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CIRCULAR POLARIZED NONLINEAR ALFVÉN WAVES—A NEW TYPE OF NONLINEAR EVOLUTION EQUATION IN PLASMA PHYSICS

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ABSTRACT. A nonlinear evolution equation is derived for Alfvén waves propagating along the magnetic field in a cold plasma. The equation provides new types of solitary waves. The phase of a solitary wave is coupled nonlinearly with its amplitude, and the propagation velocity is restricted within the range determined by the asymptotic amplitude and the wave number.

1. **Introduction.** As early as in 1942, Alfvén [1] recognized that a hydromagnetic wave propagates in an incompressible, perfectly conducting fluid in the presence of a strong magnetic field. This Alfvén wave is nothing but the low frequency limit of electromagnetic waves propagating in a plasma. Generation and propagation of the Alfvén waves in a gaseous plasma has been investigated experimentally [2], and has attracted renewed interest as one of the useful ways to heat a plasma [3]. Alfvén wave propagation in solid state plasmas provides information on the effective masses of carriers [4]. Large amplitude incompressible magnetic field perturbation observed in the solar wind has been attributed to propagation of the Alfvén wave [5].

Now, turning to the studies of nonlinear wave propagation in plasmas, we have seen the remarkable success of theoretical and experimental investigations of the ion acoustic solitary waves. Using reductive perturbation theory, Washimi and Taniuti [6] have predicted existence of the Korteweg-deVries type soliton for the ion acoustic mode. Their prediction has been experimentally confirmed by Ikezi et al. [7]. Systematic ordering of dispersive effects and nonlinear steepening effects in the reductive perturbation theory [8] provides a rigorous procedure for reducing the hyperbolic system of nonlinear partial differential equations to a single nonlinear evolution equation; namely, the Korteweg-deVries equation for a weakly dispersive system and the nonlinear Schrödinger equation for a strongly dispersive system.

In the case of Alfvén waves, however, Kakutani and Ono [9] have noticed that it is necessary to modify the expansion scheme of the reductive perturbation theory so as to be consistent with steady state solitary wave solution of Kazantsev [10]. Thus, they have been led to the conclusion that the Alfvén wave is governed by the modified Korteweg-deVries equation. We have undertaken a detailed analysis of the amplitude modulation of the nonlinear Alfvén wave, because the linear dispersion relation of the Alfvén wave differs from the form used by Kakutani and Ono if we restrict ourselves to the one-dimensional Alfvén wave propagating along the magnetic field. Independently from our studies, Mio et al. [11] have noticed that the direct application of the expansion scheme of the reductive perturbation theory is not valid for the Alfvén wave.

The purpose of the present report is twofold: first, presenting a new type of nonlinear evolution equation in the next section, we give an exact analytic steady-state solution of this new nonlinear evolution equation in the third section. Illustrating a reduction of the nonlinear Schrödinger equation from the nonlinear evolution equation in the Appendix, we present some concluding remarks in the last section.

2. **Derivation of a New Type of Nonlinear Evolution Equation.** We start from the system of equations for cold plasma studied by Kakutani and Ono [9]. Under the assumptions that the effects of displacement current and charge separation are neglected, the fundamental equations for one-dimensional propagation in dimensionless form are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\,u) = 0, \tag{1a}$$

$$\frac{du}{dt} + n^{-1}\frac{\partial}{\partial x}\frac{1}{2}(B_y^2 + B_z^2) = 0, \tag{1b}$$

$$\frac{dv}{dt} - n^{-1}\frac{\partial}{\partial x}B_y = -R_e^{-1}\frac{d}{dt}\left(n^{-1}\frac{\partial B_z}{\partial x}\right), \tag{1c}$$

$$\frac{dw}{dt} - n^{-1}\frac{\partial}{\partial x}B_z = R_e^{-1}\frac{d}{dt}\left(n^{-1}\frac{\partial B_y}{\partial x}\right), \tag{1d}$$

$$\frac{dB_y}{dt} - \frac{\partial v}{\partial x} + B_y\frac{\partial u}{\partial x} = R_i^{-1}\frac{\partial}{\partial x}\left(\frac{dw}{dt}\right), \tag{1e}$$

$$\frac{dB_z}{dt} - \frac{\partial w}{\partial x} + B_z\frac{\partial u}{\partial x} = -R_i^{-1}\frac{\partial}{\partial x}\left(\frac{dv}{dt}\right), \tag{1f}$$

where $d/dt \equiv \partial/\partial t + u\partial/\partial x$, $\vec{v} = (u, v, w)$ denotes the velocity of electrons, n the density of electrons, $\vec{B} = (B_x = 1, B_y, B_z)$ the magnetic induction vector, and R_e and R_i represent the ratios of electron and ion cyclotron frequencies to the characteristic frequency, respectively.

The linear dispersion relation for the above system is

$$\omega/k = 1 \pm \mu k, \tag{2}$$

where

(3) $$\mu = \tfrac{1}{2}(R_i^{-1} - R_e^{-1}).$$

The double sign $\pm$ designates the right (+) and left (−) polarized Alfvén waves, whose amplitudes are given as

(4a) $$\Phi_R = B_y^{(1)} - i\,B_z^{(1)},$$

(4b) $$\Phi_L = B_y^{(1)} + i\,B_z^{(1)},$$

respectively (the meaning of superscript (1) will become clear later). As mentioned in the introduction, Kakutani and Ono worked out their analysis on the basis of a linear dispersion relation of a form of $\omega/k = 1 + \beta k^2$. Hence, differing from their choice of stretching variables, we introduce the stretched space-time variables

(5a) $$\zeta = \epsilon(x - t),$$

(5b) $$\tau = \epsilon^2 t,$$

but we expand the variables in accord with Kakutani and Ono as

(6a) $$n = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \cdots,$$

(6b) $$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots,$$

(6c) $$v = \epsilon^{1/2}(v^{(1)} + \epsilon v^{(2)} + \cdots),$$

(6d) $$w = \epsilon^{1/2}(w^{(1)} + \epsilon w^{(2)} + \cdots),$$

(6e) $$B_y = \epsilon^{1/2}(B_y^{(1)} + \epsilon B_y^{(2)} + \cdots),$$

(6f) $$B_z = \epsilon^{1/2}(B_z^{(1)} + \epsilon B_z^{(2)} + \cdots).$$

Using equations (6a)–(6f) and (5a)–(5b), we obtain the following relationships among the first order quantities,

(7) $$v^{(1)} = -B_y^{(1)},\ w^{(1)} = -B_z^{(1)}$$

to order $\epsilon^{3/2}$, and

(8) $$n^{(1)} = u^{(1)} = \tfrac{1}{2}\left(B_y^{(1)^2} + B_z^{(1)^2}\right)$$

to order ϵ^2. Up to terms of order $\epsilon^{5/2}$, we get

(9a) $$\frac{\partial}{\partial \zeta} v^{(2)} + \frac{\partial}{\partial \zeta} B_y^{(2)} = \frac{\partial}{\partial \tau} v^{(1)} - R_e^{-1} \frac{\partial^2}{\partial \zeta^2} B_z^{(1)}$$

(9b) $$\frac{\partial}{\partial \zeta} w^{(2)} + \frac{\partial}{\partial \zeta} B_z^{(2)} = \frac{\partial}{\partial \tau} w^{(1)} + R_e^{-1} \frac{\partial^2}{\partial \zeta^2} B_y^{(1)}$$

(9c) $$\frac{\partial}{\partial \zeta} B_y^{(2)} + \frac{\partial}{\partial \zeta} v^{(2)} = \frac{\partial}{\partial \tau} B_y^{(1)} + \frac{\partial}{\partial \zeta}(u^{(1)} B_y^{(1)}) + R_i^{-1} \frac{\partial^2}{\partial \zeta^2} w^{(1)}$$

(9d) $$\frac{\partial}{\partial \zeta} B_z^{(2)} + \frac{\partial}{\partial \zeta} w^{(2)} = \frac{\partial}{\partial \tau} B_z^{(1)} + \frac{\partial}{\partial \zeta}(u^{(1)} B_z^{(1)}) - R_i^{-1} \frac{\partial^2}{\partial \zeta^2} v^{(1)}.$$

Eliminating the second order quantities from (9a)–(9d), we obtain the following nonlinear evolution equation for the right polarized Alfvén wave:

(10a) $$\frac{\partial}{\partial \tau} \phi_R + \frac{1}{4} \frac{\partial}{\partial \zeta} \{|\phi_R|^2 \phi_R\} - i\mu \frac{\partial^2}{\partial \zeta^2} \phi_R = 0,$$

and

(10b) $$\frac{\partial}{\partial \tau} \phi_L + \frac{1}{4} \frac{\partial}{\partial \zeta} \{|\phi_L|^2 \phi_L\} + i\mu \frac{\partial^2}{\partial \zeta^2} \phi_L = 0,$$

for the left polarized Alfvén wave. This new type of nonlinear evolution equation has been investigated numerically by Mio et al.

3. **A Steady State Solution.** Now we seek an analytic solution of (10a), restricting our attention to the right polarized Alfvén wave. A substitution of the form

(11) $$\phi_R(\zeta, \tau) = \sqrt{8}\, \psi(\zeta, \tau) \exp\{i\chi(\zeta, \tau)\}$$

with real functions ψ and χ into (10a) yields a pair of coupled equations for ψ and χ:

(12a) $$\psi_\tau + 6\psi^2 \psi_\zeta + 2\mu \psi_\zeta \chi_\zeta + \mu \chi_{\zeta\zeta} \psi = 0,$$

(12b) $$\chi_\tau \psi + 2\chi_\zeta \psi^3 - \mu \psi_{\zeta\zeta} + \mu \chi_\zeta^2 \psi = 0.$$

We assume a solution in the following form:

(13a) $$\chi(\zeta, \tau) = \mu^{-1}(K\zeta - \Omega\tau) + \theta(y),$$

(13b) $$\psi(\zeta, \tau) = \psi(y),$$

(13c) $$y = \mu^{-1}(\zeta - \lambda\tau),$$

where the wave-number K, frequency Ω and propagation velocity λ are constants to be determined from the solution of (12a) and (12b). By (13a)–(13c) we can write (12a) and (12b) as the coupled ordinary differential equations

$$(2K - \lambda)\frac{d\psi}{dy} + 6\psi^2\frac{d\psi}{dy} + 2\frac{d\psi}{dy}\frac{d\theta}{dy} + \psi\frac{d^2\theta}{dy^2} = 0, \tag{14a}$$

$$\begin{aligned}\frac{d^2\psi}{dy^2} &= -(\Omega - K^2)\psi + 2K\psi^3 \\ &\quad + (2K - \lambda + 2\psi^2)\psi\frac{d\theta}{dy} + \left(\frac{d\theta}{dy}\right)^2\psi.\end{aligned} \tag{14b}$$

Integrating (14a) once, we obtain

$$\frac{d\theta}{dy} = A\psi^{-2} + (\lambda - 2K)/2 - (3/2)\psi^2 \tag{15}$$

where A is an integration constant. Eliminating $d\theta/dy$ in (14b) by using (15), we arrive at

$$\begin{aligned}\left(\frac{d\Phi}{dy}\right)^2 &= -\Phi^4 + 2\lambda\Phi^3 - 4\{\Omega + A \\ &\quad + (\lambda - 2K)^2/4 - K^2\}\Phi^2 + 4B\Phi - 4A^2\end{aligned} \tag{16}$$

where

$$\Phi(y) = \psi^2(y) \tag{17}$$

and B is another integration constant. Equation (16) gives rise to various types of solutions. We restrict our interest to solitary wave solutions which satisfy the boundary conditions

$$\begin{aligned}&\Phi(y) \to \Phi_0 = \psi_0^2 \\ &\frac{d\theta(y)}{dy} \to 0, \quad \text{as } |y| \to \infty.\end{aligned} \tag{18}$$

These boundary conditions specify the integration constants and the shifted carrier frequency as

$$A = (3/2)\Phi_0^2 - (1/2)(\lambda - 2K)\Phi_0, \tag{19a}$$

$$B = 4\Phi_0^3 + (1/2)(12K - 5\lambda)\Phi_0^2 + 2(K - \lambda/2)^2\Phi_0, \tag{19b}$$

$$\Omega = K^2 + 2K\Phi_0. \tag{19c}$$

Here it would be worthwhile to notice that the shifted carrier frequency (19c) takes the form

$$\delta\omega = \mu k^2 + \tfrac{1}{4}\,|\Phi_R|_0^2\, k \tag{20}$$

expressed in terms of the original variables. Equation (20) is nothing but the nonlinear frequency obtained from (10a). Straightforward but lengthy calculation gives

$$\Phi(y) = \psi^2(y) = \Phi_0 + \frac{8\kappa\gamma^2}{\beta}\,[\kappa m + \cosh\{2\gamma(y - y_0)]^{-1}, \tag{21}$$

$$\begin{aligned}\theta(y) = \theta(y_0) &- 3\kappa \arctan \left\{ \sqrt{\frac{1-\kappa m}{1+\kappa m}}\tanh\{\gamma(y - y_0)\} \right\} \\ &-\gamma\delta \arctan \left\{ \sqrt{\frac{1-\kappa \ell}{1+\kappa \ell}}\tanh\{\gamma(y - y_0) \right\},\end{aligned} \tag{22}$$

where

$$\kappa = \pm 1 \tag{23a}$$

$$\ell = \alpha/\beta + 8\gamma^2/\beta\Phi_0 \quad \text{and } m = \alpha/\beta, \tag{23b}$$

$$\alpha = 2(2\Phi_0 - \lambda) \tag{23c}$$

$$\beta = 4\,\{(\Phi_0 + K)(\lambda - K - 2\Phi_0)\}^{1/2} \tag{23d}$$

$$\gamma = \tfrac{1}{2}\,\{(\lambda - \lambda_1)(\lambda_2 - \lambda)\}^{1/2} \tag{23e}$$

$$\delta = \text{sign of } (3\Phi_0 - \lambda + 2K). \tag{23f}$$

The propagation velocity λ is allowed to take a value in the region of

$$\lambda_1 < \lambda < \lambda_2, \tag{24a}$$

where

$$\lambda_1 = 2(K + 2\Phi_0) - 2\sqrt{\Phi_0(\Phi_0 + K)}\,, \tag{24b}$$

$$\lambda_2 = 2(K + 2\Phi_0) + 2\sqrt{\Phi_0(\Phi_0 + K)}\,, \tag{24c}$$

for which an obvious condition $\Phi_0 + K > 0$ is invoked.

A similar analysis is possible for the left polarized waves. In this case, solitary waves are obtained just by replacing

$$\Omega \to -\Omega \text{ and } K \to -K \tag{25}$$

in the above expressions. Then we have an extra restriction on the wave number,

(26) $$K > \Phi_0.$$

4. **Discussion.** We summarize the properties of solitary wave solutions, equations (21) and (22) with equations (23) and (24).

(1) The sign of κ classified bright ($\kappa = +1$) and dark ($\kappa = -1$) modulation of the amplitude.

(2) The sign of δ is related to fast ($\delta = +1$) and slow ($\delta = -1$) modulation of the phase of solitary waves.

(3) The propagation velocity λ of the solitary wave is bounded by the asymptotic amplitude and the wave number as indicated in (24).

These properties are novel and could be detected by experiment.

Besides the solitary waves discussed above, (16) admits a variety of nonlinear waves, such as cnoidal waves and algebraic solitary waves. Also it is interesting to notice that further application of reductive perturbation to (10a) yields the conventional nonlinear Schrödinger equation derived by Hasegawa [12]. This will be illustrated in the Appendix. Comparing the well-known envelope soliton solution of (A.11) with the present solitary wave solution, we observe that the solitary wave solution obtained in the preceding section indeed has peculiar properties not known for any other types of nonlinear evolution equations.

Appendix. Reduction of the nonlinear Schrödinger equation from (10a).

Using a small parameter ϵ, we introduce the slow variables

(A1) $$\eta = \epsilon(\zeta - \lambda\tau)$$

(A2) $$\sigma = \epsilon^2\tau$$

where λ will be determined later.

Now we seek an oscillatory solution of (7) by expanding ϕ_R as follows

(A3) $$\phi_R = \sum_{m=1} \sum_{\ell=1} \epsilon^m \psi_\ell^{(m)}(\eta, \sigma) e^{i\ell(k\zeta - \delta\tau)},$$

where the quantity δ is a frequency shift to be determined later.

Corresponding to the introduction of the stretched variables (A1) and (A2), we have the following transformation

(A4) $$\frac{\partial}{\partial\tau} \to \frac{\partial}{\partial\tau} - \epsilon\lambda \frac{\partial}{\partial\eta} + \epsilon^2 \frac{\partial}{\partial\sigma},$$

(A5) $$\frac{\partial}{\partial\zeta} \to \frac{\partial}{\partial\zeta} + \epsilon \frac{\partial}{\partial\eta}.$$

From the first order ℓ-component we have

(A6) $$-i\ell(\delta-\mu\ell k^2)\psi_\ell^{(1)}=0$$

For $\ell=\pm 1$, (A6) determines the frequency shift δ as

(A7) $$\delta=\mu k^2$$

and for $|\ell|\neq 1$, in view of (A6) and (A7), we have

(A8) $$\psi_\ell^{(1)}=0.$$

The second order term with $\ell=1$ takes the following form

(A9) $$(\lambda-2\mu k)\frac{\partial}{\partial\eta}\psi_1^{(1)}=0.$$

In order that this equation have a nontrivial solution, the quantity λ must satisfy the relation

(A10) $$\lambda=2\mu k\equiv\frac{\partial\delta}{\partial k}.$$

Finally the $\ell=1$ component of the third order perturbation yields the following nonlinear Schrödinger equation

(A11) $$i\frac{\partial}{\partial\sigma}\psi_1^{(1)}+\mu\frac{\partial^2}{\partial\eta^2}\psi_1^{(1)}-\frac{1}{4}k|\psi_1^{(1)}|^2\psi_1^{(1)}=0$$

which is exactly the one derived by Hasegawa.

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