ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 3, Summer 1978

LARGE ABELIAN SUBGROUPS OF SOME INFINITE GROUPS, II

V. FABER

1. **Introduction.** This paper extends the work of [2], [3], and [4]. The main theorems are:

THEOREM 1. *If G is an infinite group of cardinality m^+ having a (strictly increasing) normal series (G_α) with $|G_\alpha| \leqq m$ for all α, then every maximal abelian subgroup A of G satisfies $m^{|A|} \geqq |G|$.*

THEOREM 6. *Every infinite FCN* group has an abelian subgroup A such that* $\exp |A| \geqq |G|$.

THEOREM 9. *Every maximal abelian subgroup A of an uncountable ZA group G satisfies* $\exp |A| \geqq |G|$.

THEOREM 13. *Every infinite SI* group G has an abelian subgroup $A \lhd^2 G$ such that* $\exp^2|A| \geqq |G|$.

Theorem 6 sharpens Theorem 8 in [3]. Theorem 9 sharpens Corollary 1 in [2]. Theorem 13 sharpens Theorem 2 in [10]. Some of the results of this paper were previously announced in [5].

2. **Notation.** Let S and T be sets. $S < T$ always means strict inclusion. The cardinality of S is denoted by $|S|$. If m is an infinite cardinal, $\Omega(m)$ is the initial ordinal of cardinality m; m^+ is the first cardinal greater than m; $\exp^1 m = \exp m = 2^m$; $\exp^{n+1} = \exp \exp^n m$; and $\Omega(\aleph_\beta) = \omega_\beta$. The *cofinality* of an ordinal γ (cardinal m) is the first cardinal n such that γ (m) is the sum of n smaller ordinals (cardinals); we denote this by $n = cf(\gamma)$ ($cf(m)$); m is a *regular* cardinal if $cf(m) = m$ and *singular* otherwise. A *stationary subset* of ω_α is a subset which meets every closed unbounded subset.

If G is a group and H is a subgroup, we write $H \lhd^n G$ if there is an ascending normal series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

from H to G. If H is characteristic in G write $H \,\square\, G$. $C(H) = C_G(H) = C(H \leqq G)$ denotes the centralizer of H in G, while $N(H) = N_G(H) = N(H \leqq G)$ denotes the normalizer of H in G. The automorphism group of G is denoted by Aut(G). $Z_\alpha(G)$ will denote the α^{th} member of

Received by the editors on February 25, 1975, and in revised form on April 26, 1976.

the ascending central series. G is a *ZA group* if it is the union of all the terms of its ascending central series. The *class* of a ZA group is the first α such that $Z_\alpha(G) = G$. G is *nilpotent* if it is a ZA group with finite class. If $x, y \in G$, the commutator of x and y is $[x, y] = x^{-1}y^{-1}xy$. We denote the derived (commutator) group of G by G^1. Let $G^{n+1} = (G^n)^1$. If G is solvable, the *length* of G is the first n such that $G^n = E$, where $E = \{1\}$ denotes the identity subgroup.

Let $F_1(G)$ be the set of all elements in G which have at most a finite number of conjugates in G. Following [7], the *ascending FC series* of G is the series

$$E = F_0(G) \square F_1(G) \square \cdots \square F_\alpha(G) \square \cdots$$

where $F_{\alpha+1}(G)/F_\alpha(G) = F_1(G/F_\alpha(G))$, and if β is a limit ordinal, then $F_\beta = \bigcup_{\alpha<\beta} F_\alpha(G)$. If $F_1(G) = G$, G is an *FC group*. If $F_\alpha(G) = G$ for some α, then G is a *ZFC group*; if α is an integer, G is *FC nilpotent*.

If m is an infinite cardinal, let $M_1(G)$ be the set of all elements in G which have at most m conjugates in G. By analogy with the ascending FC series, we define the *ascending mC series* of G to be the series

$$E = M_0(G) \square M_1(G) \square \cdots \square M_\alpha(G) \square \cdots$$

where $M_{\alpha+1}(G)/M_\alpha(G) = M_1(G/M_\alpha(G))$, and if β is a limit ordinal, then $M_\beta(G) = \bigcup_{\alpha<\beta} M_\alpha(G)$. If $M_1(G) = G$, G is an *mC group*; if $M_\alpha(G) = G$ for some α, then G is a *ZmC group*.

We assume the terminology of § 57 and § 63 of [11] to denote various classes of generalized solvable and nilpotent groups. In addition, if $\mathfrak{X}$ is a class of groups, $\mathfrak{X}\, I^*$ is the class of all groups having an ascending invariant series with factors in $\mathfrak{X}$; $\mathfrak{X}\, J^*$ is the class of all groups having an ascending subnormal series with factors in $\mathfrak{X}$; and $\mathfrak{X}\, N^*$ is the class of all groups having an ascending normal series with factors in $\mathfrak{X}$. Additional terminology and facts concerning infinite groups can be found in [14] and [15].

3. Large abelian subgroups.

Lemma 1. *Let G be an infinite group and H a subgroup. If $H \leqq N < G$ and $|N|^{|H|} < |G|$, then $|C(H)| = |G|$.*

Proof. If N is finite, $|C(N)| = |G|$. If N is infinite, $[G : C(H)] = [G : N(H)]\,[N(H) : C(H)] \leqq |\mathrm{Cl}(H)|\,|\mathrm{Aut}\, H| \leqq |N|^{|H|} < |G|$.

Theorem 1. *If G is a group of uncountable regular cardinality having a (strictly increasing) normal series (G_α) with $|G_\alpha| < |G|$ and if H is a subgroup of G with $|G_\alpha|^{|H|} < |G|$ for all α, then $|C(H)| = |G|$. In particular, every maximal abelian subgroup A of G satisfies $m^{|A|} \geqq |G|$ for some $m < |G|$.*

PROOF. Let $\theta = \Omega(|G|)$. Well-order $G = \{x_\alpha \mid \alpha < \theta\}$ so that (1) $x_\epsilon \in G_{\epsilon+1}$ and (2) $x_\rho \in G_\epsilon$ implies that $x_\sigma \in G_\epsilon$ for all $\sigma \leqq \rho$. Since $|H| < |G|$, $H \leqq \{x_\alpha \mid \alpha < \gamma\}$ for some $\gamma < \theta$. For each $h \in H$ and ϵ such that $x_\epsilon \notin G_{\gamma+1}$, let $\mu_h(\epsilon)$ be defined by $x_{\mu_h(\epsilon)} = [h, x_\epsilon]$. Let σ be the first ordinal such that $x_\epsilon \in G_{\sigma+1}$. Since $\{x_\alpha \mid \alpha < \gamma\} \leqq G_\sigma \lhd G_{\sigma+1}$, $[h, x_\epsilon] \in G_\sigma$. If $\epsilon \leqq \mu_h(\epsilon)$, then $x_\epsilon \in G_\sigma$, a contradiction. It follows that $\mu_h(\epsilon) < \epsilon$. Let $S = \{\epsilon \mid \epsilon > \gamma \text{ and } \mathrm{cf}(\epsilon) > |H|\}$. Let

$$\mu(\epsilon) = \sup_{h \in H} \mu_h(\epsilon);$$

$\mu(\epsilon) < \epsilon$ for $\epsilon \in S$ since $\mathrm{cf}(\epsilon) > |H|$. Now $|H|^+ < |G|$, otherwise $2^{|H|} = |H|^{|H|} \geqq |G|$, so $S \supseteq \{\alpha < \theta \mid \mathrm{cf}(\alpha) = |H|^+ \text{ and } \alpha > \gamma\}$. If C is any closed unbounded subset of θ and if α is the first member of C such that $|\{\beta \in C \mid \gamma < \beta < \alpha\}| = |H|^+$, then α is the sum of the ordinals $\beta \in C$ with $\beta < \alpha$ since C is closed and hence $\mathrm{cf}(\alpha) = |H|^+$ since $|H|^+$ is regular. Thus $\alpha \in S$ and it follows that S is a stationary subset of θ. By [**13**; p. 260], there exists $T \leqq S$ such that μ is constant on T and $|T| = |G|$. (In fact, by [**6**, p. 141], T can be taken to be a stationary subset of G.) Hence there exists $\tau < \theta$ such that $[h, x_\epsilon] \in G_\tau$ for all $\epsilon \in T$ and $h \in H$. Define an equivalence relation $\sim$ on T by $\rho \sim \sigma$ if and only if for all $h \in H$, $[h, x_\rho] = [h, x_\sigma]$. Since the number of equivalence classes is less than or equal to the number of subsets of G_τ of cardinality $|H|$, there are at most $|G_\tau|^{|H|} < |G|$ equivalence classes and one must have cardinality $|G|$, that is, there exists $U \leqq T$ with $|U| = |G|$ such that for all $\rho, \sigma \in U$ and for every $h \in H$, $[h, x_\rho] = [h, x_\sigma]$. Then $x_\sigma x_\rho^{-1} h x_\rho x_\sigma^{-1} = h$ for all $\rho, \sigma \in U$ and $h \in H$, so $\langle x_\sigma x_\rho^{-1} \mid \sigma, \rho \in U\rangle \leqq C(H)$. This shows that $|C(H)| = |G|$. If A is a maximal abelian subgroup with $|A| < |G|$ and $|G_\alpha|^{|A|} < |G|$ for all α, then $|A| = |C(A)| = |G|$, a contradiction.

REMARK 1. R. Laver and the author have observed that if one assumes the generalized continuum hypothesis (G.C.H.), one gets the following stronger theorem.

THEOREM 1′. *Let G be an uncountable group with a (strictly increasing) normal series (G_α) with $|G_\alpha| < |G|$. If H is any subgroup of G with $|H| < \mathrm{cf}|G|$ and $|G_\alpha|^{|H|} < |G|$ for all α, then $|C(H)| = |G|$. In particular, every maximal abelian subgroup A of G satisfies either (1) $|A| \geqq \mathrm{cf}|G|$ or (2) $m^{|A|} \geqq |G|$ for some $m < |G|$.*

PROOF. Suppose G is a counterexample of smallest (singular) cardinality. Since $|H| < \mathrm{cf}|G|$, there is some α such that $H \leqq G_\alpha$, $|H| < |G_\alpha|$ and $|C(H)| < |G_\alpha|$. There are two cases. If there exists a $\gamma \geqq \alpha$ such that $|G_{\gamma+1}| \geqq |G_\gamma|^{++}$, then by G.C.H. $|G_\gamma|^{|H|} < |G_{\gamma+1}|$. Thus Lemma 1 applies and $|C(H)| \geqq |G_{\gamma+1}| \geqq |G_\alpha|$, a contradiction. On the other hand, if no such γ exists, there is a first ordinal μ such that $|G_\mu| = |G_\alpha|^{++}$. By

G.C.H., if $\beta < \mu$, then $|G_\beta|^{|H|} \leqq (|G_\alpha|^+)^{|H|} = |G_\alpha|^+ < |G_\mu|$, so Theorem 1 applies and yields $|C(H)| \geqq |G_\mu| \geqq |G_\alpha|$, a contradiction.

THEOREM 2. *Let G be a group with* $|G| > m$. *Suppose G has a normal series* (G_α) *such that* $[G_{\alpha+1} : G_\alpha] \leqq m$ *for all* $\alpha + 1$. *Then G has an abelian subgroup A such that* $m^{|A|} \geqq |G|$.

PROOF. Let $\aleph = \sup\{|A| \mid A \text{ abelian}\}$ and suppose $n = \sup\{(m^{|A|})^+ \mid A$ abelian$\} \leqq |G|$. If there exists A such that $|A| = \aleph$, then $n = (m^\aleph)^+$. By Theorem 1, $G_{\Omega(n)}$ has an abelian subgroup A such that $|G_{\Omega(n)}| \leqq (m^\aleph)^{|A|} \leqq m^\aleph$, contradicting $|G_{\Omega(n)}| = (m^\aleph)^+$. On the other hand, if $\sup_\alpha m_\alpha = \aleph$, every maximal abelian subgroup A of $G_{\Omega((m^{m_\alpha})^+)}$ has $|A| > m_\alpha$ so we can find an abelian subgroup of power $\aleph$.

QUESTION 1. Suppose G is a group with $|G| > m$. If G has a normal series $(G_\alpha \mid \alpha < \lambda)$ with λ a limit ordinal and $[G_{\alpha+1} : G_\alpha] \leqq m$, does every maximal abelian subgroup satisfy $m^{|A|} \geqq |G|$?

THEOREM 3. *Every infinite SN* group G has an abelian subgroup A such that* $\exp|A| \geqq |G|$.

PROOF. G has a normal series with countable factors. If G is countable, it has an infinite abelian subgroup by [**10**; p. 243]. If G is uncountable, Theorem 2 yields the result.

EXAMPLE 1. ([**15**, p. 454]) For every cardinal $\aleph_\alpha$ there is a two-step solvable group G_α with $|G_\alpha| = \aleph_\alpha$ and having a finite maximal abelian subgroup. Let $H_\epsilon = \langle x_\epsilon \rangle \cdot \langle y_\epsilon \rangle$ with $x_\epsilon^{\,2} = 1 = y_\epsilon^{\,2}$. Then $H_\alpha = \sum_{\epsilon < \omega_\alpha} H_\epsilon$ has an automorphism a_α of order three given by $a_\alpha(x_\epsilon) = y_\epsilon$ and $a_\alpha(y_\epsilon) = x_\epsilon y_\epsilon$. If G_α is the split extension $\langle a_\alpha \rangle H_\alpha$, $\langle a_\alpha \rangle$ is a maximal abelian subgroup.

THEOREM 4. [4, p. 31] *If G is an infinite FC group, every maximal abelian subgroup A has* $\exp|A| \geqq |G|$. *If G is an infinite mC group, every maximal abelian subgroup A has* $m^{|A|} \geqq |G|$.

PROOF. Since $A \leqq A^G \triangleleft G$, Lemma 1 applies. If $|A| < |G|$, then $|G| \leqq |A^G|^{|A|} \leqq |\langle \mathrm{Cl}(x) \mid x \in A \rangle|^{|A|}$. If G is FC, A cannot be finite since G is infinite. Thus $|G| \leqq (|A| \cdot \aleph_0)^{|A|} = \exp|A|$. If G is mC, $|G| \leqq (m \cdot |A|)^{|A|} = m^{|A|}$.

THEOREM 5. *Every mCN* group G has a normal series* (G_α) *such that* $[G_{\alpha+1} : G_\alpha] \leqq m$ *for all* $\alpha + 1$.

PROOF. Let (G_α) be any ascending mC series for G. Let $G_{\alpha,0} = G_\alpha$ for all α. We form an invariant series $(G_{\alpha,\beta}/G_\alpha)$ for the mC group $G_{\alpha+1}/G_\alpha$ by choosing $G_{\alpha,\beta+1}/G_{\alpha,\beta}$ to be any normal subgroup of $G_{\alpha+1}/G_{\alpha,\beta}$ of cardinality at most m. It follows that $(G_{\alpha,\beta})$ is the desired normal series.

THEOREM 6. *Every infinite FCN* group G has an abelian subgroup A such that* $\exp|A| \geqq |G|$. *Every mCN* group G has an abelian subgroup A such that* $m^{|A|} \geqq |G|$.

PROOF. This theorem follows directly from Theorems 2 and 5 except in the case where G is a countable FCN^* group. In that case, we may suppose that G has an FC series $(G_\alpha \mid \alpha \leqq \beta)$ of minimal length. Then G_α is finite for all $\alpha < \beta$. If $\beta = \omega$, G is locally finite and has an infinite abelian subgroup by [15, p. 453]. If $\beta = n + 1$, G/G_n is FC and thus has an infinite abelian subgroup H/G_n by Theorem 4. Since $H^1 \leqq G_n$ is finite, H is FC and the theorem follows from Theorem 4.

LEMMA 2. *If A is an abelian subgroup of G, then* $AZ_{\lambda+1}(G)$ *is at most a class* $\lambda + 1$ *ZA group.*

PROOF. Note that $AZ_{\lambda+1}(G)/Z_\lambda(G)$ is abelian and that $Z_\lambda(G) \leqq Z_\lambda(AZ_{\lambda+1}(G))$.

THEOREM 7. *Let A be a maximal abelian subgroup of an infinite nilpotent group G. Then*

$$\exp|A| \geqq |G|.$$

PROOF. We suppose G is a counter-example of smallest class λ. Since $G^1 \leqq Z_{\lambda-1}(G)$, if $Z_{\lambda-1}(G)$ is finite, G is FC and the result follows from Theorem 4. Thus $\exp|A| \geqq |AZ_{\lambda-1}| \geqq |Z_{\lambda-1}(G)| \geqq |G^1|$, so G is $(\exp|A|)C$. Thus

$$|G| \leqq (\exp|A|)^{|A|} = \exp|A|.$$

THEOREM 8. *Let G be an infinite ZA group with a finite maximal abelian subgroup A. Then G is a direct sum of Černikov p-groups for a finite number of distinct primes p. In particular, G is countable.*

PROOF. First we show that G is periodic. Suppose $\alpha + 1$ is the first ordinal such that there exists an element x with infinite order in $Z_{\alpha+1}$. Then Z_β is periodic for $\beta \leqq \alpha$. If $a \in A$, then $[x, a] \in Z_\alpha$. Suppose for all n, $[x^n, a] \neq 1$. Then let $\lambda + 1$ be the first ordinal such that there exists n, $[x^n, a] \in Z_{\lambda+1}$. Then $\lambda < \alpha$, so $Z_{\lambda+1}$ is periodic and there exists n such that $[x^n, a] \in Z_{\lambda+1}$. Thus $a^{-1}x^n a = x^n z$, $z \in Z_{\lambda+1}$. Since $[z, x] \in Z_\lambda$, if m is the order of z, $(a^{-1}x^n a)^m Z_\lambda = (x^n z)^m Z_\lambda = x^{nm} z^m Z_\lambda = x^{nm} Z_\lambda$. Therefore $[x^{nm}, a] \in Z_\lambda$, a contradiction. This shows that for each $a \in A$ there exists $n(a)$ such that $x^{n(a)} \in C(a)$. Let $n = \prod_{a \in A} n(a)$. A cannot be finite since $\langle x^n \rangle \leqq C(A) = A$. This proves that G is periodic.

Now suppose G is a p-group. Let A_0 be a maximal normal abelian subgroup. A_0 is an infinite maximal abelian subgroup of G by [3, p. 681]. Let $V = \{x \in A_0 \mid x^p = 1\}$. Suppose, by way of contradiction, that V is infinite. Since V is a characteristic subgroup of A_0, we may suppose $G = AV$, $V \lhd G$. Suppose W is a finite subgroup of V and $W^A \leqq W$. If $H/W = Z(G/W)$ is infinite, AH is an infinite FC group. By Theorem 4, A cannot be a maximal abelian subgroup of AH, a contradiction. Thus $Z(G/W)$ must be finite. We claim that V has a minimal infinite subgroup V_0 such that $V_0{}^A \leqq V_0$. Let $\{V_i\}_{i \in I}$ be a descending chain of infinite subgroups such that $V_i{}^A \leqq V_i$. Suppose $W = \bigcap_{i \in I} V_i$ is finite. Then $Z(G/W) \neq E$ is finite. We also know that $V_i/W \cap Z(G/W) \neq E$ (see [14, p. 14]). Thus $E \neq \bigcap_{i \in I}(V_i/W \cap Z(G/W)) = W/W \cap Z(G/W)$, a contradiction. Thus W is infinite, and the claim follows by Zorn's Lemma. Consider $(AV_0)^i \leqq V_0$. It is easy to calculate that $(AV_0)^i \leqq (A-1)V_0 \equiv \langle v^\alpha - v \mid v \in V_0, \alpha \in A\rangle$. Since for every $\alpha \in A$ there exists k such that

$$(\alpha - 1)^{pk} = \alpha^{pk} - 1 = 0,$$

where $\alpha - 1$ is the endomorphism on V defined by $(\alpha - 1)(v) = v^\alpha - v$, $(\alpha - 1)V_0$ is a proper subgroup of V_0 such that $[(\alpha - 1)V_0]^A \leqq (\alpha - 1)V_0$. Thus $(\alpha - 1)V_0$ is finite and so is $(A - 1)V_0$. It follows that $(AV_0)^1$ is finite. Hence AV_0 is an infinite FC group, contradicting Theorem 4. This shows that V is finite. Hence $A_0 = D + R$ where D is a finite sum of p^∞ groups and R is reduced. Since every infinite reduced group contains a cyclic summand, if R is infinite it contains an infinite direct sum of cyclic groups, contradicting the fact that V is finite. Thus R is finite and A_0 satisfies Min. Since A_0 is maximal, G/A_0 is a periodic group of automorphisms of A_0. By a theorem of Baer (see [14, p. 54]), G/A_0 satisfies Min. It follows that G is a Černikov p-group.

Now since G is periodic, G is the direct sum of non-trivial p-groups for different primes p, $G = \sum G_p$. A is also the direct sum of p-groups, $A = \sum A_p$, where each A_p must be a maximal abelian subgroup of G_p. Hence G is a finite sum of Černikov p-groups, and G is countable.

Example 2. Let H be the direct sum of Z_{3^∞} groups generated by $\{x_i \mid 3x_{i+1} = x_i\}$ and $\{y_i \mid 3y_{i+1} = y_i\}$. Let α be the automorphism of order 3 on H defined by $\alpha(x_i) = -x_i + y_i$ and $\alpha(y_i) = -x_i$. The split extension $\langle\alpha\rangle H$ is a Černikov 3-group. Suppose $z \in H$ and $\alpha(z) = z$. Then since $z = ax_i + by_i$ for some i with $a, b \in Z_{3^i}$, we have $3ax_i = 3by_i = 0$. Thus $z \in \langle x_1 + y_1\rangle$ and so $\langle\alpha\rangle \cdot \langle x_1 + y_1\rangle$ is a finite maximal abelian subgroup.

Theorem 9. *Let A be a maximal abelian subgroup of an uncountable ZA group G. Then*

$$\exp|A| \geqq |G|.$$

Proof. We suppose that G is a counter-example of smallest class $\lambda + n$, where λ is a limit ordinal. By hypothesis,

$$\exp|A| \geqq |AZ_{\alpha+1}(G)| \geqq |Z_{\alpha+1}(G)|$$

for all $\alpha + 1 < \lambda + n$ such that $Z_{\alpha+1}(G)$ is uncountable. Also, by Theorem 8, we may suppose that $|G| > \aleph_1$ and that A is infinite. Hence $\exp|A| \geqq |Z_{\alpha+1}(G)|$ for all $\alpha + 1 < \lambda + n$. If $n = 0$, $G = \bigcup_{\alpha<\gamma} Z_{\alpha+1}(G)$. Thus $\exp|A| \geqq \lim_{\alpha<\lambda}|Z_{\alpha+1}(G)|$. If $n = 1$, $\exp|A| \geqq \lim_{\alpha<\lambda} |Z_{\alpha+1}(G)| \geqq |Z_\lambda(G)| \geqq |G^1|$. If $n > 1$, $\exp|A| \geqq |Z_{\lambda+n-1}(G)| \geqq |G^1|$. In either case, G is $(\exp|A|)C$ and by Theorem 4,

$$|G| \leqq (\exp|A|)^{|A|} = \exp|A|.$$

Remark 2. The group G_α of Example 1 is a two-step *FC* nilpotent group. Thus Theorem 9 cannot be extended to *FC* nilpotent groups.

Question 2. Is there a nilpotent group G with a maximum abelian subgroup A such that $\exp|\text{core } A| < |G|$?

Question 3. Does every infinite FCN^* group have an equipotent SN^* subgroup?

Theorem 10. *Every infinite solvable group G of length n has a characteristic nilpotent subgroup N of class at most n such that* $\exp|N| \geqq |G|$.

Proof. We induct on n. The theorem is clearly true for $n = 1$. Let $n + 1$ be the length of a counter-example G of shortest length. We must have $\exp|G^n| < |G|$. Thus

$$[G : C(G^n)] \leqq |\text{Aut } G^n| \leqq \exp|G^n| < |G|.$$

Hence $H = C(G^n)$ □ G has the same cardinality as G. Since $G^n \subseteq Z(H)$ □ H, we must have $|Z(H)| < |H|$. Since $H/Z(H)$ has shorter solvable length, it has a characteristic nilpotent subgroup $N/Z(H)$ of class at most n such that $\exp|N/Z(H)| \geqq |H/Z(H)| = |H|$. Thus $[N, N, \cdots, N] \leqq Z(H)$. Since $[N, Z(H)] = E$, N is a nilpotent group of class at most $n + 1$. In addition, since $N/Z(H)$ □ $H/Z(H)$ and $Z(H)$ □ H, N □ H □ G and

$$\exp|N| \geqq \exp|N/Z(H)| \geqq |H| = |G|.$$

Theorem 11. *An infinite solvable group G of length n has a characteristic two-step nilpotent subgroup N such that* $\exp^n|N| \geqq |G|$.

Proof. As above, let $n + 1$ be the length of a counter-example G of shortest length. If $\exp|G^1| < |G|$, we have

$$[G : C(G^1)] \leqq |\mathrm{Aut}(G^1)| \leqq \exp|G^1| < |G|,$$

so $|C(G^1)| = |G|$. Since $C(G^1)$ is a characteristic two-step nilpotent subgroup, we must have $\exp|G^1| \geqq |G|$. Since G^1 has a characteristic two-step nilpotent subgroup N such that $\exp^n|N| \geqq |G^1|$, we have $\exp^{n+1}|N| \geqq \exp|G^1| \geqq |G|$.

REMARK 3. There is an infinite two-step solvable group all of whose normal abelian subgroups are finite. See [8].

LEMMA 3. [14, p. 14]. *Let $E \neq H \lhd G$. If G is SI^*, H contains a non-trivial abelian subgroup normal in G. If G is FCI^*, H contains a non-trivial FC subgroup normal in G.*

PROOF. We shall prove only the second statement. Let (G_α) be an invariant FC series for G. Let α be the least ordinal such that $H \cap G_\alpha \neq E$. Since α is not a limit ordinal, $H \cap G_{\alpha-1} = E$ and $H \cap G_\alpha \cong (H \cap G_\alpha)G_{\alpha-1}/G_{\alpha-1} \leqq G_\alpha/G_{\alpha-1}$. Thus $H \cap G_\alpha \lhd G$ is the desired group.

THEOREM 12. *Every infinite SI^* group G has a normal ZA subgroup H such that* $\exp|H| \geqq |G|$. *Every infinite FCI^* group G has a normal ZFC subgroup H such that* $\exp|H| \geqq |G|$.

PROOF. We shall prove only the second statement. Let $H_0 = G$ and let $A_0 = E$. If possible, let $A_{\alpha+1}$ be a normal subgroup of G such that $A_{\alpha+1}/A_\alpha$ is a non-trivial normal FC subgroup of G/A_α contained in H_α/A_α. Then let $H_{\alpha+1}$ be the normal subgroup of G such that $H_{\alpha+1}/A_\alpha = (A_{\alpha+1}/A_\alpha)C(A_{\alpha+1}/A_\alpha \leqq H_\alpha/A_\alpha) = (A_{\alpha+1}/A_\alpha)(C(A_{\alpha+1}/A_\alpha) \cap H_\alpha/A_\alpha)$. If β is a limit ordinal, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and let $H_\beta = \bigcap_{\alpha<\beta} H_\alpha$. Since G/A_α is an FCI^* group, if H_α/A_α is not trivial, then it contains a non-trivial FC subgroup normal in G/A_α by Lemma 3. Hence $A_{\alpha+1} > A_\alpha$ unless $H_\alpha = A_\alpha$. Thus there is a first ordinal γ such that $H_\gamma = A_\gamma$.

Since $A_{\alpha+1}/A_\alpha$ is an FC group, using the definition of $H_{\alpha+1}$, we see that each element in $A_{\alpha+1}/A_\alpha$ has only a finite number of conjugates in A_γ/A_α. Thus A_γ is a ZFC group. We have (see [3, Theorem 1])

$$\begin{aligned}
[G : H_\gamma] &\leqq \prod_{\alpha<\gamma} [H_\alpha : H_{\alpha+1}] = \prod_{\alpha<\gamma} [H_\alpha/A_\alpha : H_{\alpha+1}/A_\alpha] \\
&= \prod_{\alpha<\gamma} [N(A_{\alpha+1}/A_\alpha \leqq H_\alpha/A_\alpha) : (A_{\alpha+1}/A_\alpha)C(A_{\alpha+1}/A_\alpha \leqq H_\alpha/A_\alpha)] \\
&\leqq \prod_{\alpha<\gamma} [N(A_{\alpha+1}/A_\alpha \leqq H_\alpha/A_\alpha) : C(A_{\alpha+1}/A_\alpha \leqq H_\alpha/A_\alpha)] \\
&\leqq \prod_{\alpha<\gamma} \mathrm{Aut}(A_{\alpha+1}/A_\alpha).
\end{aligned}$$

It follows (see [3, § 3]) that $[G : H_\gamma] \leqq \prod_{\alpha<\gamma} \aleph_0^{|A_{\alpha+1}/A_\alpha|} \leqq \aleph_0^{\Sigma_{\alpha<\gamma}|A_{\alpha+1}/A_\alpha|} \leqq \aleph_0^{|A_\gamma|} = \exp|A_\gamma|$. Thus $|G| \leqq |H_\gamma| \exp|H_\gamma|$. Therefore $|G| \leqq \exp|H_\gamma|$.

THEOREM 13. *Every infinite SI* group G has an abelian subgroup* $A \lhd^2 G$ *such that* $\exp^2|A| \geqq |G|$.

PROOF. This follows from Theorem 12 and the fact [3, p. 681] that every maximal normal abelian subgroup A of a ZA group H satisfies $\exp|A| \geqq |H|$.

REMARK 4. (See also Remark 3.) There is a non-abelian SI^* group which has no nontrivial characteristic subgroups. (See [**12**] and [**14**, p. 102].) There is an SJ^* group which has no non-trivial normal abelian subgroups. (See [**1**]).

QUESTION 4. Can Theorem 13 be generalized to FCI^* groups?

QUESTION 5. Can Theorem 13 be improved to read $\exp|A| \geqq |G|$?

THEOREM 14. *Every infinite mCI* group G has a normal ZmC subgroup H such that* $\exp|H| \geqq |G|$.

PROOF. The proof is similar to that of Theorem 12; details are left to the reader.

REMARK 5. M. J. Tomkinson [**16**] informs us that the construction used in § 4 of [**9**] can be used to construct without the continuum hypothesis a two-step nilpotent group of cardinality $2^{\aleph_0}$ all of whose maximal abelian subgroups have cardinality $\aleph_0$. This construction does not seem to generalize to higher cardinalities and (unlike [**2**] where the continuum hypothesis is used) does not yield FC groups.

4. **Acknowledgement.** The author wishes to express his gratitude to M. Magidor and R. McKenzie who provided many helpful ideas which were incorporated in this work. Part of the work of the author was performed under a Summer Research Initiation Faculty Fellowship from the University of Colorado.

REFERENCES

1. R. S. Dark, *A prime Baer group*, Math. Zeit. **105** (1968), 294–298.

2. A. Ehrenfeucht and V. Faber, *Do infinite nilpotent groups always have equipotent abelian subgroups*? Indag. Math. 34 (1972), 202–209.

3. V. Faber, *Large abelian subgroups of some infinite groups*, Rocky Mountain Journal of Mathematics 1 (1971), 677–685.

4. ——, *Chain index theorems and their applications in infinite groups*, Ph.D. Thesis, Wash. Univ., St. Louis (1971).

5. ——, *Large abelian subgroups of infinite solvable and nilpotent groups,* AMS Notices **19** (1972), A-386.

6. G. Fodor, *Eine Bemerkung zur Theorie der regessiven Funktionen,* Acta Sci. Math. (Szeged) **17** (1956), 139–142.

7. F. Haimo, *The FC-chain of a group,* Canad. J. Math. **5** (1953), 498–511.

8. H. Heineken and I. J. Mohamed, *A group with trivial centre satisfying the normalizer condition,* Journal of Alg. **10** (1968), 368–376.

9. L. G. Kovács, B. H. Neumann and H. de Vries, Proc. Royal Soc. of London **A260** (1961), 304–316.

10. C. R. Kulatilaka, *Infinite abelian subgroups of some infinite groups,* J. London Math. Soc. **39** (1964), 240–244.

11. A. G. Kuroš, *The Theory of Groups, vol.* II, translated and edited by K. A. Hirsch, Chelsea Publishing Co., New York (1960).

12. D. H. McLain, *A characteristically-simple group,* Proc. Cambridge Philos. Soc. **50** (1954), 641–642.

13. W. Neumer, *Verallgemeinerung eines Satzes von Alexandroff und Urysohn,* Math. Zeit. **54** (1951), 254–261.

14. D. J. S. Robinson, *Infinite Soluble and Nilpotent Groups,* Queen Mary College Mathematics Notes, London (1967).

15. W. R. Scott, *Group Theory,* Prentice Hall, Englewood Cliffs (1964).

16. M. J. Tomkinson, private communication.

UNIVERSITY OF COLORADO, DENVER, COLORADO 80202