an R_δ-set. The proof of the same result for the non-autonomous case (1) involves no essential change in the argument.

PROPOSITION 1. *S is a compact subset of $C[T]$, and $S' = \{x' \mid x \in S\}$ is bounded (by M) and equicontinuous.*

PROOF. It is well known that S is compact, and the boundedness of S' was established in the paragraph preceding the proposition. To prove S' is equicontinuous, let $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in B_{MT}$ and $|x - y| < \delta$. Clearly $|x(t) - x(s)| \leqq M|t - s|$ for all $s, t \in I$. Hence

$$\begin{aligned} |t - s| < \delta/M &\Rightarrow |x(t) - x(s)| < \delta \\ &\Rightarrow |x'(t) - x'(s)| = |f(x(t)) - f(x(s))| < \varepsilon. \end{aligned}$$

Now let $f_n : R^n \to R^n$ be a Lipschitzian function bounded by M such that $|f_n(x) - f(x)| \leqq 1/n$ for all $x \in B_{MT}$. Then define a set U_n of control functions $u: I \to R^n$ by

$$U_n = \overline{\text{co}}\, A_n,$$

where

$$A_n = \{u \mid \text{for some } x \in S,\ u(t) = f(x(t)) - f_n(x(t)) \text{ on } I\}.$$

PROPOSITION 2. i) *$|u(t)| \leqq 1/n$ for all $u \in U_n$, $t \in I$;*
ii) *U_n is a compact convex subset of $C[I]$;*
iii) *For each $x \in S$ there exists $u \in U_n$ such that x is a solution of $x'(t) = f_n(x(t)) + u(t)$ on I.*

PROOF. (i) and (iii) are obvious. To prove (ii), it is sufficient (since U_n is bounded, by (i)) to prove A_n is equicontinuous. So let $\varepsilon > 0$ and choose

$\delta > 0$ such that $|t - s| < \delta \Rightarrow |x'(t) - x'(s)| < \varepsilon$ for all $x \in S$. Suppose $|f_n(x) - f_n(y)| \leqq k_n |x - y|$ for all $x, y \in B_{MT}$, let $u \in A_n$, and let $x \in S$ be such that $u(t) = f(x(t)) - f_n(x(t))$ on I. Then for $s, t \in I$ we have

$$\begin{aligned} |u(t) - u(s)| &\leqq |f(x(t)) - f(x(s))| + |f_n(x(t)) - f_n(x(s))| \\ &= |x'(t) - x'(s)| + k_n|x(t) - x(s)| \\ &< \varepsilon + k_n M|t - s| \\ &< 2\varepsilon \end{aligned}$$

for small enough $|t - s|$, independently of u.

The following purely topological proposition was proved by Aronszajn in **[1]**.

PROPOSITION 3. *Let (S_n) be a sequence of compact absolute retracts in a metric space X and let S be a compact subset of X such that*
i) *$S \subset S_n$ for all n, and*
ii) *for all $\varepsilon > 0$, $S_n \subset N_\varepsilon(S)$ for almost all n.*
Then S is an R_δ-set.

THEOREM 1. *The solution set S of* (1a) (*or of* (1)) *is an R_δ-set.*

PROOF. Define a function $\varphi_n \colon U_n \to C[I]$ by $\varphi_n(u) =$ the unique solution x of $x'(t) = f_n(x(t)) + u(t)$, $x(0) = 0$. Then define $S_n = \varphi_n(U_n)$. It is well known that φ_n is continuous and easy to see that it is one-to-one. Hence S_n is homeomorphic to the compact convex set U_n. It follows that S_n is a compact absolute retract. By Proposition 2, $S \subset S_n$. Using Proposition 3, we can prove S is an R_δ set by proving for all $\varepsilon > 0$ that $S_n \subset N_\varepsilon(S)$ for almost all n. Suppose to the contrary that there exists $\varepsilon > 0$ and an infinite sequence $n_1 < n_2 < \ldots$ such that $S_{n_j} \not\subset N_\varepsilon(S)$ for all j, and choose $x_{n_j} \in S_{n_j} - N_\varepsilon(S)$ for all j.

By an easy application of Ascoli's theorem, $\mathrm{cl}(\bigcup_n S_n)$ is compact, and so we may assume without loss of generality that (x_{n_j}) converges, say to x. We will now obtain a contradiction by proving $x \in S$.

For each j, let $u_{n_j} \in U_{n_j}$ be such that $\varphi_{n_j}(u_{n_j}) = x_{n_j}$, i.e.,

$$x_{n_j}(t) = \int_0^t (f_{n_j}(x_{n_j}(s)) + u_{n_j}(s))\, ds.$$

Now (x_{n_j}) is coconvergent with the sequence (y_{n_j}) defined by

$$y_{n_j}(t) = \int_0^t (f(x_{n_j}(s)) + u_{n_j}(s))\, ds,$$

since $|x_{n_j}(t) - y_{n_j}(t)| \leqq T/n_j$ for all $0 \leqq t \leqq T$. Moreover,

$$y_{n_j}(t) \to \int_0^t f(x(s))\, ds,\ 0 \leqq t \leqq T.$$

So

$$x(t) = \lim_{j\to\infty} x_{n_j}(t) = \lim_{j\to\infty} y_{n_j}(t) = \int_0^t f(x(s))\, ds,\ 0 \leqq t \leqq T.$$

This means $x \in S$.

3. **The Solution Set of (2) is an R_δ if $n = 1$.** As in §2, we consider only the autonomous case

$$x' \in F(x),\ x(0) = 0. \tag{2a}$$

We assume that F is a continuous set valued function on R^1 whose values are closed bounded intervals in R^1, and that F is bounded (i.e., for some M, $F(x) \subset [-M, M]$ for all x). The continuity of F means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x) \subset N_\varepsilon(F(y))$ and $F(y) \subset N_\varepsilon(F(x))$ whenever $|x - y| < \delta$.

A solution x to (2a) is defined to be any absolutely continuous function satisfying (2a) almost everywhere on $I = [0, T]$. As in §1, any solution of (2a) satisfies $|x(t)| \leqq MT$ for all $t \in I$.

THEOREM 2. *The solution set S of* (2a) (*or of* (2)) *is an R_δ-set.*

PROOF. First note that the functions f, g: $R^1 \to R^1$ such that $F(x) = [f(x), g(x)]$ for all $x \in R^1$ are continuous. This follows from the fact that the inequalities $|f(x) - f(y)| < \varepsilon$ and $|g(x) - g(y)| < \varepsilon$ are implied by the inclusions

$$[f(x), g(x)] \subset [f(y) - \varepsilon, g(y) + \varepsilon]$$

and

$$[f(y), g(y)] \subset [f(x) - \varepsilon, g(x) + \varepsilon].$$

Let f_n, g_n; $R^1 \to R^1$ be Lipschitzian functions bounded by M such that

$$\begin{aligned} f(x) - 2^{-n} < f_n(x) < f(x) - 2^{-n-1} < f(x) \\ \leqq g(x) < g(x) + 2^{-n-1} < g_n(x) < g(x) + 2^{-n} \end{aligned}$$

for all $x \in B_{MT}$. Then define

$$h_n(x, u) = f_n(x) + u(g_n(x) - f_n(x)) \text{ if } x \in R^1,\ u \in [0, 1].$$

Next define U to be the set of all measurable functions $u: I = [0, T] \to [0, 1]$. U is a subset of $L^2[I]$ and we assign to U the subspace topology induced by the weak topology on $L^2[I]$. Since $L^2[I]$ is reflexive and separable, U is compact and metrizable (see **[2,** V.4.6.8 and V.6.3.3**]**). Since U is also convex, it is an absolute retract.

Define φ_n: $U \to C[I]$ by $\varphi_n(u) = x$, where x is the unique function such that

$$x'(t) = f_n(x(t)) + u(t)(g_n(x(t)) - f_n(x(t))) = h_n(x(t), u(t)),$$

and define $S_n = \varphi_n(U)$. Clearly φ_n is one to one, and, by first observing that S_n is the solution set of

$$x'(t) \in [f_n(x(t)), g_n(x(t))], \ x(0) = 0,$$

it is easy to see that $S_1 \supset S_2 \supset \ldots$ and that $S = \bigcap_n S_n$. Thus to prove S is an R_δ-set, there remains only to prove that φ_n is continuous. For then it follows that φ_n is a homeomorphism onto S_n and S_n is a compact absolute retract for each n.

Let $u \in U$, let (u_k) be a sequence in U converging weakly to u, and let $x_k = \varphi_n(u_k)$ for all k. Since

$$x_k(t) = \int_0^t h_n(x_k(s), u_k(s))\, ds,$$

the sequence (x_k) is uniformly bounded and equicontinuous on T, and so has a uniformly convergent subsequence (x_{k_j}) converging, say, to x. If we prove

$$x(t) = \int_0^t h_n(x(s), u(s))\, ds, \tag{4}$$

then it follows that $x = \varphi_n(u)$ and that every uniformly convergent subsequence of (x_k) converges to x. Consequently $\varphi_n(u_k) \to \varphi_n(u)$ as $k \to \infty$ and φ_n is continuous at u.

To establish (4), observe that it is equivalent to

$$\lim_{j \to \infty} \int_0^t [h_n(x_{k_j}(s), u_{k_j}(s)) - h_n(x(s), u(s))]\, ds = 0, 0 \leqq t \leqq T.$$

But

$$\begin{aligned} h_n(x_{k_j}, u_{k_j}) - h_n(x, u) &= [h_n(x_{k_j}, u_{k_j}) - h_n(x, u_{k_j})] \\ &\quad + [h_n(x, u_{k_j}) - h_n(x, u)]. \end{aligned}$$

Clearly the integral of the first term on the right tends to 0. For the second term we have

$$\begin{aligned} &\int_0^t [h_n(x(s), u_{k_j}(s)) - h_n(x(s), u(s))] \\ &\quad = \int_0^1 \chi_{[0,t]}(s)[g_n(x(s)) - f_n(x(s))](u_{k_j}(s) - u(s))\, ds \\ &\quad \to 0 \text{ as } j \to \infty, \end{aligned}$$

since u_{k_j} tends weakly to u.

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