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THE GRADED RING $R[X_1, \cdots, X_n]$

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0. **Introduction.** Let R be a commutative ring which is $\mathbf{Z}$ or $\mathbf{Z}^+$ graded. If $\{X_i\}_{i=1}^n$ are indeterminates over R which are assigned homogeneous degrees $\{d_i\}_{i=1}^n$, $d_i \in \mathbf{Z}$, then $R[X_1, \cdots, X_n]$ is also a graded ring. We shall be investigating the structure of graded prime ideals in $R[X_1, \cdots, X_n]$. In § 1 we develop a graded version of valuation rings with a corresponding valuation theory. In § 2 we study $R[X_1, \cdots, X_n]$ by developing weak and strong versions of Jaffard's Special Chain Theorem (see [5]). In § 3 we apply § 1 and § 2 to obtain a graded version of Arnold's Formula (see [1], [3]) and note an interesting fact about adding indeterminates to graded rings.

Throughout, our notational conventions will be the same as in [6] with the following additions:
(1) The set of graded primes is ordered by inclusion. Maximal elements in this set are called *maximal homogeneous* or *maximal graded* ideals. A graded ring with a unique maximal homogeneous ideal is said to be homogeneously local.

(2) If R is a non-trivially graded integral domain and $S = \{$homogeneous elements in $R\backslash 0\}$, then we call R_S the *homogeneous quotient ring* of R and denote it HQR(R). It is known [10, p. 157] homogeneous quotient rings are fields or of the form $k[u, 1/u]$ where k is a field and u is transcendental over k.

We refer the reader to the appropriate sections of [2] and [10] for an account of the basic ring-theoretic properties of graded rings.

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1. **Graded Valuation Rings.** This section is devoted to developing a "graded" valuation theory by adapting the theory to graded rings. We begin with:

Definition 1.1. If R is a graded integral domain then R is a *gr. valuation ring* if for each homogeneous element x in HQR(R), either x or $1/x$ is in R.

The proofs for Proposition 1.2, Remark 1.3, and Proposition 1.4 will be omitted as there are only minor (graded prime instead of prime,

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etc.) changes in the proofs of the standard "non-graded" expressions of these results. The statements and proofs that these were derived from can be found in [4].

PROPOSITION 1.2. *If R is a $\mathbf{Z}$, $\mathbf{Z}^+$ or trivially graded integral domain then the following are equivalent:*

(1) *R is a gr. valuation ring;*

(2) *If a and b are homogeneous elements in R then either a/b or $b/a \in R$;*

(3) *The homogeneous ideals in R are linearly ordered.*

REMARK 1.3. If R is a graded integral domain, k a field and u an indeterminate so that R is a graded subring of $k[u, 1/u]$, and P is a graded prime ideal in R then there exists a gr. valuation ring V with maximal homogeneous ideal M so that $M \cap R = P$ with $\mathrm{HQR}(V) = k[u, 1/u]$.

PROPOSITION 1.4. *Let $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_s$ be a finite chain of proper graded primes in a graded ring R. Then there exists a gr. valuation ring V so that $M_1 \subset \cdots \subset M_s$ is a chain of graded prime ideals such that M_s is the maximal graded ideal of V and $M_i \cap R = P_i$ for $1 \leqq i \leqq s$.*

We now wish to establish a connection between valuation rings and gr. valuation rings by showing that if V^* is a gr. valuation ring with $\mathrm{HQR}(V^*) = k[u, 1/u]$, k a field, and with quotient field of V^* equal to $k(u)$, then there exists a valuation ring V in $k(u)$ so that $V \cap k[u, 1/u] = V^*$, rank $V =$ gr. dim V^*, the value group Γ of V will be the "value group" of V^* and in some sense (Proposition 1.9) V is unique.

PROPOSITION 1.5. *Let V be a valuation ring with quotient field $k(x)$, k a field, x an indeterminate, and let $k[x, 1/x]$ be graded with* $\deg = 1$. *Then there exists a gr. valuation ring $V^* \subseteq V \cap k[x, 1/x]$.*

PROOF. Let V^* be the ring generated by the homogeneous elements in $V \cap k[x, 1/x]$.

LEMMA 1.6. *Let $D = \sum D_i$ be a graded integral domain with quotient field $k(x)$ and let Γ be an ordered abelian group. If $f: D \to \Gamma$ is defined so that f and $f|_{D_i} = f_i$ have the properties*

(1) $f_i(d_i + g_i) \geqq \min\{f_i(d_i), f_i(g_i)\}$;

(2) $f_{i+j}(d_i d_j) = f_i(d_i) + f_j(d_j)$ *for homogeneous elements d_i, $g_i \in D_i$, $d_j \in D_j$; and*

(3) *for $r = \sum r_i$, $r_i \in D_i$, $f(r) = \inf\{f_i(r_i)\}$;*

then f can be extended to a valuation v on $k(x)$.

PROOF. We show that f satisfies the properties of a valuation on D. Let $g, h \in D$ with $g = \sum_{i \in \Lambda} g_i$, $h = \sum_{i \in \Lambda} h_i$, Λ a finite subset of $\mathbf{Z}$. Then $f(g + h) = f(\sum g_i + \sum h_i) = f(\sum_{i \in \Lambda}(g_i + h_i)) = \inf_{i \in \Lambda}\{f_i(g_i + h_i)\} \geqq \inf\{\inf_{i \in \Lambda} f_i(g_i), \inf_{j \in \Lambda}\{f_j(h_j)\}\} = \inf\{f(g), f(h)\}$. Thus, $f(g + h) \geqq \inf\{f(g), f(h)\}$. To show that $f(gh) = f(g) + f(h)$, write $g = G_0 + G_1$, $h = H_0 + H_1$ where G_0 is the sum of the homogeneous components of g which have minimal f value (i.e., if $f_1(g_1) \leqq f_i(g_i)$ for each i, then g_1 is a summand of G_0), and G_1 the remainder. Similarly, we define H_0 and H_1. We now write $gh = (G_0 + G_1)(H_0 H_1) = H_0G_0 + H_1G_0 + G_1H_0 + G_1H_1$. Note that $f(H_1G_0 + G_1H_0 + G_1H_1) \geqq f(G_0H_0)$ if $G_0H_0 \neq 0$. It follows that $f(gh) = f(G_0H_0)$ since D is a domain. Using the graded structure and (3), one obtains

$$f(gh) = f(G_0H_0) = f(G_0) + f(H_0) = f(g) + f(h).$$

THEOREM 1.7. *Let V^* be a nontrivially graded gr. valuation ring. Then there exists a valuation ring V in the quotient field of V^* so that $V \cap \mathrm{HQR}(V^*) = V^*$.*

PROOF. First of all, Proposition 1.2(3) implies that V^* has a maximal homogeneous ideal which we shall call M^*. Since V^* is nontrivially graded, $\mathrm{HQR}(V^*) = k[x, 1/x]$ for some field k. Consider the group H/E where H is the set of all homogeneous elements in $k[x, 1/x]\setminus\{0\}$ with multiplicative operation and E is the set of homogeneous units in V^*. Then H/E can be ordered by $g = \alpha + E > 0$ iff $\alpha \in M^*$. Define a map $v' : k[x, 1/x] \to H/E \cup \{\infty\}$ by $v'(\alpha) = \alpha + E$ for all $\alpha \neq 0$ homogeneous in $k[x, 1/x]$, $v'(0) = \infty$ and if $r = \sum r_i$, $v'(r) = \min\{v'(r_i)\}$. We have defined v' so as to satisfy the hypotheses of Lemma 1.6 and so v' will extend to a valuation v on $k(x)$ with valuation ring V. We claim that $V \cap k[x, 1/x] = V^*$. Clearly $V \cap k[x, 1/x] \supseteq V^*$. If $f/g \in V \cap k[x, 1/x]$, then we may assume that g is homogeneous and $f = \sum f_i$. Since $f/g \in V$ and $v(f) = \inf\{v(f_i)\}$ we have $\inf\{v(f_i)\} \geqq v(g)$. Therefore, $f_i/g \in V$ for each i so $f/g \in V^*$. Thus, $V \cap k[x, 1/x] = V^*$.

DEFINITION 1.8. If v is a valuation on a field K, R is a graded ring with quotient field K and v satisfies $v(\sum r_i) = \inf\{v(r_i)\}$ for r_i homogeneous of degree i in R, then v is a *homogeneously defined valuation.* If V is the valuation ring for v then V is a *homogeneously defined valuation ring.*

PROPOSITION 1.9. *Let V_1 and V_2 be homogeneously defined valuation rings in $k(x)$ so that $V_1 \cap k[x, 1/x] = V_2 \cap k[x, 1/x] = V^*$, a gr. valuation ring. Then $V_1 = V_2$.*

PROOF. Let $f = \sum f_i$, $g = \sum g_i$ be in V^*. If $f/g \in V_1$, then $V_1(f) \geqq V_1(g)$. Thus, $V_1(f) = \inf\{V_i(f_i)\} \geqq \inf\{V_1(g_i)\}$. For some g_j, $V_1(g_j)$ is minimal in $\{V_1(g_i)\}$. Let g' be one of these, then $V_1(f) \geqq V_1(g')$ and $f/g' \in V_1$. Since g' is homogeneous, $f/g' \in V_2$. Thus $v_2(f) \geqq v_2(g')$ $\geqq \inf\{v_2(g_i)\} = v_2(g)$. Thus, $f/g \in V_2$. Hence, $V_1 \subseteq V_2$. Similarly, $V_2 \subseteq V_1$.

DEFINITION 1.10. The homogeneously defined valuation ring V obtained from Theorem 1.7 is called the *canonical valuation ring* for V^*. (Thus every graded valuation ring is the contraction of a homogeneously defined valuation ring, its canonical valuation ring.)

REMARK 1.11. We are able to bring over the ideas of value group, rank and isolated subgroups of the value group to their appropriate graded versions so that we now have a value group for a gr. valuation ring V^*, gr. rank V^* (length of graded primes in V^*) and the isolated subgroups correspond to graded primes (see [**10**, Ch. VI, § 10]).

With Theorem 1.7 and Remark 1.11 we are able to show:

PROPOSITION 1.12. *If V^* is a gr. valuation ring of* gr. rank $= n$, $\mathrm{HQR}(V^*) = L[x, 1/x]$ *and $k[u, 1/u]$ is a graded subring of $L[x, 1/x]$ such that k is a field and* $\operatorname{tr\,deg}_{k(u)} (L(x)) = d$, *then* gr. rank $V^* \cap k[u, 1/u] \geqq n - d$.

All of valuation theory does not carry over to graded versions. For example, the Theorem of Independence of Valuations in [**8**, p. 38] does not have analogous results in gr. valuation rings as the following example shows.

EXAMPLE 1.13. Let $R = \mathbf{Z}[x]$ be graded with $\deg x = 1$. Then $\mathrm{HQR}(R) = \mathbf{Q}[x, 1/x]$. We define a gr. valuation ring V_1 by $v_1(2) = 1$, $v_1(x) = \alpha > 0$, $\alpha \in \mathbf{R}$ transcendental over $\mathbf{Q}$, and extend it linearly using the minimal condition on sums (Lemma 1.6(3)). Similarly, define V_2 by $v_2(2) = 1$, $v_2(x) = \beta > 0$, β transcendental over $\mathbf{Q}(\alpha)$, $\alpha > \beta$, and extend it as before. From the choice of α and β there exist positive integers n and m so that $\beta < n/m < \alpha$. Hence, $n - \alpha m < 0$ and $n - \beta m > 0$, which imply that $2^n x^{-m} \in V_1 \setminus V_2$ and $2^{-n} x^m \in V_2 \setminus V_1$. The homogeneous units of V_1 and the homogeneous units of V_2 are of degree 0 and are units in $Z_{(2)}$. Hence there exists only one maximal homogeneous ideal of $V_1 \cap V_2$.

We conclude this section with an example—the class of all gr. valuation rings in $\mathbf{Q}[x, 1/x]$ containing $\mathbf{Z}[x]$, $\deg x = 1$ with value group $\mathbf{Z}$.

EXAMPLE 1.14. If V is a homogeneously defined DVR in $\mathbf{Q}(x)$ containing $\mathbf{Z}[x]$ then it is completely determined by $v(x)$ and $v(p)$ for p some prime in $\mathbf{Z}$. Thus, if $v(x) = \alpha_1$, $v(p) = \alpha_2$, $(\alpha_1, \alpha_2) = 1$, and s and t are positive integers so that $\alpha_2 s - \alpha_1 = 1$ then $V \cap \mathbf{Q}[x, 1/x] = \mathbf{Z}_{(p)}[x, p^{\alpha_1}/x^{\alpha_2}, x^{\alpha_2}/p^{\alpha_1}, p^s/x^t]$.

2. **The Graded Ring $R[X_1, \cdots, X_n]$.** This section is a natural extension of [6] in that we shall be investigating gr. dim $R[X_1, \cdots, X_n]$. A result from [6] that we shall need is that if R is a graded ring, X an indeterminate assigned nonzero homogeneous degree, then gr. dim $R[X] = \operatorname{rank} M + 1$, where M is a graded prime ideal of R of maximal rank.

We begin our investigation by looking at a graded version of Jaffard's Special Chain Theorem (see [4, p. 368]). In the next three results, $R = \sum R_i$ is a graded ring and $X_1, \cdots, X_n$ are indeterminates assigned degrees in $\mathbf{Z}$. The proofs of these are essentially the same as those found in [4] where they are credited to Brewer, Montgomery, Rutter and Heinzer.

LEMMA 2.1. *Let Q be a graded prime ideal of $R[X_1]$ and let $P = Q \cap R$. If $Q \supsetneq P[X_1]$, then* gr. rank Q = gr. rank $(P[X_1]) + 1$, *and for each integer $n > 1$,* gr. rank $(Q[X_2, \cdots, X_n])$ = gr. rank $P[X_1, \cdots, X_n] + 1$.

THEOREM 2.2. *If Q is a proper graded prime ideal of $R[X_1, \cdots, X_n]$ so that $R \cap Q = P$, then* gr. rank Q = gr. rank $P[X_1, \cdots, X_n]$ + gr. rank $(Q/P[X_1, \cdots, X_n]$.

Recall that a *special chain* of primes in $R[X_1, \cdots, X_n]$ is a chain of primes $\mathscr{C}$: $P_0 \subset P_1 \subset \cdots \subset P_n$ so that $(P_i \cap R)[X_1, \cdots, X_n]$ is an element of the chain $\mathscr{C}$ for each i.

COROLLARY 2.3. (*Special Chain Theorem for Graded Rings*). *If P is a proper graded prime ideal of R of finite gr. rank, then* gr. rank P *can be realized as the length of a special chain of graded primes of $R[X_1, \cdots, X_n]$ with terminal element P. In particular, if R has finite gr. dim, then* gr. dim $R[X_1, \cdots, X_n]$ *can be realized as the length of a special chain of graded primes of $R[X_1, \cdots, X_n]$.*

LEMMA 2.4. *Let R be a graded ring which is H-local. The map $\Gamma : \mathbf{Z}^{(n)} \to \mathbf{Z}^+$, given by $\Gamma(d_1, \cdots, d_n)$ =* gr. dim $R[X_1, \cdots, X_n]$, *where $\deg X_i = d_i$, is a constant on $\mathbf{Z}^{(n)} \setminus (0, 0, \cdots, 0)$.*

An example of Γ having two values on $\mathbf{Z}^{(n)}$ is the ring in Example 3.3 of the last section when $n = 1$.

PROOF. We shall consider $R[X_1, \cdots, X_n]$ as $R[X_1, \cdots, X_{n-1}]X_n]$ with $\deg X_n \neq 0$. Let the maximal homogeneous ideal in R be denoted by M. Then the maximal graded ideal in $R[X_1, \cdots, X_{n-1}]$ is $(M, X_1, \cdots, X_{n-1})$ since the only homogeneous elements that it does not contain are in $R \setminus M$ and they are units in R and hence units in $R[X_1, \cdots, X_{n-1}]$. Thus, $(M, X_1, \cdots, X_n)$ will have maximal rank among the graded primes in $R[X_1, \cdots, X_{n-1}]$ independent of the degrees that the X_i's are assigned. Since $\deg X_n \neq 0$, gr. $\dim R[X_1, \cdots, X_n] = \operatorname{rank}(M, X_1, \cdots, X_{n-1}) + 1$, a constant.

We shall now apply this lemma to obtain a stronger version of Corollary 2.3.

THEOREM 2.5. *Let R be as in Lemma 2.4 and denote* $\deg X_i = d_i$ *for* $1 \leqq i \leqq n$ *by* $(d_1, \cdots, d_n)$. *Let* $\mathscr{C} : 0 \subset P_1 \subset \cdots \subset P_M = (M, X_1, \cdots, X_n)$ *be a maximal special chain of graded primes in* $R[X_1, \cdots, X_n]$ *with* $(d_1, \cdots, d_n) = (1, \cdots, 1)$. *Let* $\mathscr{B} = \{P_i \text{ in } \mathscr{C} \mid P_i \text{ is of the form } q_i[X_1, \cdots, X_n]\}$. *Then for each* $(d_1, \cdots, d_n) \neq (0, \cdots, 0)$ *there exists a special chain of graded primes* $\mathscr{S}(d_1, \cdots, d_n)$ *in* $R[X_1, \cdots, X_n]$ *so that all of the primes in* $\mathscr{B}$ *are in* $\mathscr{S}_{(d_1,\cdots,d_n)}$. *Moreover,* $P_i \in \mathscr{B}$ *has graded rank i in* $\mathscr{S}_{(d_1,\cdots,d_n)}$.

PROOF. By induction on gr. $\dim R$ we may assume that R is a domain. If gr. $\dim R = 0$ then $\mathscr{B} = \{0\}$ and $\mathscr{C} = 0$.

We assume the theorem is true when gr. $\dim R < m \neq 0$. Let gr. $\dim R = m$ and $P_t \neq 0$ be the prime ideal in $\mathscr{B}$ with the smallest t. Let $q_t = P_t \cap R$. Then R_S, where S is the homogeneous elements in $R \setminus q_t$, has one maximal graded ideal $q = q_t R_S$. In $R_S[X_1, \cdots, X_n]$, the chain $0 \subset P_1 \subset \cdots \subset P_t$ is preserved with $P_t = q[X_1, \cdots, X_n]$ in the $(1, \cdots, 1)$ case for $R_S[X_1, \cdots, X_n]$. But gr. $\dim R_S[X_1, \cdots, X_n] =$ gr. rank $(q, X_1, \cdots, X_n)$ and is independent of $(d_1, \cdots, d_n) \neq (0, \cdots, 0)$ by Lemma 2.4. The Special Chain Theorem for Graded Rings implies that there exists a maximal special chain of graded primes $\mathscr{C}_{(d_1,\cdots,d_n)}$ for each $(d_1, \cdots, d_n)$ passing through $q[X_1, \cdots, X_n]$. The ring $R_S[X_1, \cdots, X_n]/q[X_1, \cdots, X_n]$ will be of the form $k[u, 1/u][X_1, \cdots, X_n]$ or $k[X_1, \cdots, X_n]$ for k a field and both of these have graded dimension independent of $(d_1, \cdots, d_n)$. Hence, $q[X_1, \cdots, X_n]$ has graded rank $= t$ independent of $(d_1, \cdots, d_n) \neq (0, \cdots, 0)$.

Let $R' = R/q_t$. Then gr. $\dim R' \leqq m - 1$, so $R'[X_1, \cdots, X_n]$ satisfies the inductive hypothesis with the chain $0 < \overline{P}_{t+1} < \cdots < \overline{P}_m$ and a set $\overline{\mathscr{B}}$ obtained from $0 < P_1 < \cdots < P_t < P_{t+1} < \cdots < P_m$ and $\mathscr{B}$. Let $(d_1, \cdots, d_n)$ be given and $\mathscr{S}'_{(d_1,\cdots,d_n)} : 0 < \overline{P'_{t+1}} < \cdots < \overline{P'_m}$ the chain given by the induction hypothesis with $\overline{P}_i \in \overline{\mathscr{B}}$ having graded rank

$i - t$ in $R/q[X_1, \cdots, X_n]$. From $\mathscr{S}'_{(d_1,\cdots,d_n)}$ we get a chain $P_t < P'_{t+1} < \cdots < P'_m$ in $R[X_1, \cdots, X_n]$ from P_t to $P'_m = P_m$ satisfying the desired properties. We patch this chain with the chain obtained from $\mathscr{C}_{(d_1,\cdots,d_n)}$ to obtain a chain $\mathscr{S}_{(d_1,\cdots,d_n)} : 0 < P'_1 < \cdots < P_t < \cdots < P'_m = P_m$ which satisfies the conclusion of the theorem for gr. dim $R = m$.

We conclude this section by looking at a special case of $R[X_1, \cdots, X_n]$.

Proposition 2.6. *If $k[u, 1/u]$ is a homogeneous quotient ring and $\theta_1, \cdots, \theta_n$ are homogeneous elements of a graded extension of $k[u, 1/u]$, then* gr. dim $k[u, 1/u]$ $[\theta_1, \cdots, \theta_n] = d$ *if and only if d is the transcendence degree of $k(u, \theta_1, \cdots, \theta_n)$ over $k(u)$.*

Proof. If $\deg \theta_i = d_i$, then by Lemma 1 of [**6**], gr. dim $k[u, 1/u]$ $[\theta_1, \cdots, \theta_n = d$ iff the 0^{th} component $k[\theta_1/u^{d_1}, \cdots, \theta_n/u^{d_n}]$ has dimension d iff $\operatorname{tr.deg}_k(k(\theta_1/u^{d_1}, \cdots, \theta_n/u^{d_n})) = d$ iff $\operatorname{tr.deg}_{k(u)}(k(u, \theta_1, \cdots, \theta_n)) = d$.

3. **Applications.** It is the purpose of this section to use some of the results of the first two sections to show that if R has finite valuative dimension and R has a maximal homogeneous ideal which is also maximal, then the graded dimension and the Krull dimension can be made equal by adding enough indeterminates. We also show that a graded version of Arnold's Formula exists (see [1] and [3]).

Jaffard, in [5], defined the *valuative dimension* of a domain R, written $\dim_v R$, to be the smallest integer n such that each valuation overring of R has dimension at most n. If no such n exists then the valuative dimension is said to be infinite. Arnold [1] was able to extend some of the results of Jaffard concerning valuative dimension. Specifically [1, Th. 6 (4)], he was able to show that if $\dim_v R = n$, then $\dim R$ $[X_1, \cdots, X_{n-1}] + 1 = \dim R$ $[X_1 \cdots X_n]$. We shall use this fact in the proof of Proposition 3.1.

Proposition 3.1. *If $\dim_v R = n$, R is a graded integral domain, R has one graded maximal ideal and $\{X_i\}_{i=1}^n$ are assigned degrees $\{d_i\}_{i=1}^n$ not all 0, then* gr. dim $R[X_1, \cdots, X_n] = \dim R[X_1, \cdots, X_n]$.

Proof. We shall be using the same notation as in the proof of Lemma 2.4. By Lemma 2.4 we may assume that each X_i has degree 1. We have $\dim R[X_1, \cdots, X_{n-1}] = \operatorname{rank}(M, X_1, \cdots, X_{n-1})$ so we obtain $\dim R[X_1, \cdots, X_n] = \dim R[X_1, \cdots, X_{n-1}] + 1 = \operatorname{rank}(M, X_1, \cdots, X_{n-1}) + 1 =$ gr. dim $R[X_1, \cdots, X_n]$ with the first equality a result of Arnold and the last equality from [**6**].

A result of Arnold [1, Theorem 5, p. 323] can be stated as follows: If R is a commutative integral domain with quotient field K and $X_1, \cdots, X_n$ are indeterminates over R then there exist $\theta_1, \cdots, \theta_n$ in K so that $\dim R[X_1, \cdots, X_n] = \dim R[\theta_1, \cdots, \theta_n] + n$. This result will be referred to as Arnold's Formula. Eakin later gave a proof of this result in [3] which corrected an error in Arnold's proof of the result. The remainder of this section will be devoted to stating a graded version of Arnold's Formula (Theorem 3.2) and adapting Eakin's proof to the graded case.

THEOREM 3.2. *If R is a graded integral domain and $X_1, \cdots, X_n$ are indeterminates assigned arbitrary degrees in* $\mathbf{Z}$ *not all zero, then there exist homogeneous elements $\theta_1, \cdots, \theta_n$ in* HQR(R) *so that* $\text{gr.}\dim R[X_1, \cdots, X_n] \leqq n + \text{gr.}\dim R[\theta_1, \cdots, \theta_n]$.

PROOF. The proof follows the one given by Eakin in [3]. We note that we may assume R to be H-local and $\mathbf{Z}$ or $\mathbf{Z}^+$ graded since if R is trivially graded, HQR(R) is the quotient field of R and Arnold's Formula states that there exist $\theta_1, \cdots, \theta_n$ in the quotient field of R so that $\dim R[\theta_1, \cdots, \theta_n] = \dim R[X_1, \cdots, X_n] - n$. But $\text{gr.}\dim R[\theta_1, \cdots, \theta_n] = \dim R[\theta_1, \cdots, \theta_n]$ and $\text{gr.}\dim R[X_1, \cdots, X_n] \leqq \dim R[X_1, \cdots, X_n]$. Thus, $\text{gr.}\dim R[\theta_1, \cdots, \theta_n] + n \geqq \text{gr.}\dim R[X_1, \cdots, X_n]$.

Suppose there were a counterexample $R[X_1, \cdots, X_n]$. As in Eakin's argument, we may assume that n is minimal and that for this fixed n R has minimal graded dimension.

Let the indeterminates $X_1, \cdots, X_n$ be assigned degrees $(d_1, \cdots, d_n)$ with not all $d_i = 0$. Let $0 < P_1 < \cdots < P_s$ be a special chain of graded primes of maximal length in $R[X_1, \cdots, X_n]$. We may assume that $P_s = (M, X_1, \cdots, X_n)$, M the maximal graded prime in R. Let t be minimal such that $P_t \cap R = q \neq 0$. Then q is graded and $t - 1 \leqq n$. Since the chain is special, $P_t = q[X_1, \cdots, X_n]$ and $\text{gr.rank}\, P_t = t$. Letting S be the set of homogeneous elements in $R - q$, then $T = R_S[X_1, \cdots, X_n]$ has graded dimension $n + t$. Define $\overline{T}$ to be $R_S[X_1, \cdots, X_n]/P_{t-1}R_S[X_1, \cdots, X_n]$. This implies that $\text{gr.}\dim \overline{T} = n + 1$ so there is a chain

$$(*) \qquad 0 < q\overline{T} < \overline{P_{t+1}} < \cdots < \overline{P_{t+n}} < \overline{T}.$$

By Proposition 1.4 there exists a gr. valuation ring V^* which contains $\overline{T}$, is contained in HQR($\overline{T}$), and is centered on the chain $(*)$. Let $K_0[u, 1/u] = \text{HQR}(R)$. If $V = V^* \cap K_0[u, 1/u]$, then by Proposition 1.12, $\text{rank}\, V \geqq t$. Since each graded prime of $(*)$ contains q, each graded prime of V contains q so each graded prime of V meets R_S at qR_S. Let $M_1 \subsetneqq \cdots \subsetneqq M_t$ be a chain of graded primes in V so that $M_j \cap R_S = qR_S$. For each $1 \leqq i \leqq t - 1$, choose θ_i homogeneous in $M_{i+1} \setminus M_i$ with degree $\neq 0$ is possible (don't choose any if $t = 1$).

Consider the canonical mapping $\sigma : V \to V/M_1$. Under this mapping $R[\theta_1, \cdots, \theta_n]$ maps into V/M_1. Denote HQR(R_S/qR_S) by $L[v, 1/v]$ and $\sigma(\theta_j)$ by $\bar{\theta}_j$. Then under σ, $(D/q)[\overline{\theta_1}, \cdots, \theta_{t-1}] \subset L[v, 1/v][\theta_1, \cdots, \theta_{t-1}] \subset V/M_1$. By the choice of the θ_i's the primes $\underline{M_i/M_1}$ for $1 \leqq i \leqq t-1$ lie over distinct primes of $L[v, 1/v][\overline{\theta_1}, \cdots, \overline{\theta_{t-1}}]$. Thus, gr. dim $L[v, 1/v][\overline{\theta_1}, \cdots, \overline{\theta_{t-1}}] \geqq t-1$ and by Proposition 2.6 the $\overline{\theta_i}$ are algebraically independent over $L(v)$.

Let I be the kernel of $\tau : R[X_1, \cdots, X_{t-1}] \to R[\theta_1, \cdots, \theta_{t-1}]$ given by $X_j \to \theta_j$. If we follow τ b σ we get a mapping:

$$R[X_1, \cdots, X_{t-1}] \xrightarrow{\sigma \circ \tau} \frac{R}{q} \quad [\overline{\theta_1}, \cdots, \theta_{t-1}] \cong \quad \frac{R}{q}[X_1, \cdots, X_{t-1}]$$

which clearly has kernel $q[X_1, \cdots, X_{t-1}]$. Thus, $I \subset q[X_1, \cdots, X_{t-1}]$ and $I \cap D = 0$ so the containment is strict. Note here that I is not necessarily graded and what we wish to show is that I can be made into a graded prime so that

$$\frac{D[X_1, \cdots, X_n]}{I[X_t, \cdots, X_n]} \cong D[\theta_1, \cdots, \theta_{t-1}][X_t, \cdots, X_n]$$

has graded dimension $\geqq s - (t-1)$ with equality when $t \neq n+1$.

We begin by showing that when $t = n+1$ we may get the inequality. If $t = n+1$, then choose the X_i's so that $\deg X_i = \deg \theta_i$. This makes I into a graded prime ideal. Note that if one of the X_i's has nonzero degree, there is a maximal special chain going through $q[X_1, \cdots, X_n]$ by Theorem 2.5, and $q[X_1, \cdots, X_n]$ has graded rank $t = n+1$. Thus,

$$\begin{aligned} \text{gr. dim} \frac{R[X_1, \cdots, X_n]}{I} &> \text{gr. dim} \frac{R[X_1, \cdots, X_n]}{q[X_1, \cdots, X_n]} \\ &= s - t \\ &= s - (n+1). \end{aligned}$$

But $R[X_1, \cdots, X_n]/I \underset{\text{graded}}{\cong} R[\theta_1, \cdots, \theta_n]$, so gr. dim $R[\theta_1, \cdots, \theta_n] \geqq s - n =$ gr. dim $R[X_1, \cdots, X_n] - n$, a contradiction. On the other hand, if all of the θ_i's and hence the X_i's are degree 0, then q must contain all of the nonzero degree homogeneous elements. As before, we have $q[X_1, \cdots, X_n]$ and I graded primes in $R[X_1, \cdots, X_n]$ and $(R/q)[X_1, \cdots, X_n]$ is graded with R/q trivially graded and $\deg X_i = 0$. In this case, Krull dimension and graded dimension are the same. Since

Krull dimension $\geqq$ graded dimension when the indeterminates are not all of degree 0, then $\text{gr.}\dim (R/q)[X_1, \cdots, X_n] \geqq s - (n+1)$. Thus, $\text{gr.}\dim R[X_1, \cdots, X_n]/I > s - (n+1)$ and, as before, $R[X_1, \cdots, X_n]/I \cong R[\theta_1, \cdots, \theta_n]$. Thus, $\text{gr.}\dim R[\theta_1, \cdots, \theta_n] \geqq s - n = \text{gr.}\dim R[X_1, \cdots, X_n] - n$, a contradiction.

Thus, we may assume that $t < n + 1$. This means that X_n may be taken to be of nonzero degree and it will not affect the choices of the degrees of the X_i, $i < n$, that we shall make. As before, we choose the degrees of the X_i to correspond to the degrees of the θ_i and we get I to be a graded prime. The proof from this point on follows Eakin's proof with only minor modifications (graded rank for rank, graded homomorphism for homomorphism, etc.) and so we omit it.

We conclude with one example showing that we may choose an R and an n so that there exist $\theta_1, \cdots, \theta_n$ in HQR(R) so that $\text{gr.}\dim R[\theta_1, \cdots, \theta_n] + n > \text{gr.}\dim [X_1, \cdots, X_n]$. We use a Seidenberg example [9].

Example 3.3. Let R be a local domain of Krull dimension 1 so that $\dim R[X] = 3$. Then Arnold's Formula states that there exists a θ in the quotient field of R so that $\dim R[\theta] = 3 - 1 = 2$. If R is trivially graded, θ will be homogeneous of degree 0. Thus, $\text{gr.}\dim R[\theta] = \dim R[\theta] = 2$. If X is assigned degree 1, then $\text{gr.}\dim R[X] = 2$ and $\text{gr.}\dim R[\theta] + 1 = 3 > 2$.

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