ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 4, Summer 1977

UNIFORM FINITE GENERATION OF LIE GROUPS LOCALLY-ISOMORPHIC TO SL(2, R)

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ABSTRACT. Let G be a connected Lie group with Lie algebra g, $\{X_1, \cdots, X_\ell\}$ a minimal generating set for g. The order of generation of G with respect to $\{X_1, \cdots, X_\ell\}$ is the smallest integer n such that every element of G can be written as a product of n elements taken from $\exp(tX_1), \cdots, \exp(tX_\ell)$; n may equal ∞. We find all possible orders of generation for all Lie groups locally isomorphic to $SL(2, R)$.

1. **Introduction.** A connected Lie group G is generated by one-parameter subgroups $\exp(tX_1), \cdots, \exp(tX_\ell)$ if every element of G can be written as a finite product of elements chosen from these subgroups. In this case, define the order of generation of G to be the least positive integer n such that every element of G possesses such a representation of length at most n; if no such integer exists, let the order of generation of G be infinity. The order of generation will, of course, depend upon the one-parameter subgroups.

Computation of the order of generation of G for given $X_1, \cdots, X_\ell$ is equivalent to finding the greatest wordlength needed to write each element of a finite group in terms of generators $g_1, \cdots, g_\ell$. In both cases it is natural to restrict attention to minimal generating sets. From now on, therefore, suppose that no subset of $\{\exp(tX_1), \cdots, \exp(tX_\ell)\}$ generates G.

It is easy to see that $\exp(tX_1), \cdots, \exp(tX_\ell)$ generate G just in case $X_1, \cdots, X_\ell$ generate the Lie algebra g of G. If σ is an automorphism of G, the order of generation of G with respect to $X_1, \cdots, X_\ell$ is clearly the same as the order of generation of G with respect to $\sigma_*(X_1), \cdots, \sigma_*(X_\ell)$. Call two generating sets $\{X_1, \cdots, X_\ell\}$ and $\{Y_1, \cdots, Y_\ell\}$ *equivalent* if it is possible to find an automorphism σ of G, a permutation τ of $\{1, 2, \cdots, \ell\}$, and non-zero constants $\lambda_1, \cdots, \lambda_\ell$ such that $X_i = \lambda_i \sigma_*(Y_{\tau(i)})$; the order of generation of G depends only on the equivalence class of the generating set.

In a series of previous papers [2, 3, 4, 5, 6], the possible orders of generation for all two and three dimensional *linear* Lie groups were found. The remaining nonlinear groups are

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in R \right\} \Big/ \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, n \in Z \right\}$$

Received by the editors on December 2, 1975.

AMS (MOS) subject classification (1970). Primary 22E15, 22E60; Secondary 20F05.

and $\widetilde{SL}(2, R)/kZ$ for $k = 0, 3, 4, 5, 6, \cdots$, where $\widetilde{SL}(2, R)$ is the universal covering group of $SL(2, R)$. The first of these groups, however, is easily handled by the methods of [2] (see remark A at the end of that paper). We wish now to finish the calculations for Lie groups of dimension $\leqq 3$ by discussing $\widetilde{SL}(2, R)/kZ$.

2. **Results.** The group $\widetilde{SL}(2, R)/kZ$ is locally isomorphic to $\widetilde{SL}(2, R)/2Z = SL(2, R)$; we always identify its Lie algebra with $\mathfrak{sl}(2, R)$, the set of 2×2 real matrices of trace zero.

THEOREM. *The following is a list of all minimal generating sets for $\widetilde{SL}(2, R)/kZ$ up to equivalence, and the corresponding orders of generation of $\widetilde{SL}(2, R)/kZ$. When a group has order of generation n, the last column lists those expressions of length n which give the entire group (for instance, XYX means that every element of the group can be written in the form* $\exp(t_1X)\exp(t_2Y)\exp(t_3X)$).

PROOF. We can suppose $k \neq 1, 2$, for $PSL(2, R) = \widetilde{SL}(2, R)/Z$ was considered in [3] and $SL(2, R) = \widetilde{SL}(2, R)/2Z$ was considered in [6].

In [2] we classified minimal generating sets for $SL(2, R)$. This classification remains valid for $\widetilde{SL}(2, R)/kZ$ since each automorphism of the Lie algebra $\mathfrak{sl}(2, R)$ comes from an automorphism of $\widetilde{SL}(2, R)/kZ$. Indeed if σ_* is an automorphism of $\mathfrak{sl}(2, R)$, σ_* induces an automorphism σ of $\widetilde{SL}(2, R)$ which takes the center Z of $SL(2, R)$ back to itself; hence σ takes kZ to kZ and induces an automorphism of $\widetilde{SL}(2, R)/kZ$.

It is easy to dispose of the first three generating sets on our list. Consider first the elliptic-elliptic case. There is a canonical map $\widetilde{SL}(2, R)/kZ \rightarrow \widetilde{SL}(2, R)/Z = PSL(2, R)$, so the order of generation of $\widetilde{SL}(2, R)/kZ$ must be greater than or equal to the corresponding order of generation of $PSL(2, R)$; this order is ∞ by [3].

Consider next the elliptic-parabolic and elliptic-hyperbolic cases. Expressions of the form YXY do not give all of $PSL(2, R)$ [3], so they cannot give all of $\widetilde{SL}(2, R)/kZ$. It suffices to show that every element of $\widetilde{SL}(2, R)$ can be written in the form XYX. Let $g \in \widetilde{SL}(2, R)$ and call the natural map from $\widetilde{SL}(2, R)$ to $PSL(2, R)$ "π". Then $\pi(g)$ can be written in the form $\exp(t_1X)\exp(t_2Y)\exp(t_3X)$ by [3]. Of course exp is the usual map from $\mathfrak{sl}(2, R)$ to $PSL(2, R)$; if by abuse of notation we let it also denote the map from $\mathfrak{sl}(2, R)$ to $\widetilde{SL}(2, R)$, then $\pi(g) = \pi(\exp(t_1X)\exp(t_2Y)\exp(t_3X))$ and so $g = n\exp(t_1X)\exp(t_2Y)\exp(t_3X)$ where $n \in \operatorname{Ker}\pi$. However, we will show in the next paragraph that every element of $\operatorname{Ker}\pi$ can be written in the form $\exp(tX)$ for some t, so $g = \exp(tX)\exp(t_1X)\exp(t_2Y)\exp(t_3X) = \exp([t + t_1]X)\exp(t_2Y)\exp(t_3X)$.

If G is an arbitrary connected Lie group with universal covering group $\tilde{G}$ and covering map $\pi : \tilde{G} \to G$, there is a canonical isomorphism $\Psi : \pi_1(G) \to \operatorname{Ker} \pi$; if $\nu : [0, 1] \to G$ represents $\xi \in \pi_1(G)$ and $\tilde{\nu} : [0, 1] \to \tilde{G}$ is the lift of ν to $\tilde{G}$, $\Psi(\xi) = \tilde{\nu}(1)$. In our case the injection

$$\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\} / \pm I \to PSL(2, R)$$

induces an isomorphism of fundamental groups, so $\nu_n(t) = \exp(\pi n t X) : [0, 1] \to PSL(2, R)$ represents $n \in Z = \pi_1(PSL(2, R))$; by the same abuse of notation used in the previous paragraph, $\exp(\pi n X)$ equals $n \in Z = \operatorname{Ker} \pi$.

The remaining cases require more thought. Recall that $PSL(2, R)$ acts on the projective line $P^1 = R \cup \{\infty\}$ by $x \to (ax + b)/(cx + d)$. Call an ordered triple (x_1, x_2, x_3) in $P^1 \times P^1 \times P^1$ *oriented* if there is a cyclic permutation σ such that $-\infty < x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)} \leqq \infty$. Whenever (x_1, x_2, x_3) and (y_1, y_2, y_3) are oriented triples, $PSL(2, R)$ contains a unique element mapping x_i to y_i.

Fix a point $A \in P^1$. The map $g \to g(A)$ from $PSL(2, R)$ to P^1 induces an isomorphism of fundamental groups; indeed it is well known that

$$\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\} / \pm I \to PSL(2, R)$$

induces an isomorphism of fundamental groups, and

$$\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\} / \pm I \to PSL(2, R) \to P^1$$

is a homeomorphism.

Recall the isomorphism $\operatorname{Ker}[\pi : \widetilde{SL}(2, R) \to PSL(2, R)] \cong \pi_1(PSL(2, R))$ discussed earlier. Combining it with the above isomorphism, we find a canonical isomorphism $\operatorname{Ker} \pi \cong \pi_1(P^1) \cong Z$. If $\tilde{\nu}$ is a path in $\widetilde{SL}(2, R)$ starting at the identity and ending at $n \in \operatorname{Ker} \pi$, $(\pi \circ \nu)(A)$ goes around P^1 n times.

The universal covering space L of P^1 is, of course, homeomorphic to the real line. Without describing the covering map $\tau : L \to P^1$ in detail, let us imagine it so chosen that $\tau^{-1}(\infty)$ consists of all integers and $x \to x + n$ is a covering map whenever n is an integer.

Fix an oriented triple $(A, B, C) \in P^1 \times P^1 \times P^1$ and a point $A_L \in L$ over A. Whenever $\nu : [0, 1] \to \widetilde{SL}(2, R)$ is a path starting at the identity, $\pi \circ \nu(1)$ maps (A, B, C) to a unique oriented triple (a, b, c), and $(\pi \circ \nu(t))(A)$ is a path in P^1 from A to a; this path uniquely lifts to a path in L from A_L to a point a_L over a. Occasionally we write $a_L(\nu)$ to indicate the dependence of a_L on ν.

Suppose $\mu : [0, 1] \to \widetilde{SL}(2, R)$ is a second path starting at the identity. Then $\pi \circ \nu(1) = \pi \circ \mu(1)$ if and only if ν and μ are associated with the same triple (a, b, c). In this case $\nu(1) = n\mu(1)$ where $n \in \operatorname{Ker}\pi$, and $a_L(\nu) = a_L(\mu) + \hat{n}$; we claim $n = \hat{n}$. In fact, let $\sigma(t)$ be the path in $\widetilde{SL}(2, R)$ obtained by tracing $\nu(t)$ and then tracing $n\mu(t)$ backward; σ starts at the identity and ends at n. Therefore $(\pi \circ \sigma(t))(A)$ goes around P^1 n times and its lift to L starts at A_L and ends at $A_L + n$. But $(\pi \circ \sigma(t))(A)$ is just $(\pi \circ \nu(t))(A)$ followed by $(\pi \circ n\mu(t))(A)$ traced backward. The lift of the first path begins at A_L and ends at $a_L(\nu)$; the lift of the second path begins at $a_L(\mu)$ and ends at A_L. Equivalently we can lift the second path so that it begins at $a_L(\nu) = a_L(\mu) + \hat{n}$ and ends at $A_L + \hat{n}$, so $\hat{n} = n$.

Consider the expression $\exp(t_1X_1) \cdots \exp(t_\ell X_\ell)$ in $\widetilde{SL}(2, R)$, where $X_1, \cdots, X_\ell$ are elements of $s\ell(2, R)$, not necessarily distinct. There is an obvious path from the identity to this element obtained by setting $t_1 = \cdots = t_\ell = 0$ initially, then gradually changing t_ℓ to its final value, then changing $t_{\ell-1}$ from 0 to its final value, etc. Therefore, $\exp(t_1X_1) \cdots \exp(t_\ell X_\ell)$ is associated with an oriented triple (a, b, c) and a point $a_L \in L$. Indeed, (A, B, C) is mapped to (a, b, c) by moving it first via X_ℓ to a triple $(a_{\ell-1}, b_{\ell-1}, c_{\ell-1})$, then moving $(a_{\ell-1}, b_{\ell-1}, c_{\ell-1})$ to $(a_{\ell-2}, b_{\ell-2}, c_{\ell-2})$ by $X_{\ell-1}$, and so forth, until finally (a_1, b_1, c_1) is moved to (a, b, c) by X_1. Moreover, A_L is simultaneously moved to a_L by the lifted actions of the $\exp(tX_i)$ on L.

If we are given a family of expressions $\{\exp(t_1X_1) \cdots \exp(t_\ell X_\ell), \cdots\}$ every element of $\widetilde{SL}(2, R)/kZ$ can be written in one of these forms just in case (A, B, C) can be carried to any oriented triple (a, b, c) by a series of motions "X_ℓ, then $X_{\ell-1}, \cdots$, then X_1", etc., in at least k ways so that the resulting points $a_{L_1}, \cdots, a_{L_k}$ are inequivalent modulo kZ.

After these general remarks, let us turn to a specific example to see how everything works out in practice! Consider the parabolic-parabolic case:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since

$$\exp(tX) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

$\exp(tX)(p) = p + t$; similarly $\exp(tY)(p) = p/(pt + 1)$. Notice that $\exp(tX)$ leaves ∞ fixed and acts transitively on R; $\exp(tY)$ leaves 0 fixed and acts transitively on $P^1 - \{0\}$. Choose the covering map $\tau : L \to P^1$ so that $\tau(1/2) = 0$.

Lemma 1. *The order of generation of* $\widetilde{SL}(2, R)$ *with respect to* X, Y *is* ∞; *if* $k \geqq 2$ *is even, the order of generation of* $\widetilde{SL}(2, R)/kZ$ *with respect to* X, Y *is at least* $k + 2$.

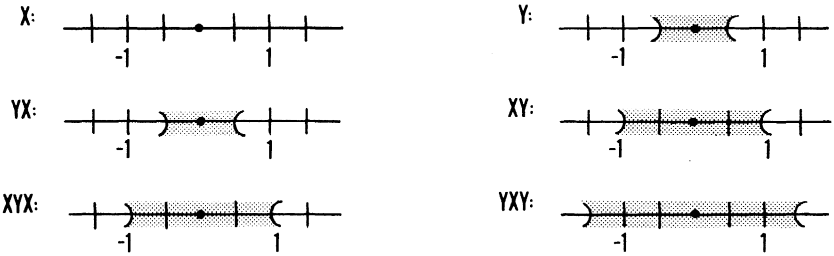

Figure 1

Proof. We will show that it takes at least $k + 2$ terms to produce k points in $\pi^{-1}(e)$. Choose $(A, B, C) = (\infty, 0, 1)$, $A_L = 0$; then (a, b, c) also equals $(\infty, 0, 1)$. The successive images of A_L in L must belong to the shaded regions above. The only way we can get k integral points in the union of the shaded regions associated with expressions with fewer than $k + 2$ terms is to use at least one of the integral points at the extremes of the shaded region $(-(k + 1)/2, (k + 1)/2)$ belonging to the expression $Y \cdots Y$ with $k + 1$ terms. However, neither of these points can come from an expression mapping $(\infty, 0, 1)$ to $(\infty, 0, 1)$. For instance, consider the point at the right of the region; let $B_L = 1/2$ in L and watch B_L move under the series of maps being considered. Each map preserves order in L, so B_L must move even further to the right than $k/2$. But Y leaves B_L fixed, XY moves it into $(0, 1) \subset (-1, 1)$, YXY moves it into $(-3/2, 3/2)$, etc., so the image of B_L is in $(-(k + 1)/2, (k + 1)/2)$ and there is no point in $(k/2, (k + 1)/2)$ equivalent to $1/2$.

Lemma 2. *If* $k \geqq 3$ *is odd, the order of generation of* $\widetilde{SL}(2, R)/kZ$ *with respect to* X, Y *is at least* $k + 2$.

Proof. We will show that it takes at least $k + 2$ terms to produce k points in

$$\pi^{-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Choose $(A, B, C) = (\infty, 0, 1)$, $A_L = 0$; then $(a, b, c) = (0, \infty, -1)$. The only way we can get k half integral points in the union of the shaded regions associated with expressions with fewer than $k + 2$ terms is to use at least one of the half integral points at the extremes of the region $(-(k + 1)/2, (k + 1)/2)$ belonging to the expression $X \cdots Y$ with $k + 1$

terms. Exactly as before, neither of these can come from an expression mapping $(\infty, 0, 1)$ to $(0, \infty, -1)$.

LEMMA 3. *Let (a, b, c) be an oriented triple such that $a \neq \infty$. There is an expression of the form XYX taking $(\infty, 0, 1)$ to (a, b, c) and $A_L = 0$ to $a_L \in (-1, 0)$, and a second such expression taking A_L to $a_L \in (0, 1)$.*

PROOF. It is easier to work backward. Note that X applied to (a, b, c) given $(a - \lambda, b - \lambda, c - \lambda)$. If eventually (a, b, c) is to go to $(\infty, 0, 1)$, Y must take $a - \lambda$ to ∞ since ∞ is a fixed point of X. Therefore

$$Y(p) = \frac{p}{1 - \dfrac{p}{a - \lambda}}$$

and YX maps (a, b, c) to $(\infty, (a - \lambda)(b - \lambda)/(a - b), (a - \lambda)(c - \lambda)/(a - c))$. A final translation can carry this to $(\infty, 0, 1)$ just in case

$$\left| \frac{(a - \lambda)(b - \lambda)}{a - b} - \frac{(a - \lambda)(c - \lambda)}{a - c} \right| = 1$$

(remember that all triples are oriented). So we want to choose λ such that $|a - \lambda|^2 \, |(b - c)/(a - b)(a - c)| = 1$; this is possible in exactly two ways. For one of the two ways $a - \lambda < 0$, so $a_L \in (0, 1)$; for the other $a - \lambda > 0$ and $a_L \in (-1, 0)$.

LEMMA 4. *If $k \geqq 2$, every element of $\widetilde{SL}(2, R)/kZ$ can be written in the form $\cdots XYX$ using $k + 2$ terms.*

PROOF. As usual let $(A, B, C) = (\infty, 0, 1)$, $A_L = 0$. Let (a, b, c) be an arbitrary oriented triple. The earlier picture shows that we can map A_L to k elements a_L in L covering a, inequivalent modulo kZ, by expressions $\cdots XYX$ with at most $k + 2$ terms. Consider a typical such expression and assume that no term is the identity. Its inverse carries (a, b, c) to (∞, β, ν) and its last three terms XYX carry (∞, β, ν) to $(\alpha_1, \beta_1, \nu_1)$. The element $a_{L,1}$ in L over α_1 belongs to $(-1, 1)$. Note that $\alpha_1 \neq \infty$, for otherwise XYX carries A_L back to itself and this would require Y to be the identity. Now by lemma 3 there is a second expression $\widetilde{XYX}$ carrying $(\infty, 0, 1)$ to $(\alpha_1, \beta_1, \nu_1)$ and A_L to $a_{L,1}$; replacing $\cdots Y(XYX)$ by $\cdots Y(\widetilde{XYX})$, we obtain an expression that maps $(\infty, 0, 1)$ to (a, b, c) and A_L to a_L.

LEMMA 5. *If $k \geqq 2$, every element in $\widetilde{SL}(2, R)/kZ$ can be written in the form $\cdots YXY$ with $k + 2$ terms.*

PROOF. There is an automorphism σ of $s\mathfrak{L}(2, R)$ interchanging X and Y up to scalars; indeed $\sigma(A) = -A^T$. Thus lemma 4 implies lemma 5.

Remark. Next consider the parabolic-hyperbolic case:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Notice that $\exp(tY)$ leaves two points ± 1 in P^1 fixed and acts transitively on each of the connected components of $P^1 - \{\pm 1\}$. We can suppose $\tau(1/3) = -1$, $\tau(2/3) = 1$.

Lemma 6. *The order of generation of $\widetilde{SL}(2, R)$ with respect to X, Y is ∞; if $k \geqq 2$ is even, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is at least $k + 2$.*

Proof. As before, we will show that it takes at least $k + 2$ terms to produce k points in $\pi^{-1}(e)$. Choose $(A, B, C) = (\infty, -1, 1)$, $A_L = 0$. The successive images of A_L in L must belong to the shaded regions below.

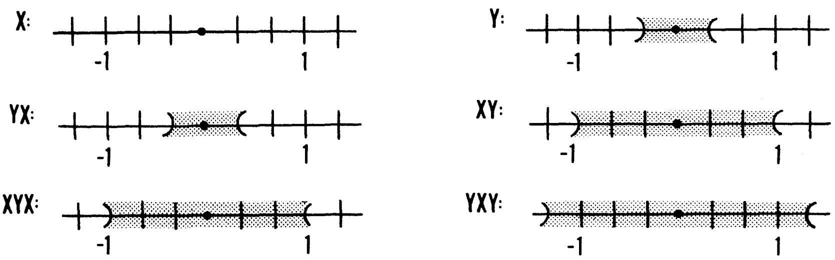

Figure 2

The rest of the argument is exactly as in the proof of lemma 1.

Lemma 7. *If $k \geqq 3$ is odd, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is at least $k + 2$. Indeed, even the expression $Y \cdots Y$ of length $k + 2$ cannot give all of $\widetilde{SL}(2, R)/kZ$.*

Proof. It suffices to prove the last statement, for any expression of length $k + 1$ can be made to look like the expressions $Y \cdots Y$ of length $k + 2$ by adding Y at the beginning or the end. Let $(A, B, C) = (-1, 1, \infty)$, $A_L = 1/3$, and let $g \in PSL(2, R)$ map this triple to $(1, -1, 0)$. Then $Y(1/3) = 1/3$, $Y(2/3) = 2/3$, $YXY(1/3)$ and $YXY(2/3)$ are contained in $(-1/3, 4/3)$, $YXYXY(1/3)$ and $YXYXY(2/3)$ are contained in $(-4/3, 7/3)$, etc., so $Y \cdots Y(1/3)$ and $Y \cdots Y(2/3)$ belong to a region D which contains exactly k points congruent to $2/3$. However, the largest of these points cannot come from a map taking $(-1, 1, \infty)$ to $(1, -1, 0)$ because $Y \cdots Y(2/3)$ would be larger than $Y \cdots Y(1/3)$ and congruent to $1/3$, and there is no such point in D.

REMARK. To finish this case, it is enough to prove that whenever $k \geqq 2$, the expression $\cdots YXYX$ of length $k+2$ generates $\widetilde{SL}(2, R)/kZ$. Indeed if k is even and $g \in \widetilde{SL}(2, R)/kZ$, write $g^{-1} = Y(t_1)X(t_2) \cdots Y(t_{k+1})X(t_{k+2})$; then $g = X(-t_{k+2})Y(-t_{k+1}) \cdots X(-t_2)Y(-t_1)$, so $\cdots XYXY$ also generates $\widetilde{SL}(2, R)/kZ$.

LEMMA 8. *Let (a, b, c) be an oriented triple with $a \neq \infty$ and let $a_L \in (-1, 1)$ cover a. There is an expression of the form $YXYX$ mapping $(\infty, -1, 1)$ to (a, b, c) and $A_L = 0$ to a_L.*

PROOF. Notice that in L, $YXYX(0) \in (-4/3, 4/3)$. Since every element of $SL(2, R)$ can be written in the form $YXYX$ [6], for each oriented triple (a, b, c) there exist two expressions of the form $YXYX$ mapping $(\infty, -1, 1)$ to (a, b, c) and taking $A_L = 0$ to a_1 and a_2 respectively, such that a_1 and a_2 are inequivalent modulo $2Z$. But points in $[-2/3, 2/3]$ are equivalent modulo $2Z$ only to themselves in $(-4/3, 4/3)$, so each element of $[-2/3, 2/3]$ occurs in this way. We must investigate the intervals $(-1, -2/3)$ and $(2/3, 1)$; by symmetry it suffices to study $(2/3, 1)$.

We shall work backward from (a, b, c) to $(\infty, -1, 1)$; since elements in $\exp(tX)$ are translations preserving ∞, it suffices to find an expression of the form YXY mapping (a, b, c) to $(\infty, \tilde{b}, \tilde{c})$ and a_L to 0 such that $|\tilde{c} - \tilde{b}| = 2$. We are already supposing $a_L \in (2/3, 1)$; it is easy to see that after application of a suitable expression in $\exp(tY)$, we can also suppose that b and c are represented by b_L and c_L in L such that $0 < b_L < c_L < a_L$. Applying a suitable X, we can assume $0 < b_L < c_L < a_L < 1/3$. Having now used the first Y and X available to us, we must show that whenever (a, b, c) is an oriented triple covered by a_L, b_L, c_L in L and $0 < b_L < c_L < a_L < 1/3$, there is an expression of the form YX mapping a_L to 0 and (a, b, c) to $(\infty, \tilde{b}, \tilde{c})$ such that $|\tilde{b} - \tilde{c}| = 2$.

For each $x \in (0, 1/3)$, there is a unique t_1 such that $\exp(t_1X)$ maps a_L to x; let this element map b_L and c_L to $b_L(x)$ and $c_L(x)$ covering $b(x)$ and $c(x)$ in P^1. Notice that $0 < b_L(x) < c_L(x) < x < 1/3$, so $-\infty < b(x) < c(x) < \tau(x) < -1$. For each $x \in (0, 1/3)$, there is a unique t_2 such that $\exp(t_2Y)$ maps x to 0; let this element map $b_L(x)$ and $c_L(x)$ to $\tilde{b}_L(x)$ and $\tilde{c}_L(x)$ covering $\tilde{b}(x)$ and $\tilde{c}(x)$ in P^1. Notice that $-1/3 < \tilde{b}_L(x) < \tilde{c}_L(x) < 0$, so $1 < \tilde{b}(x) < \tilde{c}(x) < \infty$. Clearly $|\tilde{c}(x) - \tilde{b}(x)|$ is a continuous function of x. To complete the proof, it is enough to show that $|\tilde{c}(x) - \tilde{b}(x)| \to 0$ as $x \to 1/3$ and $|\tilde{c}(x) - \tilde{b}(x)| \to \infty$ as $x \to 0$.

Whenever a, b, c and d are four distinct points in P^1, the cross ratio $\langle a, b; c, d\rangle$ is by definition $(a - c)/a - d) \cdot (b - d)/(b - c)$; recall that the action of $PSL(2, R)$ on P^1 preserves cross ratios. Hence $\langle 1, -1; \tau(x), c(x)\rangle = \langle 1, -1; \infty, \tilde{c}(x)\rangle$ and $(1 - \tau(x)/(1 - c(x)) \cdot (-1 - c(x)/(-1 - \tau(x)) = (-1 - \tilde{c}(x))/(1 - \tilde{c}(x))$. As $x \to 1/3$, $\tau(x) \to -1$; moreover $c(x) - \tau(x) = c - a$ because $\exp(tX)$ is a translation so $c(x) \to c - a - 1 \neq -1$. Hence $(-1 - \tilde{c}(x))/(1 - \tilde{c}(x)) \to \infty$, so $\tilde{c}(x) \to 1$. Since $1 < \tilde{b}(x) < \tilde{c}(x) < \infty$, $|\tilde{c}(x) - \tilde{b}(x)| \to 0$.

Next we study the situation as $x \to 0$. Then $\langle 1, -1; \tau(x), c(x)\rangle = \langle 1, -1; \infty, \tilde{c}(x)\rangle$ so $(1 - \tau(x))/(1 - c(x)) \cdot (-1 - c(x))/(-1 - \tau(x)) = (-1 - \tilde{c}(x))/(1 - \tilde{c}(x))$. If $x \to 0$, $c_L(x) \to 0$ since $0 < c_L(x) < x$, so $\tau(x)$ and $c(x)$ approach ∞, $(1 - \tau(x))/(-1 - \tau(x)) \cdot (-1 - c(x))/(1 - c(x)) = (-1 - \tilde{c}(x))/(1 - \tilde{c}(x))$ approaches 1, and thus $\tilde{c}(x)$ approaches ∞. Then $\langle 1, b(x); \tau(x), c(x)\rangle = \langle 1, \tilde{b}(x); \infty, \tilde{c}(x)\rangle$, so $(1 - \tau(x))/(1 - c(x)) \cdot (b(x) - c(x))/(b(x) - \tau(x)) = (\tilde{b}(x) - \tilde{c}(x))/(1 - \tilde{c}(x))$. But each element of $\exp(tX)$ acts on P^1 by translation, so $(b(x) - c(x))/(b(x) - \tau(x))$ is a non-zero constant independent of x. Similarly $\tau(x) = \tau(a_L) + \lambda(x)$ and $c(x) = \tau(c_L) + \lambda(x)$ where $\lambda(x) \to \infty$ as $x \to 0$, so $(1 - \tau(x))/(1 - c(x)) \to 1$ as $x \to 0$. Consequently $(\tilde{b}(x) - \tilde{c}(x))/(1 - \tilde{c}(x))$ approaches a non-zero constant as $x \to 0$; since $\tilde{c}(x) \to \infty$, $|\tilde{c}(x) - \tilde{b}(x)| \to \infty$.

LEMMA 9. *Let $k \geqq 3$. There is an expression $\cdots YX$ with $k + 2$ terms mapping $(\infty, -1, 1)$ to (a, b, c) and $A_L = 0$ to a_L provided $a_L \in [-k/2, k/2]$ if k is even, $a_L \in (-(k + 1)/2, (k + 1)/2)$ if k is odd. In particular $\cdots YX$ generates $\widetilde{SL}(2, R)/kZ$.*

PROOF. We prove this by induction on k. Lemma 8 suffices to begin the induction because our proof of the step $k \to k + 1$ for k even will only require the induction hypothesis when $a_L \in (-k/2, k/2)$.

Suppose the theorem is known for an even k; we prove it for $k + 1$. Let (a, b, c) and $a_L \in (-(k/2) - 1, (k/2) + 1)$ be given. It is possible to map a_L into the region $(-k/2, k/2)$ by an expression of the form $Y_1^{-1}X_1^{-1}$; suppose that a_L goes to $\tilde{a}_L$ and (a, b, c) goes to $(\tilde{a}, \tilde{b}, \tilde{c})$. When $k = 2$, we can assume $\tilde{a} \neq \infty$. By induction there is an expression $YX \cdots YX$ of length $k + 2$ taking $(\infty, -1, 1)$ to $(\tilde{a}, \tilde{b}, \tilde{c})$ and A_L to $\tilde{a}_L$. Hence $(X_1Y_1)(YX \cdots YX) = X_1(Y_1Y)X \cdots YX$ carries $(\infty, -1, 1)$ to (a, b, c) and A_L to a_L.

If the theorem is known for an odd k, (a, b, c) is a given triple, and $a_L \in [-(k + 1)/2, (k + 1)/2]$, we can find Y_1 carrying a_L to $\tilde{a}_L$ in $(-(k + 1)/2, (k + 1)/2)$ and (a, b, c) to $(\tilde{a}, \tilde{b}, \tilde{c})$; by induction there is an expression $X \cdots YX$ taking $(\infty, -1, 1)$ to $(\tilde{a}, \tilde{b}, \tilde{c})$ and A_L to $\tilde{a}_L$, so $Y_1X \cdots YX$ carries $(\infty, -1, 1)$ to (a, b, c) and A_L to a_L.

REMARK. Consider next the hyperbolic-hyperbolic (fixed points interlacing) case:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} \alpha & 1 \\ 1 & -\alpha \end{pmatrix}, \quad \alpha \geqq 0.$$

Notice that $\exp(tX)$ leaves two points $0, \infty$ fixed and $\exp(tY)$ leaves two points $\alpha \pm (\alpha^2 + 1)^{1/2}$ fixed; both $\exp(tX)$ and $\exp(tY)$ act transitively on the connected components of the complements of their fixed point sets. We can suppose $\tau(1/4) = \alpha - (\alpha^2 + 1)^{1/2}$, $\tau(1/2) = 0$, $\tau(3/4) = \alpha + (\alpha^2 + 1)^{1/2}$. Although we will refrain from drawing orbit pictures from now on, the reader will often find it useful to do so.

Since $\exp(tX)(p) = e^{2t}p$, $\exp(tX)(p)$ approaches 0 as $\tau \to -\infty$ and ∞ as $t \to \infty$; a similar statement holds for Y.

Lemma 10. *The order of generation of $\widetilde{SL}(2, R)$ with respect to X, Y is ∞. If $k \geqq 2$ is even, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is at least $2k + 4$.*

Proof. We will show that it takes at least $2k + 4$ terms to produce k points in $\pi^{-1}(e)$. Notice that any expression giving e in $PSL(2, R)$ must act on L by $x \to x + n$, $n \in Z$, since the lift to L of a motion of P^1 is uniquely determined up to covering transformations and the identity on L is one lift of the identity map on P^1. Any non-trivial motion of L induced by $\exp(tX)$ or $\exp(tY)$ maps one of 0, 1/4, 1/2, 3/4 left and one right; for instance $\exp(tX)$ for $t > 0$ acts as follows:

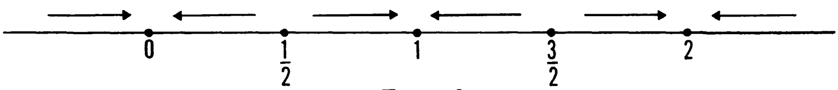

Figure 3

Suppose we are given an expression with fewer than $2k + 4$ terms. Without loss of generality we can suppose that 0 is initially left fixed and then moved left. Thus the expression begins with X, and $X(0) = 0$, $YX(0) < 0$, $XYX(0) < 0$, $YXYX(0) < 1/4$, $XYXYX(0) < 1/2$, etc., so that eventually the image of 0 is smaller than $k/2$. Hence the only translations of L, $x \to x + n$, that can be achieved are those with $n < k/2$. Similarly n must be larger than $-k/2$; there are only $k - 1$ integers in the interval $(-k/2, k/2)$.

Lemma 11. *If $k \geqq 1$ is odd, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is at least $2k + 4$.*

Proof. We will show that it takes at least $2k + 4$ terms to produce k points in

$$\pi^{-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Notice that the map $x \to -1/x$ interchanges 0 and ∞ and also interchanges $\alpha - (\alpha^2 + 1)^{1/2}$ and $\alpha + (\alpha^2 + 1)^{1/2}$. A little thought shows that we can choose the covering map $\tau : L \to P^1$ so that $\tau(x + 1/2) = -1/\tau(x)$; thus the lift of

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

to L has the form $x \to x + 1/2 + n$ for some integer n.

The proof of lemma 11 is exactly like the proof of lemma 10. Given an expression with fewer than $2k + 4$ terms, we can choose one of 0, 1/4, 1/2, 3/4, say 0, so that the initial term of the expression leaves 0 fixed and the next moves it to the left; then the final image of 0 is smaller than $k/2$. Similarly, the final image of 0 is larger than $-k/2$. The map on L thus has the form $x \to x + 1/2 + n$ where $|1/2 + n| < k/2$, and there are only $k - 1$ such integers n.

LEMMA 12. *If b in P^1 is not ∞, there is an expression of the form YXY taking ∞ to ∞ and 0 to b.*

PROOF. There is an action of Y on L taking 1/2 to 3/8; denote the image of 0 by δ and note that $0 < \delta < 1/4$. For each $t \geqq 0$, $(\exp tX)(\delta)$ belongs to the interval $(0, \delta] \subseteq (0, 1/4)$ and there is a unique $u(t)$ such that $(\exp u(t)Y \exp tX)(\delta) = 0$. Let $(\exp u(t)Y \exp tX)(3/8) = b_L(t)$; $b_L(t)$ is continuous in t and $b_L(0) = 1/2$. Notice that as $t \to \infty$, $b_L(t) \to 0$. Consequently there is an expression YXY taking 0 to 0 and 1/2 to any $b_L \in (0, 1/2]$.

Similarly there is an expression of the form YXY taking 0 to 0 and $-1/2$ to any $b_L \in [-1/2, 0)$. The lemma follows immediately by projection of these results from L to P^1.

REMARK. The orbit picture shows that the expression YXY whose existence is guaranteed by this lemma preserves $\tau^{-1}(\infty)$ in L pointwise.

LEMMA 13. *Every element of $\widetilde{SL}(2, R)/kZ$ can be written in terms of the expression $YX \cdots X$ with $2k + 4$ terms.*

PROOF. Choose $(A, B, C) = (\infty, 0, \alpha + (\alpha^2 + 1)^{1/2},)$, $A_L = 0$. Let (a, b, c) be an oriented triple, a_L an element in $[-k/2, k/2]$ covering a. The orbit picture shows that there is an expression $YX \cdots XY$ with $2k + 1$ terms mapping A_L to a_L. Let the inverse of this expression map (a, b, c) to $(\infty, \tilde{b}, \tilde{c})$. By the previous lemma, there is an expression $\widetilde{YXY}$ mapping $(\infty, 0)$ to $(\infty, \tilde{b})$. Hence the expression $(YX \cdots XY)(\widetilde{YXY}) = (YX \cdots X)(Y\tilde{Y})(\widetilde{XY})$ with $2k + 3$ terms maps A_L to a_L and $(\infty, 0, \hat{c})$ to (a, b, c) for some $\hat{c} \in (0, \infty)$. There is an $\tilde{\tilde{X}}$ taking $\alpha + (\alpha^2 + 1)^{1/2}$ to $\hat{c}$; then $(YX \cdots X)(Y\tilde{Y})\ (\widetilde{XY})\tilde{\tilde{X}}$ takes A_L to a_L and

$(\infty, 0, \alpha + (\alpha^2 + 1)^{1/2})$ to (a, b, c).

LEMMA 14. *Every element of $\widetilde{SL}(2, R)/kZ$ can be written in terms of the expression $XY \cdots Y$ with $2k + 4$ terms.*

PROOF. If $g \in \widetilde{SL}(2, R)/kZ$, write $g^{-1} = Y(t_1)X(t_2) \cdots X(t_{2k+4})$; then $g = X(-t_{2k+4}) \cdots X(-t_2)Y(-t_1)$.

REMARK. Next we consider the hyperbolic-hyperbolic (fixed points noninterlacing) case:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} \alpha & -1 \\ 1 & -\alpha \end{pmatrix}, \; \alpha > 1.$$

The fixed points of $\exp(tX)$ are 0, ∞ and those of $\exp(tY)$ are $\alpha \pm (\alpha^2 - 1)^{1/2}$. We suppose $\tau(1/4) = 0$, $\tau(1/2) = \alpha - (\alpha^2 - 1)^{1/2}$, $\tau(3/4) = \alpha + (\alpha^2 - 1)^{1/2}$.

LEMMA 15. *The order of generation of $\widetilde{SL}(2, R)$ with respect to X, Y is ∞. If $k \geqq 2$ is even, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is at least $k + 2$; indeed neither expression of length $k + 2$ can give all of $\widetilde{SL}(2, R)/kZ$.*

PROOF. Let g in $PSL(2, R)$ map $(0, 1, \infty)$ to $(\alpha - (\alpha^2 - 1)^{1/2}, 1, \alpha + (\alpha^2 - 1)^{1/2})$. We will show that the expression $YX \cdots X$ with length $k + 2$ cannot produce k points in $\pi^{-1}(g)$.

Notice that in L, $X(0) = 0$, $YX(0) \in (-1/4, 1/2)$, $XYX(0) \in (-3/4, 1)$, $YXYX(0) \in (-5/4, 3/2)$, etc., so the image of 0 under the expression with $k + 2$ terms is contained in $(-k/2 - 1/4, k/2 + 1/2)$, a region with exactly k points equivalent to $3/4$. However, the largest of these points cannot correspond to an expression giving g in $PSL(2, R)$, since the image of $1/4$ would also belong to the region described above, would be larger than the image of 0, and would be equivalent to $1/2$, and there is no such point.

Similarly we can find an element $\tilde{g} \in PSL(2, R)$ such that the expression $XY \cdots Y$ of length $k + 2$ cannot produce k points in $\pi^{-1}(\tilde{g})$.

LEMMA 16. *If $k \geqq 3$ is odd, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is at least $k + 2$; indeed neither expression of length $k + 2$ can give all of $\widetilde{SL}(2, R)/kZ$.*

PROOF. Let g in $PSL(2, R)$ map $(\infty, 0, 1)$ to $(0, \infty, -1)$; we will show that the expression $XY \cdots X$ of length $k + 2$ cannot produce k points in $\pi^{-1}(g)$.

Notice that in L, the image of 0 under the expression in question is contained in $(-(k + 1)/2 + 1/4, (k + 1)/2)$; this region contains k points equivalent to $1/4$. However, the largest of these points cannot correspond to an expression giving g in $PSL(2, R)$, since the image of 1/4 would also belong to the region described above, would be larger than the image of 0, and would be equivalent to 0, and there is no such point.

Similarly we can find an element $\tilde{g} \in PSL(2, R)$ such that the expression $YX \cdots Y$ of length $k + 2$ cannot produce k points in $\pi^{-1}(\tilde{g})$.

REMARK. We now wish to show that the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is exactly $k + 2$. As usual, we may assume $k \geqq 3$. Whenever (a, b, c) is an oriented triple of points in P^1, there is a unique fourth point d such that the element in $PSL(2, R)$ which maps $(\infty, 0, \alpha - (\alpha^2 - 1)^{1/2})$ to (a, b, c) maps $\alpha + (\alpha^2 - 1)^{1/2}$ to d. Notice that g in $PSL(2, R)$ maps $(\infty, 0, \alpha - (\alpha^2 - 1)^{1/2})$ to (a, b, c) just in case it maps *one* of the oriented triples obtained from $(\infty, 0, \alpha - (\alpha^2 - 1)^{1/2}, \alpha + (\alpha^2 - 1)^{1/2})$ by omitting a point to the corresponding triple in (a, b, c, d). The following lemmas show that whenever (a, b, c) is an oriented triple, there is an oriented triple formed by omitting one of the points of (a, b, c, d), say for purposes of discussion (b, c, d), and an expression of length $k + 2$ taking (b, c, d) to $(0, \alpha - (\alpha^2 - 1)^{1/2}, \alpha + (\alpha^2 - 1)^{1/2})$ in k ways so that the element b_L in L covering b maps to k elements $1/4 + n_1, \cdots, 1/4 + n_k$ covering 0 and if $i \neq j$, $n_i - n_j \notin kZ$. This suffices to prove that $\widetilde{SL}(2, R)/kZ$ has order of generation $k + 2$ for the inverses of the expressions in question map $(0, \alpha - \alpha^2 - 1)^{1/2}, \alpha + (\alpha^2 - 1)^{1/2})$ to (b, c, d) and $1/4 + n_i$ to b_L or (by lifting in a different way) 1/4 to $b_L - n_i$; our previous remarks show that the resulting k elements of $\widetilde{SL}(2, R)/kZ$ are unequal and their projections to $PSL(2, R)$ map $(\infty, 0, \alpha - (\alpha^2 - 1)^{1/2})$ to (a, b, c).

Notice that the interval $(-\infty, 0)$ in P^1 is an orbit of $\exp(tX)$ which excludes the fixed points of Y; the interval $(\alpha - (\alpha^2 - 1)^{1/2}, \alpha + (\alpha^2 - 1)^{1/2})$ plays the same role for Y.

LEMMA 17. *Let n be an integer and suppose a_L and b_L in L satisfy $-1/4 + n < a_L < b_L < n$ or $1/4 + n < a_L < b_L < 1/2 + n$. There is an expression of the form YX mapping a_L to n and b_L to $1/4 + n$.*

PROOF. Without loss of generality, suppose $-1/4 + n < a_L < b_L < n$. For each $x \in (-1/4 + n, n)$, there is a unique t_1 such that $(\exp t_1X)(a_L) = x$; let $(\exp t_1X)(b_L) = b_L(x)$ and notice that $-1/4 + n < x < b_L(x) < n$. There is a unique t_2 such that $(\exp t_2Y)(x) = n$; let $(\exp t_2Y)(b_L(x)) = \bar{b}_L(x)$ and notice that $n < \bar{b}_L(x) < n + 1/2$. We shall prove that when $x \to n$, $\bar{b}_L(x) \to n$ and when $x \to n - 1/4$, $\bar{b}_L(x) \to n + 1/2$; by continuity there is an x with $\bar{b}_L(x) = n + 1/4$.

We have

$$\begin{aligned}&\langle \tau(x), \tau(b_L(x)); \alpha - (\alpha^2-1)^{1/2}, \alpha + (\alpha^2-1)^{1/2}\rangle\\ &\quad = \langle \infty, \tau(\tilde{b}_L(x)); \alpha - (\alpha^2-1)^{1/2}, \alpha + (\alpha^2-1)^{1/2}\rangle\end{aligned}$$

so

$$\begin{aligned}&\frac{\tau(x) - (\alpha - (\alpha^2-1)^{1/2})}{\tau(x) - (\alpha + (\alpha^2-1)^{1/2})} \cdot \frac{\tau(b_L(x)) - (\alpha + (\alpha^2-1)^{1/2})}{\tau(b_L(X)) - (\alpha - (\alpha^2-1)^{1/2})}\\ &\qquad = \frac{\tau(\tilde{b}_L(x)) - (\alpha + (\alpha^2-1)^{1/2})}{\tau(\tilde{b}_L(x)) - (\alpha - (\alpha^2-1)^{1/2})}.\end{aligned}$$

As $x \to n$, $\tau(x) \to \infty$ and the first factor on the left approaches 1. Since $x < b_L(x) < n$, as $x \to n$, $b_L(x) \to n$ and $\tau(b_L(x)) \to \infty$, so the second factor on the left approaches 1. It follows that as $x \to n$, $\tau(\tilde{b}_L(x)) \to \infty$ and $\tilde{b}_L(x) \to n$.

Similarly

$$\begin{aligned}&\langle \tau(x), \alpha - (\alpha^2-1)^{1/2}; \tau(b_L(x)), \alpha + (\alpha^2-1)^{1/2}\rangle\\ &\quad = \langle \infty, \alpha - (\alpha^2-1)^{1/2}; \tau(\tilde{b}_L(x)), \alpha + (\alpha^2-1)^{1/2}\rangle\end{aligned}$$

so

$$\begin{aligned}&\frac{\tau(x) - \tau(b_L(x))}{\tau(x) - (\alpha + (\alpha^2-1)^{1/2})} \cdot \frac{-2(\alpha^2-1)^{1/2}}{(\alpha - (\alpha^2-1)^{1/2}) - \tau(b_L(x))}\\ &\qquad = \frac{-2(\alpha^2-1)^{1/2}}{(\alpha - (\alpha^2-1)^{1/2}) - \tau(\tilde{b}_L(x))}.\end{aligned}$$

As $x \to n$, $\tau(x) \to \infty$ and the first factor on the left approaches 1. Since map, $\tau(b_L(x))/\tau(x) = \tau(b_L)/\tau(a_L)$. Therefore as $x \to -1/4 + n$, $\tau(b_L(x)) \to \tau(b_L)(\alpha + (\alpha^2-1)^{1/2})/\tau(a_L)$ and the cross-ratio approaches ∞; it follows that $\tau(b_L(x)) \to \alpha - (\alpha^2-1)^{1/2}$ and hence $b_L(x) \to n + 1/2$.

LEMMA 18. *If $k \geqq 3$ is odd, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is $k + 2$.*

PROOF. Let (a, b, c, d) be a 4-tuple as described above. Choose a_L and b_L in L covering a and b and suppose for a moment that $-1/4 < a_L < b_L < 1/2$. Let n be an integer satisfying $|n| \leqq (k-1)/2$. The reader can easily show that an expression with $k - 1$ terms of the form $XY \cdots XY$ exists mapping a_L and b_L to $\tilde{a}_L$ and $\tilde{b}_L$ where $-1/4 + n < \tilde{a}_L < \tilde{b}_L < n$ or $1/4 + n < \tilde{a}_L < \tilde{b}_L < 1/2 + n$. By lemma 17, there is an expression of the form $\widetilde{YX}$ mapping $\tilde{a}_L$, $\tilde{b}_L$ to n, $1/4 + n$; then $(\widetilde{YX})(XY \cdots XY) = \tilde{Y}(\tilde{X}X)(Y \cdots XY)$ is an expression of length k mapping a_L to n and (a, b) to $(\infty, 0)$. Since the image of c and $\alpha - (\alpha^2-1)^{1/2}$ belong to the same component of $P^1 - \{0, \infty\}$ (because all triples are oriented), we can find an element of $\exp(tX)$ leaving ∞ and

0 fixed and mapping the image of c to $\alpha - (\alpha^2 - 1)^{1/2}$; thus we can find an expression of length $k + 1$ mapping (a, b, c) to $(\infty, 0, \alpha - (\alpha^2 - 1)^{1/2})$ and a_L to n. By previous remarks, this proves the lemma; since only $k + 1$ terms have been used, we have an extra term at our disposal with which to force the original assumption on a_L and b_L.

Finally, suppose a and b are arbitrary. Choose a_L and b_L in $[0, 1)$ covering a and b. If $a_L < b_L$, an expression of the form $\exp(tX)$ exists mapping both into the interval $[0, 1/2)$ and the above argument takes over from there. If $0 \leqq b_L \leqq 1/4 < a_L < 1$, an expression of the form $\exp(tX)$ exists leaving b_L in $[0, 1/4]$ and mapping a_L into $(3/4, 1)$; this last point is equivalent to a point in $(-1/4, 0)$, so the previous argument takes over once more. We are done unless $0 \leqq b_L < a_L \leqq 1/4$ or $1/4 < b_L < a_L < 1$.

Similar arguments hold for the pair (c, d). In this case we choose c_L, d_L in $(-1/4, 3/4]$; we are done unless $-1/4 < d_L < c_L < 1/2$ or $1/2 \leqq d_L < c_L \leqq 3/4$. But (a, b, c, d) is an oriented 4-tuple, so $0 \leqq b_L < a_L \leqq 1/4$ implies $0 \leqq b_L < c_L < d_L < a_L \leqq 1/4$ and we are done; $1/2 \leqq d_L < c_L \leqq 3/4$ implies $1/2 \leqq d_L < a_L < b_L < c_L \leqq 3/4$ and we are again done. We can have trouble only if $1/4 < b_L < a_L < 1$ and $-1/4 < d_L < c_L < 1/2$. In this case if $c_L < 0$, $d_L + 1$ and $c_L + 1$ are the unique representatives of c and d in $[0, 1)$; then $b_L < c_L + 1 < d_L + 1 < a_L$, contradicting $d_L < c_L$. If $d_L \geqq 0$, c_L and d_L are the representatives of c and d in $[0, 1)$ and again $c_L < d_L$. Hence $-1/4 < d_L < 0$ and $0 \leqq c_L < 1/2$; then c_L and $d_L + 1$ are the unique representatives of c and d in $[0, 1)$, so $b_L < c_L < d_L + 1 < a_L$, and $b_L \in (1/4, 1/2)$, $a_L \in (3/4, 1)$. Therefore a has a second representative $\tilde{a}_L$ in $(-1/4, 0)$ and the arguments given earlier apply to (a, b).

Lemma 19. *If $k \geqq 2$ is even, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y is $k + 2$.*

Proof. Let (a, b, c, d) be a 4-tuple of the usual kind, and choose a_L and b_L covering a and b. Suppose for a moment that $1/4 < a_L < b_L < 1$. Let n be an integer satisfying $-k/2 < n \leqq k/2$. The reader can easily show that an expression of the form $XY \cdots YX$ with length $k - 1$ exists mapping a_L and b_L to $\tilde{a}_L$ and $\tilde{b}_L$ where $-1/4 + n < \tilde{a}_L < \tilde{b}_L < n$ or $1/4 + n < \tilde{a}_L < \tilde{b}_L < 1/2 + n$. By lemma 17, there is an expression of the form YX mapping $\tilde{a}_L$ and $\tilde{b}_L$ to n and $1/4 + n$; $(\widetilde{YX})(XY \cdots YX) = \tilde{Y}(\tilde{X}X)(Y \cdots X)$ is an expression of length k mapping a_L and b_L to n and $1/4 + n$. From here on, the proof follows that given for lemma 18.

Suppose next that a and b are arbitrary; choose a_L and b_L in $(-1/4, 3/4]$ covering a and b. If $a_L < b_L$, an expression of the form $\exp(tY)$ maps both into the interval $(1/4, 3/4]$ and the previous argument takes over. If $-1/4 < b_L < 1/2 \leqq a_L \leqq 3/4$ an expression of the form $\exp(tY)$ leaves a_L in $[1/2, 3/4]$ and maps b_L into $(-1/4, 0)$; this last point is equivalent to a point in $(3/4, 1)$ and the previous argument takes over again. We are done unless $-1/4 < b_L < a_L < 1/2$ or $1/2 \leqq b_L < a_L \leqq 3/4$.

Similar arguments hold for the pair (c, d). In this case we choose $c_L, d_L \in [0, 1)$ and we are done unless $0 \leqq d_L < c_L \leqq 1/4$ or $1/4 < d_L < c_L < 1$. By an argument similar to that of lemma 18, both bad situations can occur only if $a_L \in (1/4, 1/2)$ and $b_L \in (-1/4, 0)$. But then b has a second representative $b_L \in (3/4, 1)$ and earlier arguments apply to (a, b).

REMARK. Finally, consider the case

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}.$$

The fixed points of $\exp(tX)$ are $0, \infty$, those of $\exp(tY)$ are $-1, \infty$, and those of $\exp(tZ)$ are $-1, 0$. We suppose $\tau(1/3) = -1, \tau(2/3) = 0$.

Any two of X, Y, Z generate a two-dimensional subgroup; the order of generation of all two-dimensional Lie groups is two [5]. Consequently, if an expression equals g in $\widetilde{SL}(2, R)$ and contains three consecutive terms from the same pair, there is a shorter expression which also equals g. A little thought shows that we can restrict attention to expressions in which X, Y, and Z appear cyclically.

LEMMA 20. *The order of generation of $\widetilde{SL}(2, R)$ with respect to X, Y, Z is ∞. If $k \geqq 2$ is even, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y, Z is at least $(3k + 6)/2$. Moreover, no expression of length $(3k + 6)/2$ can give all of $\widetilde{SL}(2, R)/kZ$.*

LEMMA 20. *The order of generation of $SL(2, R)$ with respect to*

PROOF. Consider the expression $YX \cdots ZYX$ of length $(3k + 6)/2 - 1$ and suppose it gives $e \in PSL(2, R)$. In L, $X(0) = 0$, $YX(0) = 0$, $ZYX(0) < 1/3$, $XZYX(0) < 2/3$, etc., so the image of 0 is smaller than $k/2$. Similarly $ZYX(0) > -1/3$, $XZYX(0) > -1/3$, $YXZYX(0) > -2/3$, $ZYXZYX(0) > -2/3$, etc., so the image of 0 is considerably larger that $-k/2$. Hence the expression acts on L by $x \to x + n$ where $|n| < k/2$. The same result holds for any cyclic expression of length $(3k + 6)/2 - 1$. It follows that no combination of expressions of length less than $(3k + 6)/2$ can give k points in $\pi^{-1}(e)$.

Even the expression $ZYX \cdots ZYX$ of length $(3k + 6)/2$ does not give every element of $\widetilde{SL}(2, R)/kZ$. Indeed the image of 0 in L under this expression belongs to the interval $(-k/2 + 1/3, k/2 + 1/3)$; this interval contains only $k - 1$ points equivalent to 1/3. Thus if $g \in PSL(2, R)$ maps $(\infty, -1, 0)$ to $(-1, 0, \infty)$, the expression $\cdots YXZYX$ of length $(3k + 6)/2$ gives at most $k - 1$ points in $\pi^{-1}(g)$.

LEMMA 21. *If $k \geqq 3$ is odd, the order of generation of $\widetilde{SL}(2, R)/kZ$ with respect to X, Y, Z is at least $(3k + 5)/2$. Moreover, no expression of length $(3k + 5)/2$ can give all of $\widetilde{SL}(2, R)/kZ$.*

PROOF. Let g in $PSL(2, R)$ map $(\infty, -1, 0)$ to $(0, \infty, -1)$. Some thought shows that we can choose τ so the map from L to L given by $x \to x + 2/3$ covers g. Consider the expression $ZYX \cdots ZYX$ of length $(3k + 5)/2 - 1$ and suppose it gives g in $PSL(2, R)$. In L, the image of 0 is contained in $(-(k - 1)/2 - 1/3,\ (k - 1)/2 + 1/3)$. Hence the action of the expression on L is $x \to x + \lambda$ where $|\lambda| < (k - 1)/2 + 1/3$. A similar result holds for any cyclic expression of length $(3k + 5)/2 - 1$. But there are only $k - 1$ numbers in $(-(k - 1)/2 - 1/3, (k - 1)/2 + 1/3)$ equivalent to $2/3$.

Even the expression $XZYX \cdots ZYX$ of length $(3k + 5)/2$ does not give k points in $\pi^{-1}(g)$, for in L the image of 0 is, in fact, less than $(k - 1)/2 + 2/3$ and larger than $-(k - 1)/2$, and this interval contains only $k - 1$ points equivalent to $2/3$.

LEMMA 22. *Let (a, b, c) be an oriented triple, and let a_L in L cover a. Suppose there is an expression of length ℓ taking $A_L = 0$ to a_ℓ. Then there is an expression of length $\ell + 2$ taking A_L to a_L and $(\infty, -1, 0)$ to (a, b, c).*

PROOF. Let the inverse of the expression in question map (a, b, c) to $(\infty, \tilde b, \tilde c)$; it is enough to find an expression with two terms fixing A_L and mapping $(\infty, -1, 0)$ to $(\infty, \tilde b, \tilde c)$.

If $-1 < c$, there is an element in $\exp(tY)$ mapping 0 to $\tilde c$. If this expression maps $\tilde{\tilde b}$ to $\tilde b$, it maps $(\infty, \tilde{\tilde b}, 0)$ to $(\infty, \tilde b, \tilde c)$; since all triples are oriented, $\tilde{\tilde b} < 0$ and there is an element in $\exp(tX)$ mapping -1 to $\tilde{\tilde b}$, so YX maps $(\infty, -1, 0)$ to $(\infty, \tilde b, \tilde c)$.

If $\tilde c \leqq -1$, $\tilde b < \tilde c < 0$ and there is an element in $\exp(tX)$ mapping -1 to $\tilde b$. Let this expression map $\tilde{\tilde c}$ to $\tilde c$; then $(\infty, -1, \tilde{\tilde c})$ maps to $(\infty, \tilde b, \tilde c)$, so $-1 < \tilde{\tilde c}$. Hence there is an element in $\exp(tY)$ mapping 0 to $\tilde{\tilde c}$ and XY maps $(\infty, -1, 0)$ to $(\infty, \tilde b, \tilde c)$.

LEMMA 23. *Let $k \geqq 2$ be even and let $|a_L| \leqq k/2$; there is an expression of length $3k/2 + 1$ mapping $A_L = 0$ to a_L. Hence the order of generation of $\widetilde{SL}(2, R)/kZ$ is $(3k + 6)/2$.*

PROOF. If $a_L > 0$, the expression $ZYX \cdots YXZ$ suffices; indeed $Z(0)$ can be any point in $[0, 1/3)$, $XZ(0)$ can be any point in $[0, 2/3)$, etc. If $a_L < 0$, the expression $ZXY \cdots XYZ$ similarly suffices.

LEMMA 24. *Let $k \geqq 3$ be odd and let $|a_L| \leqq k/2$; there is an expression of length $(3k + 1)/2$ mapping $A_L = 0$ to a_L. Hence the order of generation of $\widetilde{SL}(2, R)/kZ$ is $(3k + 5)/2$.*

PROOF. Exactly as for lemma 23.

					Order of Generation $k = 0$	Order of Generation $k = 1$	Order of Generation $k \geqq 2$	Expressions Giving All of G
elliptic:	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	elliptic: $-1 < \alpha < 0$	$\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$	—	∞	∞	∞	—
elliptic:	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	parabolic:	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	—	3	3	3	XYX
elliptic:	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	hyperbolic: $0 < \alpha \leqq 1$	$\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$	—	3	3	3	XYX
parabolic:	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	parabolic:	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	—	∞	4	$k + 2$	$\cdots XYX$ and $\cdots YXY$
parabolic:	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	hyperbolic:	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	—	∞	4	$k + 2$	$\cdots XYX$ and $\cdots YXY$ if $k = 1$ or k even; $X \cdots X$ if $k \geqq 3$ and k odd
hyperbolic:	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	hyperbolic: (fixed points non-interlacing) $\alpha > 1$:	$\begin{pmatrix} \alpha & -1 \\ 1 & -\alpha \end{pmatrix}$	—	∞	4	$k + 2$	None
hyperbolic:	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	hyperbolic (fixed points interlacing) $\alpha \geqq 0$:	$\begin{pmatrix} \alpha & 1 \\ 1 & -\alpha \end{pmatrix}$	—	∞	6	$2k + 4$	$\cdots XYX$ and $\cdots YXY$
	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$		$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$		∞	4	$\frac{3k+5}{2}$ for k odd; $\frac{3k+6}{2}$ for k even	None

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