ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 2, Spring 1979

THE G-TRANSFORM OF GENERALIZED FUNCTIONS

R. S. PATHAK AND J. N. PANDEY

ABSTRACT. The classical G-transform is extended to generalized functions (distributions). The corresponding inversion formula due to Kesarwani is shown to be valid in the weak distributional sense. A structure forumla for a class of generalized functions whose G-transform exists is also given.

1. **Introduction.** In recent years, quite a variety of integral transforms have been extended to generalized functions. In this paper, we consider a G-transform, which encompasses a number of integral transforms as special cases, both known as well as unknown. The G-transform with its inversion formula, studied by Kesarwani [8, 9], after a change of variables

$$x^{2\gamma} = X,\ y^{2\gamma} = Y,\ v^{2\gamma} = V \text{ and } x^{-(1/2)+1/4\gamma} f(x^{1/2\gamma}) = F(X),$$

is represented by

$$(1)\qquad F(x) = \int_0^\infty G^{m,p}_{p+q,m+n}\left(xy \,\middle|\, \begin{matrix} a_p, b_q \\ c_m, d_n \end{matrix} \right) dy \cdot \int_0^\infty G^{n,q}_{p+q,m+n}\left(vy \,\middle|\, \begin{matrix} -b_q, -a_p \\ -d_n, -c_m \end{matrix} \right) F(v)\,dv,$$

where m, n, p, q are non-negative integers, a_p, b_q, c_m, d_n are complex numbers and

$$G^{m,p}_{p+q,m+n}\left(x \,\middle|\, \begin{matrix} a_p, b_q \\ c_m, d_n \end{matrix} \right)$$

is Meijer's G-function [7].

In the present paper, we extend Kesarwani's inversion theorem (cf. [9], Theorem 1) for the G-transform defined by (1) to generalized functions by interpreting convergence in the weak distributional sense. Our notations and terminology follow those of [7] and [16]. We shall need the following formulae (cf. [7], pp. 150–152):

$$(2)\qquad x^\sigma G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right) = G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_p + \sigma \\ b_q + \sigma \end{matrix} \right),$$

$$(3)\qquad G^{m,n+1}_{p+1,q+1}\left(x \,\middle|\, \begin{matrix} a, a_p \\ b_q, b \end{matrix} \right) = (-1)^r G^{m+1,n}_{p+1,q+1}\left(x \,\middle|\, \begin{matrix} a_p, a \\ b, b_q \end{matrix} \right),$$

$q \geqq m,\ a - b = r,\ r = 0, \pm 1, \pm 2, \cdots,$

Received by the editors on July 29, 1976, and in revised form on February 28, 1977.
This work was supported by the National Research Council Grant A5298.
Copyright © 1979 Rocky Mountain Mathematical Consortium

(4) $$x^k\left(\frac{d}{dx}\right)^k G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=G^{m,n+1}_{p+1,q+1}\left(x\,\middle|\,\begin{matrix}0,\,a_p\\ b_q,\,k\end{matrix}\right).$$

Define

(5) $$k(x)\equiv G^{m,p}_{p+q,m+n}\left(x\,\middle|\,\begin{matrix}a_p,\,b_q\\ c_m,\,d_n\end{matrix}\right)$$

and

(6) $$h(x)\equiv G^{n,q}_{p+q,m+n}\left(x\,\middle|\,\begin{matrix}-b_q,\,-a_p\\ -d_n,\,-c_m\end{matrix}\right),$$

so that by (4)

(7) $$k_r(x)\equiv x^r\left(\frac{d}{dx}\right)^r k(x)=G^{m,p+1}_{p+q+1,m+n+1}\left(x\,\middle|\,\begin{matrix}0,\,a_p,\,b_q\\ c_m,\,d_n,\,r\end{matrix}\right)$$

and

(8) $$h_r(x)\equiv x^r\left(\frac{\mathrm{d}}{dx}\right)^r h(x)=G^{n,q+1}_{p+q+1,m+n+1}\left(x\,\middle|\,\begin{matrix}0,\,-b_q,\,-a_p\\ -d_n,\,-c_m,\,r\end{matrix}\right)$$

for each $r=0,1,2,\cdots$. Notice that $k_0(x)=k(x)$ and $h_0(x)=h(x)$. It will be assumed throughout that m, n, p, q are non-negative integers satisfying $m-q=n-p>0$ and

$$a_k-c_j\neq 1,2,3,\cdots,\quad \text{for } k=1,2,\cdots,p,\ j=1,2,\cdots,m,$$

and

$$d_k-b_j\neq 1,2,3,\cdots,\quad \text{for } k=1,2,\cdots,n,\ j=1,2,\cdots,q.$$

To simplify the analysis, we introduce the following notations. Set

$$\tfrac{1}{2}\sigma=m-q=n-p,$$

$$\Delta=\mathrm{Re}\left(\sum_1^m c_j+\sum_1^n d_j-\sum_1^p a_j-\sum_1^q b_j\right),$$

(9)
$$\beta_1=\min \mathrm{Re}(c_j),\ j=1,\cdots,m,$$
$$\beta_2=\min \mathrm{Re}(-d_j),\ j=1,\cdots,n,$$
$$\sigma_1=\max \mathrm{Re}(a_j),\ j=1,\cdots,p,\ \sigma_2=\max \mathrm{Re}(-b_j),\ j=1,\cdots,q,$$
$$\eta=\max\left[\frac{1}{\sigma}\ \{\tfrac{1}{2}(1-\sigma)+\Delta\},\ \sigma_1-1\right],$$
$$\lambda_r=\max\left[\frac{1}{\sigma}\ \{\tfrac{1}{2}(1-\sigma)+\Delta+r\},\ \sigma_1-1,\ -1\right],$$
$$r=0,1,2,\cdots.$$

Now, we give asymptotic estimates of $k(x)$, and those of $h(x)$ can be derived from $k(x)$ by a simple change of parameters.

(i) If $p + q \leqq m + n$, then

$$k(x) = 0(x^{\beta_1}),\ x \to 0+ \qquad \text{(cf. [7], p. 145)} \tag{10}$$

$$h(x) = 0(x^{\beta_2}),\ x \to 0+. \tag{11}$$

(ii) If $m - q \geqq 1$, then as $x \to \infty$,

$$\begin{aligned} k_\tau(x) &= x^{1/\sigma\{(1/2)(1-\sigma)+\Delta+\tau\}}\{\cos(\sigma x^{1/\sigma} + \alpha)(A + 0[x^{-2/\sigma}]) \\ &\quad + \sin(\sigma x^{1/\sigma} + \alpha)0(x^{-1/\sigma})\} \\ &\quad + \sum_{j=0}^{p} x^{(\mathrm{Re}\, a_j - 1)}\{E_j + 0(x^{-1})\}, \end{aligned} \tag{12}$$

where A, α, E_j are certain constants and $a_0 = 0$ (cf. [7], p. 191 (9)). Hence we can also write

$$k_\tau(x) = 0(x^{\lambda_\tau}), \qquad x \to \infty \tag{13}$$

and

$$k(x) = 0(x^{\eta}), \qquad x \to \infty. \tag{14}$$

2. **An integrodifferential operator.** From [7], we know that kernel $k(\alpha x)$ satisfies the differential equation

$$\left[(-1)^\tau \alpha x \prod_{j=1}^{p} (\delta - a_j + 1) \prod_{j=1}^{q} (\delta - b_j + 1) - \prod_{j=1}^{m} (\delta - c_j) \prod_{j=1}^{n} (\delta - d_j) \right] k(\alpha x) = 0 \tag{15}$$

where $\delta \equiv x(d/dx)$ and $\tau = m - q$. Using the fact that

$$(\delta + v)f(x) = x^{1-v}D[x^v f(x)]$$

we can write the above differential equation in the integrodifferential form

$$\Delta_x k(\alpha x) = (-1)^\tau \alpha k(\alpha x) \tag{16}$$

where the integrodifferential operator Δ_x is defined by

$$\begin{aligned} \Delta_x &\equiv x^{-1} \prod_{j=1}^{q} (x^{b_j} D^{-1} x^{-1-b_j}) \prod_{j=1}^{p} (x^{a_j} D^{-1} x^{-1-a_j}) \\ &\quad \cdot \prod_{j=1}^{m} (x^{1+c_j} D x^{-c_j}) \prod_{j=1}^{n} (x^{1+d_j} D x^{-d_j}), \end{aligned} \tag{17}$$

in which we interpret

$$D = \frac{d}{dx},$$

$$D^{-1} = \int_0^x \cdots dt,$$

$$\prod_{j=1}^{p} (x^{a_j} D x^{1-a_j}) = (x^{a_1} D x^{1-a_1})(x^{a_2} D x^{1-a_2}) \cdots (x^{a_p} D x^{1-a_p}),$$

and so on. Note that the operator D^{-1} can be applied successively on $k(\alpha z)$ provided that

$$\beta_1 > \max[\operatorname{Re} a_i, \operatorname{Re} b_j],\ i = 1, \cdots, p,\ j = 1, \cdots, q. \tag{18}$$

Similarly, it can be seen that the second kernel $h(\alpha x)$ satisfies the integrodifferential equation

$$\nabla_x h(\alpha x) = (-1)^{\tau} \alpha h(\alpha x) \tag{19}$$

where τ is the same as in (15) and the operator ∇_x is defined by

$$\begin{aligned} \nabla_x \equiv x^{-1} \prod_{j=1}^{p} (x^{-a_j} D^{-1} x^{-1+a_j}) \prod_{j=1}^{q} (x^{-b_j} D^{-1} x^{-1+b_j}) \\ \cdot \prod_{j=1}^{n} (x^{1-d_j} D x^{d_j}) \prod_{j=1}^{m} (x^{1-c_j} D x^{c_j}). \end{aligned} \tag{20}$$

The operator ∇_x can be applied to $h(\alpha x)$ provided that

$$\beta_2 > \max[\operatorname{Re}(-a_i), \operatorname{Re}(-b_j)],\ i = 1, \cdots, p,\ j = 1, \cdots, q. \tag{21}$$

REMARK 1. If for a given $k(x)$ the operator Δ_x does not involve the integration operator D^{-1} then the conditions (18) and (21) are treated as empty. In case $p = q = 0$ the integration operator is absent and the aforesaid conditions do not apply. The Hankel transform corresponds to this case.

REMARK 2. The operators Δ_x and ∇_x can be applied on any $C^\infty(R)$ function φ any number of times which satisfies the asymptotic orders

$$\varphi^{(k)}(x) = 0(x^{\alpha-k}),\ x \to 0+,\ k = 0, 1, 2, \cdots$$

where

$$\alpha > \max(|\operatorname{Re} a_i|, |\operatorname{Re} b_j|),\ i = 1, \cdots, p,\ j = 1, \cdots, q.$$

Some properties of these operators are described below.

LEMMA 1. *Let* $\varphi \in C^{\infty}(R_{+})$ *with the asymptotic order*

$$\varphi^{(k)}(x) = 0(x^{\alpha-k}),\ x \to 0+,\ k = 0, 1, 2, \cdots,$$

where $\alpha > \max(|\mathrm{Re}\, a_i|, |\mathrm{Re}\, b_j|)$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$. *Then the integration operators* $(x^{-a_i}D^{-1}x^{-1+a_i})$ *and* $(x^{-a_j}D^{-1}x^{-1+a_j})$ *when acting on* φ *in succession are commutative.*

PROOF. Assume that

$$f_1 = (x^{-a_i}D^{-1}x^{-1+a_i})(x^{-a_j}D^{-1}x^{-1+a_j})\varphi(x)$$

and

$$f_2 = (x^{-a_j}D^{-1}x^{-1+a_j})(x^{-a_i}D^{-1}x^{-1+a_i})\varphi(x).$$

It is easily seen, on using the fact that the differentiation operators

$$(x^{1-a_i}Dx^{a_i} \text{ and } (x^{1-a_j}Dx^{a_j})$$

are commutative, that

$$[(x^{1-a_i}Dx^{a_i})(x^{1-a_j}Dx^{a_j})]f_1(x) = \varphi(x)$$

and

$$[(x^{1-a_i}Dx^{a_i})(x^{1-a_j}Dx^{a_j})]f_2(x) = \varphi(x).$$

Hence

$$[(x^{1-a_i}Dx^{a_i})(x^{1-a_j}Dx^{a_j})](f_1 - f_2) = 0,$$

so that

$$f_1 - f_2 = Ax^{-a_i} + Bx^{-a_j}.$$

But, $f_1 - f_2 = 0(x^{\alpha})$ as $x \to 0+$, where $\alpha > \max(|\mathrm{Re}\, a_i|, |\mathrm{Re}\, a_j|)$, $i = 1, \cdots, p$, $j = 1, \cdots, q$ and therefore $A = B = 0$. Thus, $f_1 = f_2$.

LEMMA 2. *Let* $\varphi \in C^{\infty}(R_{+})$ *with the asymptotic order*

$$\varphi^{(k)}(x) = 0(x^{\alpha-k}),\ x \to 0+,\ k = 0, 1, 2, \cdots,$$

where $\alpha > \max(|\mathrm{Re}\, a_i|, |\mathrm{Re}\, b_j|)$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$. *Then the integration operator* $(x^{-a_i}D^{-1}x^{-1+a_i})$ *and the differentiation operator* $(x^{1-d_j}Dx^{d_j})$ *when acting on* φ *in succession are commutative.*

PROOF. The proof can be given by a single computation. For details, see [13, pp. 8–9].

COROLLARY. *The differentiation and integration operators in* Δ_x *and* ∇_x *when acting on* $\varphi \in C^\infty(R_+)$ *with the asymptotic order*

$$\varphi^{(k)}(x) = 0(x^{\alpha-k}),\ x \to 0+,\ k = 0, 1, 2, \cdots,$$

where $\alpha > \max(|\mathrm{Re}\, a_i|, |\mathrm{Re}\, b_j|)$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$, *can be switched in any order.*

PROOF. Since two differentiation operators are commutative, the result follows in view of Lemmas 1 and 2.

3. **The testing function space $G_{a,b}$.** Let I denote the positive half-axis $(0, \infty)$. For $a, b \in R^1$ construct a positive continuous function $\xi_{a,b}(x)$ on R^1 as follows:

$$\xi(x) \equiv \xi_{a,b}(x) \equiv \begin{cases} x^{-a} & 0 < x < 1 \\ x^{-b} & 1 < x < \infty. \end{cases}$$

Then $G_{a,b}$ is defined as the space of all infinitely differentiable complex-valued functions $\varphi(x)$ on I such that for each non-negative integer r

$$\gamma_r(\varphi) \equiv \gamma_{a,b,r}(\varphi) \equiv \sup_{0<x<\infty} |\xi_{a,b}(x)\Delta_x^r \varphi(x)| < \infty, \tag{22}$$

where Δ_x is the integrodifferential operator defined in Section 2. The operator Δ_x can be applied to $\varphi(x)$ provided that for each $k = 0, 1, 2, \ldots$

$$\varphi^{(k)}(x) = 0(x^{\alpha-k}),\ x \to 0+, \tag{23}$$

where $\alpha > \max[|\mathrm{Re}(a_i)|, |\mathrm{Re}(b_j)|]$, $i = 1, \cdots, p$, $j = 1, \cdots, q$. In the case $p = q = 0$ condition (23) is treated as empty.

The γ_r are seminorms on $G_{a,b}$ and γ_0 is a norm. The topology over $G_{a,b}$ can be generated by the separating collection of seminorms $\{\gamma_r\}_{r=0}^\infty$, [16, p. 8] and therefore $G_{a,b}$ is a countably multinormed space. We may say that a sequence $\{\varphi_\nu\}_{\nu=1}^\infty$ where each φ_ν belongs to $G_{a,b}(I)$ converges in $G_{a,b}(I)$ to $\varphi(x)$ if for each fixed r, $\gamma_r(\varphi_\nu - \varphi)$ tends to zero as $\nu \to \infty$. We say that a sequence $\{\varphi_\nu(x)\}_{\nu=1}^\infty$ where each $\varphi_\nu(x)$ belongs to $G_{a,b}(I)$ is a Cauchy sequence in $G_{a,b}$ if $\gamma_r(\varphi_\mu - \varphi_\nu)$ tends to zero for any non-negative integer r as μ and ν both tend to infinity independently of each other. It can be readily seen that $G_{a,b}$ is sequentially complete locally convex, Hausdorff topological vector space. The space $D(I)$, i.e., the space of infinitely differentiable functions having compact supports defined over $I(0 < t < \infty)$, is a subspace of $G_{a,b}(I)$ and the topology of $D(I)$ [14, Vol. I, p. 65] is stronger than the topology induced on $D(I)$ by $G_{a,b}(I)$ and as such the restriction of any member of $G_{a,b}(I)$ to $D(I)$ is in $D'(I)$.

We may compare the space $G_{a,b}(I)$ with other testing function spaces on which Hankel transformable generalized functions are defined. Indeed, by suitable change of variables and specialization of parameters it can be shown that the inversion formulae for the Hankel transform given by Zemanian [**16**], Koh and Zemanian [**6**], Dube and Pandey [**1**] and those for the Hardy transforms established by Pathak and Pandey [**11**, **12**] are special cases of the general inversion formula proved in this paper.

LEMMA 3. *If* $m - q = n - p \geqq 1$, $b \geqq \eta$, $a \leqq \beta_1$, $1 + \beta_1 > \max[\mathrm{Re}(a_i), \mathrm{Re}(b_j)]$, $i = 1, \cdots, p$, $j = 1, \cdots, q$ *and* $t > 0$, *then for fixed* $x > 0$, $k(xt) \in G_{a,b}$, *where* $k(x)$ *is defined by* (5).

PROOF. The result can be proved by using standard technique. For details, see [**13**, p. 11].

4. **The G-transform of generalized functions.** For $f \in G'_{a,b}$, the distributional G-transform can be defined by

$$F(y) = \langle f(x), k(xy)\rangle \tag{24}$$

where x, $y > 0$ and $k(x)$ is the same as defined by (5). From Lemma 3, we know that for fixed $y > 0$, $k(xy) \in G_{a,b}$, hence the relation is meaningful.

THEOREM 1. *For* $y > 0$, *let* $F(y)$ *be the G-transform of* $f \in G'_{a,b}(I)$ *as defined by* (24). *Assume that* $a \leqq \beta_1$ *and* $b \geqq \lambda_2$ *where* β_1 *and* λ_2 *are constants defined by* (9). *Then* $F(y)$ *is differentiable and*

$$F'(y) = \left\langle f(x), \frac{\partial}{\partial y} k(xy) \right\rangle.$$

PROOF. The proof is straightforward; for details, see [**13**, p. 12].

THEOREM 2. *Let* $m - q = n - p > 1$ *and let* $F(y)$ *be the distributional G-transform of* $f \in G'_{a,b}$ *where* $b \geqq \eta$, $a \leqq \beta_1 > -1 + \max[\mathrm{Re}(a_i), \mathrm{Re}(b_j)]$, $i = 1, \cdots, p$, $j = 1, \cdots, q$.
Then

$$F(y) = \begin{cases} 0(y^{\min(a,b)}) & y \to 0+ \\ 0(y^{\max(a,b)+s}) & y \to \infty, \end{cases}$$

where s is a non-negative integer.

PROOF. The proof can be given by using the boundedness property of generalized functions. For details, see [**13**, p. 14].

LEMMA 4. *Let $a \leqq \beta_1$, $b \geqq \eta$, $\min(a, b) + \beta_2 > -1$ and $x, y > 0$, and β_1 and β_2 be constants as defined by* (9). *Let $k(x)$ and $h(x)$ be the functions as defined by* (5) *and* (6) *respectively; then for fixed $x > 0$*

$$\int_0^\epsilon k(ty)h(xy)dy \to 0$$

in $G_{a,b}$ as $\epsilon \to 0+$.

PROOF. The proof is similar to that of [11, Lemma 4] and is therefore omitted. For details, see [13, p. 15].

LEMMA 5. *Let $f \in G'_{a,b}$ under the conditions of Lemma* 1. *Then, for fixed x, $N > 0$ and $\min(a, b) + \beta_2 > -1$,*

$$\int_0^N \langle f(t), k(ty)\rangle\, h(xy)dy = \left\langle f(t), \int_0^N k(ty)h(xy)dy \right\rangle .$$

PROOF. The proof follows by using Riemann sum technique. For details see [13, p. 16].

5. **Inversion of the distributional G-transform.** Let us define

$$G_N(t, x) \equiv \int_0^N k(ty)h(xy)dy \tag{25}$$

where $k(x)$ and $h(x)$ are the same as defined in (5) and (6).

LEMMA 6. *Let $x, y, N > 0$ and $m - q = n - p \geqq 1$, $\beta_2 > -1$,*

$$\mathrm{Re}(a_j - b_h) < 1,\ j = 1, \cdots, p,\ h = i, \cdots, q,$$

$$\mathrm{Re}(c_j - d_h) > -1,\ j = 1, \cdots, m,\ h = 1, \cdots, n,$$

$$-\tfrac{1}{2}(1 + \sigma) < \min(1 + \Delta - \sigma \,\mathrm{Re}\, a_j, -\Delta + \sigma \,\mathrm{Re}\, b_h),$$

$$j = 1, \cdots, p,\ h = 1, \cdots, q.$$

Define

$$\Psi_N(x, y) \equiv \int_0^N k(xu)\,\frac{h_1(yu)}{u}\,du,$$

where

$$h_1(x) = \int_0^x h(u)du = xG^{n,q+1}_{p+q+1,m+n+1}\left(x \,\middle|\, \begin{matrix} 0, -b_q, -a_p \\ -d_n, -c_m, -1 \end{matrix}\right).$$

Then

$$\lim_{N\to\infty} \Psi_N(x, y) = \begin{cases} 1 & (x < y) \\ 0 & (x > y). \end{cases}$$

PROOF. We have

$$\lim_{N\to\infty} \Psi_N(x, y) = \int_0^\infty yG^{m,p}_{p+q,m+n}\left(xu \,\middle|\, \begin{matrix} a_p, b_q \\ c_m, d_n \end{matrix}\right) G^{n,q+1}_{p+q+1,m+n+1}$$

$$\left(yu \,\middle|\, \begin{matrix} 0, -b_q, -a_p \\ -d_n, -c_m, -1 \end{matrix}\right)du$$

$$= G^{10}_{11}\left(\begin{matrix} x \\ y \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right)$$

[7, p. 159 (1)], which is equal to 1 if $x < y$ and equal to 0 if $x > y$. For details, see [13, p. 17].

LEMMA 7. *Let c and d $(d > c)$ be two positive numbers; then under the conditions of Lemma 6,*

$$\lim_{N\to\infty} \int_c^d G_N(t, x)dx = \begin{cases} 1 & t \in (c, d) \\ 0 & t \notin [c, d]. \end{cases}$$

PROOF. In view of Lemma 6, we have

$$\int_c^d G_N(t, x)dx = \int_0^N k(ty)\frac{h_1(dy) - h_1(cy)}{y}dy$$

$$= \Psi_N(t, d) - \Psi_N(t, c).$$

If $c < d < t$, or $t < c < d$, this tends to 0, and if $c < t < d$, this tends to 1, when $N \to \infty$.

LEMMA 8. *For $0 < a \leqq t \leqq b$, $0 < c \leqq x \leqq d$ and $N > 0$, the function $G_N(t, x)$ is bounded uniformly for all x, t, $N > 0$, provided that*

$$\beta_1 \geqq \eta > -\beta_2 - 1, \ \mathrm{Re}(a_j - b_h) < 1$$

and

$$-\tfrac{1}{2}(1 + \sigma) < \min(\Delta - \sigma \,\mathrm{Re}\, a_j, -\Delta + \sigma \,\mathrm{Re}\, b_h),$$
$$j = 1, \cdots, p, h = 1, \cdots, q.$$

PROOF. Let us consider at first the case $0 < N \leqq 1$.

$$|G_N(t, x)| \leqq \int_0^N t^\eta (ty)^{-\eta} k(ty) \mid |y^\eta h(xy)|dy.$$

Since for $\beta_1 \geqq \eta$, $|z^{-\eta}k(z)|$ is bounded by a positive constant B for all $z > 0$, and for $c \leqq x \leqq d$ and $0 < y \leqq 1$, $(xy)^{-\beta_2}|h(xy)|$ is bounded by another constant D. Therefore,

$$|G_N(t, x)| \leqq B\ D \sup_{a \leqq t \leqq b} t^{\eta} x^{\beta_2} \int_0^N y^{\eta+\beta_2} dy$$

$$< B\ D \sup_{c \leqq x \leqq d} x^{\beta_2} \sup_{a \leqq t \leqq b} t^{\eta} \frac{1}{\eta + \beta_2 + 1} < M_1,$$

M_1 being independent of x, t and N. Next, consider the case $N > 1$.

$$|G_N(t, x)| \leqq \left| \int_0^1 k(ty)h(xy)dy \right| + \left| \int_1^N k(ty)h(xy)dy \right|.$$

Since the first integral is bounded by M_1 we consider the second. In view of the asymptotic expansions of $k(ty)$ and $h(xy)$ we can write

$$\int_1^N k(ty)h(xy)dy$$

$$= \int_1^N (ty)^{1/\sigma\{(1/2)(1-\sigma)+\Delta\}}\{\cos(\sigma(ty)^{1/\sigma} + \alpha)(A + 0[(ty)^{-2/\sigma}])$$
$$+ \sin(\sigma(ty)^{1/\sigma} + \alpha)0[(ty)^{-1/\sigma}]\}$$
$$+ (ty)^{\sigma_1-1}\{E + 0[(ty)^{-1}]\}$$
$$\cdot[(xy)^{1/\sigma\{(1/2)(1-\sigma)\Delta\}}\{\cos(\sigma(xy)^{1/\sigma} + \alpha')(A' + 0[(xy)^{-2/\sigma}])$$
$$+ \sin(\sigma(xy)^{1/\sigma} + \alpha')0[(xy)^{-1/\sigma}]$$
$$+ (xy)^{\sigma_2-1}\{E' + 0[(xy)^{-1}]\}]dy,$$

where A, A', E, E', α, α' are certain definite constants. Considering each of the above integrals of the right-hand side separately, it can be shown that each of them is bounded, the bound being independent of t and N provided that

$$\sigma_1 + \sigma_2 < 1,\ 1/\sigma\{-\tfrac{1}{2}(1 + \sigma) + \Delta\}$$
$$+ \sigma_2 < 0,\ 1/\sigma\{\tfrac{1}{2}(1 + \sigma) - \Delta\} + \sigma_1 < 0.$$

See also [13, p. 20].

COROLLARY. *For $0 < c \leqq \alpha \leqq \beta \leqq d$, $0 < a \leqq t \leqq b$ and $N > 0$,*

$$\int_\alpha^\beta |G_N(t, x)|\ dx < \infty.$$

LEMMA 9. *For $0 < 2\ \delta < c < t < b$, $c > 0$,*

$$\int_c^{t-\delta} G_N(t, x)dx \to 0$$

as $N \to \infty$ uniformly for $c < t < b$, provided that $\sigma_1 + \sigma_2 < 1$ and $-(1/2)(\sigma + 1) < \min(-\Delta - \sigma\sigma_2, \Delta - \sigma\sigma_1)$.

PROOF. We have

$$\begin{aligned}\int_c^{t-\delta} G_N(t, x)dx &= \int_c^{t-\delta} dx \int_0^N k(ty)h(xy)dy \\ &= \int_0^N dy \int_c^{t-\delta} k(ty)h(xy)dx \\ &= \int_0^\infty dy \int_c^{t-\delta} k(ty)h(xy)dx \\ &\quad - \int_N^\infty dy \int_c^{t-\delta} k(ty)h(xy)dx.\end{aligned}$$

Since, by Lemma 7, the first integral on the right-hand side is equal to zero,

$$\int_c^{t-\delta} G_N(t, x)dx = - \int_N^\infty dy \int_c^{t-\delta} k(ty)h(xy)dx.$$

Now, therefore, we establish the uniform convergence of the integral on the right. Using the notation of Lemma 6, we have

$$\int_N^\infty dy \int_c^{t-\delta} k(ty)h(xy)dx = \int_N^\infty k(ty) \left[\frac{h_1(t - \delta)y - h_1(cy)}{y} \right] dy.$$

Using the asymptotic estimates of $k(ty)$ and $h_1((t - \delta)y)$ and considering each of the integrals separately as in Lemma 8, it can be shown that for $0 < 2\delta < c < t < b$ and $c > 0$, the right-hand side tends to zero as $N \to \infty$, provided that

$$\sigma_1 + \sigma_2 < 1, \ -\tfrac{1}{2}(\sigma + 1) < -\Delta - \sigma\sigma_2,$$
$$-\tfrac{1}{2}(\sigma + 1) < \Delta - \sigma\sigma_1.$$

See also [13, pp. 21–22].

LEMMA 10. *Let* $0 < t < d - \delta$ *and* $\delta > 0$. *Then,*

$$\int_{t+\delta}^d G_N(t, x)dx \to 0$$

as $N \to \infty$ *uniformly for* $0 < t < d - \delta$ *provided that* $\sigma_1 + \sigma_2 < 1$ *and* $-(1/2)(\sigma + 1) < \min(-\Delta - \sigma\sigma_2, \Delta - \sigma\sigma_1)$.

PROOF. Proof is based on the conclusion of Lemma 7 and the technique is similar to that used in the proof of Lemma 9.

LEMMA 11. *Let* $\varphi(x) \in D(I)$ *and its support be contained in* $[c, d]$ *where* $0 < c < d$. *Let* $c + \delta \leqq t \leqq b$, $\delta > 0$. *Then*

$$\int_c^{t-\delta} G_N(t, x)\varphi(x)dx \to 0$$

as $N \to \infty$ *uniformly for all* $t \in [c + \delta, b]$, *provided that* $\sigma_1 + \sigma_2 < 1$ *and* $-(1/2)(\sigma + 1) < \min(-\Delta - \sigma\sigma_2, \Delta - \sigma\sigma_1)$.

PROOF. By Lemma 8, there exists a constant K such that $|G_N(t, x)| < K$ uniformly for all $x \in [c, d]$, $t \in [c + \delta, b]$ and $N > 0$.

In view of the uniform continuity of $\varphi(x)$ in $c \leqq x \leqq d$, for a given arbitrary $\epsilon > 0$, we can find a continuous function $\chi(x)$ such that

$$\int_c^{t-\delta} |\varphi(x) - \chi(x)|dx \leqq \int_c^{b-\delta} |\varphi(x) - \chi(x)|dx < \frac{\epsilon}{K}.$$

The interval $(c, t - \delta)$ may be divided into sub-intervals (c, x_1), (x_1, x_2), $\cdots$, $(x_{n-1}, t - \delta)$, so chosen that the fluctuation of $\chi(x)$ in each of these sub-intervals is less than $\epsilon/K(b - \delta - c)$. Let $\psi(x)$ be a function which, in the interior of each part (x_{r-1}, x_r), where $r = 1, 2, \cdots, n$, has the constant value $c_r = \chi(x_r + x_{r-1})/2$. At the extremities of the parts, we take $\psi(x)$ to have the value zero. Thus, $\psi(x)$ has the finite set of values $c_1, c_2, \cdots, c_n, 0$.

Since $|\chi(x) - \psi(x)| < \epsilon/K(b - \delta - c)$ everywhere except at the end points of n sub-intervals of $(c, t - \delta)$, we have

$$\int_c^{t-\delta} |\chi(x) - \psi(x)|dx < \frac{\epsilon}{K}$$

and therefore

$$\int_c^{t-\delta} |\varphi(x) - \psi(x)|dx < \frac{2\epsilon}{K}.$$

Now,

$$\begin{aligned}
&\left| \int_c^{t-\delta} \varphi(x)G_N(t, x)dx \right| \\
&\quad \leqq \left| \int_c^{t-\delta} \{\varphi(x) - \psi(x)\}G_N(t, x)dx \right| \\
&\quad\quad + \left| \sum_{r=1}^{n} c_r \int_{x_{r-1}}^{x_r} G_N(t, x)dx \right| \\
&\quad \leqq \left| \int_c^{t-\delta} \right| \varphi(x) - \psi(x) \left| \, \right| G_N(t, x) \left| dx \right. \\
&\quad\quad + \sum_{r=1}^{n} |c_r| \left| \int_{x_{r-1}}^{x_r} G_N(t, x)dx \right| \\
&\quad < 2\epsilon + \sum_{r=1}^{n} |c_r| \left| \int_{x_{r-1}}^{x_r} G_N(t, x)dx \right|.
\end{aligned}$$

Since t lies outside the interval $[x_{r-1}, x_r]$ for each $r = 1, 2, 3, \cdots$, in view of Lemma 10,

$$\left| \int_{x_{r-1}}^{x_r} G_N(t, x)dx \right| \to 0$$

independently of t for all $t \in [c + \delta, b]$ as $N \to \infty$. A positive number N_ϵ (not depending on x) can be so chosen that

$$\left| \int_{x_{r-1}}^{x_r} G_N(t, x)dx \right| < \frac{\epsilon}{\sum_{r=1}^{n} |c_r|}, \text{ for } r = 1, 2, \cdots,$$

and for all values of $t \in [c + \delta, b]$. Thus, $|\int_c^{t-\delta} \varphi(x)G_N(t, x)dx| < 3\,\epsilon$, provided $N \geqq N_\epsilon$, for all values of $t \in [c + \delta, b]$.

LEMMA 12. *Let* $\varphi(x) \in D(I)$ *and its support be contained in* $[c, d]$, *where* $0 < c < d$. *Let* $0 < t < d - \delta$, $c > 2\,\delta > 0$. *Then*

$$\int_{t+\delta}^{d} G_N(t, x)\varphi(x)dx \to 0$$

as $N \to \infty$ *uniformly for all* $t \in (0, d - \delta)$ *provided that* $\sigma_1 + \sigma_2 < 1$ *and* $(1/2)(\sigma + 1) < \min(-\Delta - \sigma\sigma_2, \Delta - \sigma\sigma_1)$.

PROOF. Assume at first that $\varphi(x)$ is an infinitely differentiable real valued function defined on $[t + \delta, d]$, $0 < t < d - \delta$. Then $\varphi(x)$ is a function of bounded variation on $[t + \delta, d]$ [17, p. 118, Ex. b]. Consequently, there exist monotonically increasing functions $p(x)$ and $q(x)$ on $[t + \delta, d]$, with $p(t + \delta) = q(d) = 0$ such that [17, p. 120, Theorem 6.27]

$$\varphi(x) = p(t + \delta) + p(x) - q(x) \qquad (t + \delta \leqq x \leqq d).$$

Hence

$$\int_{t+\delta}^{d} G_N(t, x)\varphi(x)dx = p(t + \delta) \int_{t+\delta}^{d} G_N(t, x)dx + \int_{t+\delta}^{d} p(x)G_N(t, x)dx - \int_{t+\delta}^{d} q(x)G_N(t, x)dx.$$

The result now can be proved by using mean value theorem of integral calculus followed by a variation of technique used in the proof of Lemmas 9 and 10. For further details, see [13, pp. 25–26].

The proof for infinitely differentiable complex valued function $\varphi(x)$ can be given by separating it into its real and imaginary parts.

LEMMA 13. *Suppose that $\varphi(x) \in D(I)$ and its support is contained in $[c, d]$. Let the conditions of Lemmas 6 and 8 be satisfied. Then, for $\beta_1 \geqq \eta$, $b > \eta > -\beta_2 - 1$, and $a \leqq 0$,*

$$\int_c^d G_N(t, x)\varphi(x)dx \to \varphi(t)$$

in $G_{a,b}$ as $N \to \infty$.

PROOF. Using the properties of the operators Δ_x and ∇_x, we have

$$\begin{aligned}\xi(t)\Delta_t^r \int_c^d \varphi(x)dx \int_0^N k(ty)h(xy)dy \\ &= \xi(t) \int_c^d \varphi(x)dx \int_0^N k(ty)\nabla_x^r h(xy)dy \\ &= \xi(t) \int_c^d \varphi_r(x)G_N(t, x)dx\end{aligned}$$

(integration by parts) where $\varphi_r(x) \equiv \Delta_x^r\varphi(x)$.

Therefore, in view of Lemma 7, we need only show that $I \to 0$ uniformly for all $t > 0$ as $N \to \infty$, where I is the expression defined below:

$$\begin{aligned}I \equiv \xi(t) &\left[\int_c^d \varphi_r(x)G_N(t, x)dx - \varphi_r(t) \right] \\ = \xi(t) &\int_c^d [\varphi_r(x) - \varphi_r(t)]G_N(t, x)dx.\end{aligned}$$

Now, using the standard technique as used in [11], it can be shown that $I \to 0$ as $N \to \infty$ uniformly for all $t > 0$. For further details, see [13, pp. 27–32].

THEOREM 3. *Assumptions:*

(i) $\sigma/2 \equiv m - q = n - p \geqq 1$,

(ii) $\Delta \equiv \mathrm{Re}\left(\sum_1^m c_j + \sum_1^n d_j - \sum_1^p a_j - \sum_1^q b_j \right)$,

(iii) $\mathrm{Re}(a_j - b_h) < 1$, $j = 1, \cdots, p$, $h = 1, \cdots, q$,

(iv) $\mathrm{Re}(c_j - d_h) > -1$, $j = 1, \cdots, m$, $h = 1, \cdots, n$,

(v) $\mathrm{Re}\, c_j \geqq \max[1/\sigma\{(1/2)(1 - \sigma) + \Delta\}, \mathrm{Re}(a_i - 1)]$,
$j = 1, \cdots, m$, $i = 1, \cdots, p$,

(vi) $-(1/2)(1 + \sigma) < \min(\Delta - \sigma\, \mathrm{Re}\, a_j, -\Delta + \sigma\, \mathrm{Re}\, b_h)$,
$j = 1, \cdots, p$, $h = 1, \cdots, q$,

(vii) $\min \operatorname{Re}(c_h) > \max[\operatorname{Re} a_i, \operatorname{Re} b_j]$,
$h = 1, \cdots, m,\ i = 1, \cdots, p,\ j = 1, \cdots, q,$

$\min \operatorname{Re}(-d_h) > \max[\operatorname{Re}(-a_i), \operatorname{Re}(-b_j)]$,
$h = 1, \cdots, n,\ i = 1, \cdots, p,\ j = 1, \cdots, q.$

The condition (vii) *is treated as empty in case* $p = q = 0$.

(viii) $a \leqq \min(\operatorname{Re} c_h, 0),\ h = 1, \cdots, m,$

$b > \max[1/\sigma\{(1/2)(1-\sigma) + \Delta\},$
$\operatorname{Re}(a_j - 1)] > -1 - \min \operatorname{Re}(-d_h),$
$j = 1, \cdots, p,\ h = 1, \cdots, n,$

$\min(a, b) > -\min \operatorname{Re}(-d_h) - 1,$
$\min \operatorname{Re}(-d_h) > -1,\ h = 1, \cdots, n,$

(ix) $F(y)$ *is the distributional G-transform of* $f \in G'_{a,\,b}$ *defined by*

$$F(y) = \left\langle f(x),\ G^{m,p}_{p+q,m+n}\left(xy \,\middle|\, \begin{matrix} a_p, b_q \\ c_m, d_n \end{matrix}\right)\right\rangle.$$

Conclusion: For each $\varphi(t) \in D(I)$,

$$\lim_{N\to\infty} \left\langle \int_0^N F(y) G^{n,q}_{p+q,m+n}\left(xy \,\middle|\, \begin{matrix} -b_q, -a_p \\ -d_n, -c_m \end{matrix}\right) dy,\ \varphi(x)\right\rangle = \langle f(t), \varphi(t)\rangle. \tag{26}$$

PROOF. Assume that the support of $\varphi(x)$ is contained in the interval $[c, d]$, $d > c > 0$. The result (26) will be proved by justifying the steps in the following manipulations.

$$\left\langle \int_0^N F(y)h(xy)dy,\ \phi(x)\right\rangle \tag{27}$$
$$= \int_c^d \int_0^N F(y)h(xy)\varphi(x)dydx$$

$$= \int_c^d \varphi(x)dx \int_0^N \langle f(t), k(ty)\rangle h(xy)dy \tag{28}$$

$$= \int_c^d \left\langle f(t), \int_0^N k(ty)h(xy)dy\right\rangle \varphi(x)dx \tag{29}$$

$$= \int_c^d \langle f(t), G_N(t, x)\rangle \varphi(x)dx \tag{30}$$

$$= \left\langle f(t), \int_c^d G_N(t, x)\varphi(x)dx\right\rangle \tag{31}$$

$$\to \langle f(t), \varphi(t)\rangle. \tag{32}$$

The equality of expressions (27) and (28) is obvious in view of Theorems 1 and 2. That expression (28) equals (29) follows from Lemma 5. The fact that expression (29) equals (30) is obvious. By following a technique very similar to that used in proving Lemma 5, one can show the equality of (30) and (31). Lastly, expression (31) goes to that in (32) as $N \to \infty$, by Lemma 13. This completes the proof of the theorem.

An immediate consequence of the above inversion theorem is the following uniqueness theorem.

THEOREM 4. *Let the distributional G-transforms of* $f, g \in G'_{a,b}(I)$ *be* $F(y)$ *and* $G(y)$ *respectively and assume that* $F(y) = G(y)$ *for all* $y > 0$. *Then* $f = g$ *in the sense of equality over* $D(I)$.

6. **Some special cases of the inversion Theorem 3.** By specializing the parameters in the definition of the kernel $k(x)$, a number of known as well as unknown inversion theorems can be deduced as corollaries to Theorem 3. A few of them are cited below. In all of the following cases, the definition of the space $G_{a,b}$ is to be modified according to the specialization of the orders and parameters.

Taking $p = q$, $m = n$, $a_j + b_j = 0$ for $j = 1, \cdots, p$ and $c_h + d_h = 0$ for $h = 1, \cdots, m$ in Theorem 3, we arrive at the following extension of the inversion theorem established by Fox [3] for the symmetric G-transform.

COROLLARY 1. *Assumptions:*

(i) $\sigma/2 \equiv m - p \geqq 1$,

(ii) $\mathrm{Re}(a_j) < 1/2$, $j = 1, \cdots, p$, $\mathrm{Re}(c_h) > -1/2$, $h = 1, \cdots, m$,

$$\mathrm{Re}(c_h) \geqq \max[1/2\sigma(1-\sigma), \mathrm{Re}(a_j - 1)], \quad h = 1, \cdots, m,\ j = 1, \cdots, p,$$

(iii) $\min \mathrm{Re}\, c_h > \max|\mathrm{Re}\, a_j|$, $h = 1, \cdots, m$, $j = 1, \cdots, p$.
Condition (iii) *is treated as empty if* $p = 0$.

(iv) $a \leqq \min(\mathrm{Re}\, c_h, 0)$, $h = 1, \cdots, m$,

$$b > \max[(1/2)(1/\sigma - 1), \mathrm{Re}(a_j - 1)] > -1 - \min \mathrm{Re}\, c_h, \quad j = 1, \cdots, p,\ h = 1, \cdots, m,$$

$$\min(a, b) > -\mathrm{Re}\, c_h - 1, \quad h = 1, \cdots, m,$$

(v) $F(y)$ *is the distributional G-transform of* $f \in G'_{a,b}$ *defined by*

$$F(y) = \left\langle f(x), G^{m,p}_{2p,2m}\left(xy \,\middle|\, \begin{matrix} a_p, & -a_p \\ c_m, & -c_m \end{matrix}\right)\right\rangle .$$

Conclusion: For each $\varphi(t)$ *in* $D(I)$, *we have*

$$\lim_{N\to\infty} \left\langle \int_0^N F(y) G^{m,p}_{2p,2m}\left(xy \,\middle|\, \begin{matrix} a_p, & -a_p \\ c_m, & -c_m \end{matrix} \right) dy,\ \varphi(x) \right\rangle$$
$$= \langle f(t),\ \varphi(t)\rangle.$$

An interesting special case of this corollary is a generalization of Hankel transform to distributions, which has been studied by Zemanian [**16**], Koh and Zemanian [**6**], Dube and Pandey [1] and others. This follows on setting $m = 1$, $p = 0$ and $c_1 = \nu/2$. In fact, the inversion theorems established by these authors can be deduced from the present work by a suitable change of variables.

COROLLARY 2. *Let* $f \in G'_{a,b}$ *where* $a \leqq \min((1/2)\mathrm{Re}\,\nu,\ 0)$, $b > -1/4$, $\min(a, b) > -(1/2)\mathrm{Re}\,\nu - 1$, $\mathrm{Re}\,\nu \geqq -(1/2)$ *and let* $F(y)$ *be the distributional Hankel transform of* $f \in G'_{a,b}$ *defined by* $F(y) = \langle f(x), J_\nu(2(xy)^{1/2})\rangle$. *Then, for each* $\varphi(t) \in D(I)$,

$$\lim_{N\to\infty} \left\langle \int_0^N F(y) J_\nu(2(xy)^{1/2}) dy,\ \varphi(x) \right\rangle$$
$$= \langle f(t),\ \varphi(t)\rangle.$$

Setting $m = 2$, $p = 0$, $q = 1$, $n = 1$, $b_1 = -\nu/2 - \alpha$, $c_1 = -\nu/2$, $c_2 = \nu/2$ and $\alpha_1 = -\nu/2 - \alpha$ in Theorem 3, we arrive at an extension of the Hardy transform [4] which has been given earlier by Pathak and Pandey [11].

COROLLARY 3. *Let* $f \in G'_{a,b}$ *where* $a \leqq -(1/2)|\mathrm{Re}\,\nu|$, $b > -(1/4)$, $\min(a, b) > -(1/4)\mathrm{Re}(\nu + 2\alpha) - 1$, $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\nu + \alpha) > -1$, $-1/2 \leqq \mathrm{Re}\,\nu \leqq 1/2$, $|\mathrm{Re}(\nu + 2\alpha)| < 3/2$ *and let* $F(y)$ *be the distributional Hardy transform of* $f \in G'_{a,b}$ *defined by*

$$F(y) = \langle f(x),\ C_\nu(2(xy)^{1/2})\rangle$$

where

$$C_\nu(x) = \cos(\alpha\pi) J_\nu(x) + \sin(\alpha\pi) Y_\nu(x).$$

Then, for each $\varphi(t) \in D(I)$,

$$\lim_{N\to\infty} \left\langle \int_0^N F(y) F_\nu(2(xy)^{1/2}) dy,\ \varphi(x) \right\rangle$$
$$= \langle f(t),\ \varphi(t)\rangle,$$

where

$$F_\nu(x) = (x/2)^{\nu+2\alpha}\ {}_1F_2\left({}^{\ \ \ \ \ \ 1}_{1+\alpha,1+\alpha+\nu};\ -\tfrac{1}{4}x^2\right).$$

Setting $m = p = 1$, $q = 0$, $n = 2$, $a_1 = \nu/2 + \alpha$, $c_1 = \nu/2 + \alpha$, $d_1 = \nu/2$ and $d_2 = -\nu/2$ in Theorem 3 leads to the following extension of the Hardy transform which has also been studied by Pathak and Pandey [**12**].

COROLLARY 4. *Let* $f \in G'_{a,b}$ *where* $a \leqq \min((1/2)\mathrm{Re}(\nu + 2\alpha), 0)$, $b > \max[-1/4, (1/2)\mathrm{Re}(\nu + 2\alpha) - 1]$, $\min(a, b) > (1/2)|\mathrm{Re}\,\nu| - 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\nu + \alpha) > 0$, $-1/2 \leqq \mathrm{Re}(\nu + 2\alpha) < 3/2$, $-3/2 \leqq \mathrm{Re}\,\nu \leqq 3/2$, *and let* $F(y)$ *be the distributional Hardy transform of* f *defined by*

$$F(y) = \langle f(x), F_\nu(2(xy)^{1/2})\rangle.$$

Then, for each $\varphi(t) \in D(I)$,

$$\lim_{N\to\infty} \left\langle \int_0^N F(y)C_\nu(2(xy)^{1/2})dy, f(x) \right\rangle$$

$$= \langle f(t), \varphi(t)\rangle.$$

By taking $\alpha = 1/2$ in Corollaries 3 and 4, we can obtain inversion formulae for the distributional Y_ν – and H_ν – transforms respectively.

COROLLARY 5. *Let* $f \in G'_{a,b}$ *where* $a \leqq -(1/2)|\mathrm{Re}\,\nu|$, $b > -1/4$, $\min(a, b) > -(1/2)\,\mathrm{Re}\,\nu - 3/2$, $-1/2 \leqq \mathrm{Re}\,\nu \leqq 1/2$, *and let* $F(y)$ *be the distributional Y-transform of* $f \in G'_{a,b}$ *defined by*

$$F(y) = \langle f(x), Y_\nu(2(xy)^{1/2})\rangle.$$

Then, for each $\varphi(t) \in D(I)$.

$$\lim_{N\to\infty} \left\langle \int_0^N F(y)H_\nu(2(xy)^{1/2})dy, \varphi(x) \right\rangle$$

$$= \langle f(t), \varphi(t)\rangle.$$

COROLLARY 6. *Let* $f \in G'_{a,b}$ *where* $a \leqq \min((1/2)\mathrm{Re}(\nu + 1), 0)$, $b > \max[-1/4, (1/2)\mathrm{Re}(\nu - 1)]$, $\min(a, b) > (1/2)|\mathrm{Re}\,\nu| - 1$, $-1/2 \leqq \mathrm{Re}\,\nu \leqq 1/2$, *and let* $F(y)$ *be the distributional H-transform of* f *defined by*

$$F(y) = \langle f(x), H_\nu(2(xy)^{1/2})\rangle.$$

Then, for each $\varphi(t) \in D(I)$,

$$\lim_{N\to\infty} \left\langle \int_0^N F(y)Y_\nu(2(xy)^{1/2})dy, \varphi(x) \right\rangle$$

$$= \langle f(t), \varphi(t)\rangle.$$

7. In the following theorem, a structure formula for the restriction of an element $f \in G'_{a,b}(I)$ to $D(I)$ is given.

THEOREM 5. *Let f be an arbitrary element of $G'_{a,b}(I)$. Then there exist bounded measurable functions $g_i(x)$ defined for $x > 0$ for $i = 0, 1, 2, \cdots, r$ where r is some non-negative integer depending upon f such that for arbitrary $\varphi \in D(I)$ we have*

$$\langle f, \varphi\rangle = \left\langle \sum_{i=0}^{r} (-1)^i \nabla_x^i \left[\xi(x) D_x^2 \int_0^x g_i(t)dt \right], \varphi(x) \right\rangle,$$

where ∇_x is the integrodifferential operator defined by (20).

PROOF. The proof can be given by using standard technique [1]. For details, see [13, p. 39].

REFERENCES

1. L. S. Dube and J. N. Pandey, *On the Hankel transform of distributions*, Tôhoku Math. Journ. **27** (1975), 337–354.

2. A. Erdelyi et al, *Higher transcendental functions*, Vol. 1, McGraw-Hill, New York, 1953.

3. C, Fox, *The G and H functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc. **98** (1961), 395–429.

4. G. H. Hardy and E. C. Titchmarsh, *A class of Fourier kernels*, Proc, London Math, Soc. **35** (1933), 116–155.

5. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series* Vols. I and II, Harren Press, Washington, D.C. 1950.

6. E. Koh and A. H. Zemanian, *The complex Hankel and I transformations of generalized functions*, SIAM J. Appl. Math. **16** (1968), 945–957.

7. Y. L. Luke, *The special functions and their approximations*, Vol. 1, Academic Press, 1969.

8. R. Narain (Kesarwani), *The G-functions as unsymmetrical Fourier kernels* I, Proc. Amer. Math. Soc. **13** (1962), 950–959.

9. ———,*The G-functions as unsymmetrical Fourier kernels* II, Proc. Amer. Math, Soc. **14** (1963), 18–28.

10. J. N. Pandey and A. H. Zemanian, *Complex inversion for the generalized convolution transformation*, Pacific J. Math **25** (1968), 147–157.

11. R. S. Pathak and J. N. Pandey, *A distributional Hardy transformation*, Proc. Camb. Phil. Soc. **76** (1974), 247–262.

12. ———, *A distributional Hardy transformation* II, Communicated for publications.

13. ———, *The G-transform of generalized functions*, Carleton Math. Series.

14. L. Schwartz, *Theorie des distributions*, Vols. I and II, Hermann, Paris, 1957, 1959.

15. E. C. Titchmarsh, *A pair of inversion formulae*, Proc. London Math. Soc. **22** (1923), xxxiv-xxxv.

16. A. H. Zemanian, *Generalized integral transformations*, Interscience, New York, 1968.

17. W. Rudin, *Principles of Mathematical Analysis*, 2nd ed., McGraw-Hill, New York, 1964.

BANARAS HINDU UNIVERSITY, VARANASI 221005, INDIA
CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA