ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 3, Summer 1977

PROJECTION METHODS FOR NONLINEAR NODAL PROBLEMS

P. W. BATES AND G. B. GUSTAFSON*

ABSTRACT. The work produces some degree-theoretic and operator-theoretic tools for the study of projection methods associated with nonlinear nodal problems. The numerical procedure is outlined for the case of sublinear and superlinear nodal problems.

1. **Introduction.** This work investigates equations of the form

$$y''(t) + f(t, y(t), y'(t)) = 0, 0 < t < 1, \tag{1.1}$$

with the boundary and nodal condition

$$y(0) = y(1) = 0, y \text{ has exactly } n \text{ distinct zeros in } (0, 1). \tag{1.2}$$

The objective is to prove existence of solutions to (1.1), (1.2) and to approximate such solutions by solving finite dimensional problems.

To accomplish the above task, the problem (1.1), (1.2) is converted to a coupled pair of operator equations (Section 3) to which a projection method is applied (Sections 10, 11). Existence of solutions to the operator equations is demonstrated by fixed point techniques which use topological degree (Section 4). In addition the method enables one to deduce the existence of solutions to the "approximate equations" resulting from the projection scheme and, further, to show convergence of a sequence of solutions of the "approximate equations" to a solution of the original operator equation.

The projection method employed is tailored to the nodal properties of solutions. In addition, the method may be applied to the operator equations without modification.

Equation (1.1) will be investigated for two general cases, (i) the superlinear case, (Section 14), for example

$$\begin{aligned} &y''(t) + bt\sin^2(1/t)(1 + |y'(t)|^k)|y(t)|^\alpha y(t) = 0, \\ &0 < t < 1, \text{ where } 0 \leqq k \leqq 2 \text{ and } \alpha, b > 0, \end{aligned} \tag{1.3}$$

(ii) the sublinear case, (Section 15), for example

$$y''(t) + bt\sin^2(1/t)[1 - |y(t)|)|y(t)|^\beta(|y'(t)|^2 + 1) + \sin^2 y'(t) + 1]$$

$$y(t) = 0, |y(t)| \leqq 1, 0 < t < 1, \tag{1.4}$$

*The second author was supported by the U.S. Army under Grant Number DAHC-04-74-G-0208.

$$y''(t) + bt\sin^2(1/t)(\sin^2 y'(t) + 1)y(t) = 0, |y(t)| \geqq 1, 0 < t < 1,$$
$$\text{where } -1 < \beta < 0 < b < 4.$$

During the latter part of the 1950's Nehari [11], [12], studied the problem (1.1), (1.2) using variational techniques, minimizing a nonlinear functional over a certain class of admissible functions. The conditions which he imposed upon the function f were rather specialized and many of his results are extended in this work. In the decade which followed, very little progress was made, then a new method of attack was introduced by P. Rabinowitz [1], [15], [16]. His technique was to write (1.1) as an integral equation, and to seek solutions of the integral equation in the open set $S_n{}^+ \subseteq C^1([0, 1])$ consisting of functions positive near zero and having exactly n simple zeros in (0, 1). More recently, R. E. L. Turner [20], [21] has employed similar methods to extend the work of Rabinowitz. In the Rabinowitz and Turner papers, nonlinear Sturm-Liouville problems are studied and continua of solutions to (1.1), (1.2) are obtained. However, the growth conditions imposed on the nonlinear terms are somewhat special. Earlier papers by G. Pimbley [13], [14] investigated (1.1), (1.2) from the bifurcation standpoint; many of Pimbley's results are extended in Turner [21].

The formulation of the existence problem used in this work is due to G. B. Gustafson [2].

2. **Preliminaries.** This section gives the basic notation and definitions used throughout.

The Banach spaces to be used are as follows:

$C([a, b])$, the space of continuous real-valued functions x on $[a, b]$ with norm $\|x\|_{a,b} = \max\{|x(t)| : a \leqq t \leqq b\}$.

$C^1([a, b]) = \{x \in C([a, b]) : x' \in C([a, b])\}$ with norm $|||x|||_{a,b} = \|x\|_{a,b} + \|x'\|_{a,b}$,

$L_p([a, b])$, $1 \leqq p < \infty$, the space of measurable functions given by $L_p([a, b]) = \{x : \|x\|_p \equiv (\int_a^b |x(t)|^p\, dt)^{1/p} < \infty\}$,

$L_\infty([a, b]) = \{x : \|x\|_\infty \equiv \text{essential } \sup\{|x(t)| : a \leqq t \leqq b\} < \infty\}$, and $L_p{}^1([a, b]) = \{x \in AC([a, b]) : x' \in L_p([a, b])\}$ with norm $\|x\| = \|x\|_{a,b} + \|x'\|_p$; $AC([a, b])$ is the linear space of absolutely continuous functions on $[a, b]$.

When the interval [a, b] = [0, 1] the notation $\|x\|_c \equiv \|x\|_{a,b}$ will be used, and E will denote either of the spaces $C([0, 1])$ and $L_p{}^1([0, 1])$ when no confusion is possible.

The characteristic function χ_E of the set E will be used in various formulas, it is defined by $\chi_E(t) = 1$ for $t \in E$, $= 0$ for $t \notin E$.

We proceed to record some definitions. The following parallels M. A. Krasnosel'skii [8].

DEFINITION 2.2. A subset K of a Banach space E is called a *cone* if
(i) K is closed in E,
(ii) $x, y \in K, \alpha, \beta \geqq 0$ implies $\alpha x + \beta y \in K$
(iii) $x \in K, -x \in K$ implies $x = 0$.

DEFINITION 2.3. An operator $B : X \to Y$, X, Y Banach spaces, is called *completely continuous* if it is both continuous and compact (i.e., maps bounded sets into precompact sets).

It is worth noting that (1.1) is no less general than

$$(\sigma(t)y'(t))' + f(t, y(t), y'(t)) = 0, a < t < b, \tag{2.1}$$

where $-\infty < a < b < \infty$ and σ is a continuous positive function. To convert (2.1) to (1.1) see Hartman [4].

$$\begin{cases} y''(t) = h(t), \\ y(a) = y(b) = 0. \end{cases}$$

Explicitly,

$$G(t, s; a, b) = \begin{cases} (t - a)(b - s)(b - a)^{-1}, t \leqq s \\ (s - a)(b - t)(b - a)^{-1}, s \leqq t. \end{cases} \tag{2.2}$$

One observes that $0 \leqq G(t, s; a, b) \leqq (b - a)/4$, $|G_t(t, s; a, b)| \leqq 1$ and $\int_a^b G(t, s; a, b)\, ds \leqq (b - a)^2/8$. These estimates will be used in subsequent sections.

One observes that $0 \leqq G(t, s; a, b) \leqq (b - a)/4$, $|G_t(t, s; a, b)| \leqq 1$ and $\int_a^b G(t, s; a, b)\, ds \leqq (b - a)^2/8$. These estimates will be used in subsequent sections.

3. **Equivalent Operator Equations.** The nonlinear nodal problem will be converted to a pair of nonlinear operator equations acting in a suitable space. The operators will be of the form $B: \bar{T} \times K \to K$, $F : \bar{T} \times K \to R^n$, where K is the cone of nonnegative functions in a certain Banach space of functions, E, and T is the open "tetrahedron" in R^n given by $T = \{a \in R^n, a = (a_1, a_2, \cdots, a_n) : 0 < a_1 < a_2 < \cdots < a_n < 1\}$.

To convert (1.1)–(1.2) to a cone-valued fixed-point problem in a Banach space, the solution y is replaced by $x = |y|$, and on each interval $[a_i, a_{i+1}]$ between the zeros $\{a_i\}$ of y, the problem (1.1)–(1.2) is inverted via Green's function kernels. The result is an operator equation $x = B(a, x)$. The transition from x back to y requires the identity $x'(a_i +) + x'(a_i -) = 0$, which generates a finite-dimensional operator equation $F(a, x) = 0$. The explicit assumptions, and formulas for B and F, shall be given below.

Consider the differential equation (1.1) with nodal conditions (1.2). It shall be assumed that $f\colon [0, 1] \times R^2 \to R$ is continuous, that $yf(t, y, u) \geqq 0$ for all $t \in [0, 1]$ and $y, u \in R$, and that $f(\,\cdot, y(\,\cdot\,), y'(\,\cdot\,)) \in I_q([0, 1])$ for some $q > 1$ whenever $y \in E$. Finally, $f(t, y, u) = 0$ if and only if $y = 0$, $t -$ a.e. on $0 \leqq t \leqq 1$.

The *nodal operator* B will be defined on $0 \leqq t \leqq 1$ by

$$B(a, x)(t) = \sum_{i=0}^{n} (-1)^i \int_{a_i}^{a_{i+1}} G(t, s; a_i, a_{i+1}) \cdot f(s, (-1)^i x(s), (-1)^i x'(s))\, ds \chi_{[a_i, a_{i+1}]}(t), \tag{3.1}$$

$a_0 \equiv 0$ and $a_{n+1} \equiv 1$, for $a = (a_1, \cdots, a_n) \in \bar{T}$, $x \in K$. The formula makes sense whenever $x \in K$ and $x'(t)$ exists $t -$ *a.e.* on $[0, 1]$.

The *matching operator* F will be defined by $F = (F_1, \cdots, F_n)$ where

$$\begin{aligned}(-1)^{i-1} F_i(a, x) = &\int_{a_{i-1}}^{a_i} G_t(a_i, s; a_{i-1}, a_i) \\ &\cdot f(s, (-1)^{i-1} x(s), (-1)^{i-1} x'(s))\, ds \\ &- \int_{a_i}^{a_{i+1}} G_t(a_i, s; a_i, a_{i+1}) \\ &\cdot f(s, (-1)^i x(s), (-1)^i x'(s))\, ds, \\ &\qquad i = 1, \cdots, n.\end{aligned} \tag{3.2}$$

Since all integrands are in $L_1([0, 1])$ we may take any of the integrals above to be zero whenever the interval becomes degenerate.

LEMMA 3.1. *The coupled pair of operator equations*

$$B(a, x) = x, \quad F(a, x) = 0, \tag{3.3}$$

for $(a, x) \in T \times K$ *is equivalent to the nodal problem* (1.1), (1.2) *in the following sense:*

(i) *If* $(a, x) \in T \times K$ *is a solution to* (3.3), *then*

$$y(t) \equiv \sum_{i=0}^{n} (-1)^i x(t) \chi_{[a_i, a_{i+1}]}(t)$$

is a solution to (1.1), (1.2).

(ii) *If* y *is a solution to* (1.1), (1.2) *with exactly* $n + 2$ *zeros at* $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$, *then*

$$(a, x) \in T \times K$$

is a solution to (3.3), *where* $a = (a_1, \cdots, a_n)$ *and*

PROOF. It suffices to prove that

$$\bar{F}_i(a, x) \equiv \int_{a_i}^{a_{i-1}} G_t(a_i, s; a_i, a_{i+1})(-1)^i f(s, (-1)^i x(s))\, ds$$

is continuous. The proof for $F_i - \bar{F}_i$ is similar.

Consider $(a, x), (b, y) \in \bar{T} \times K$. Then

$$\begin{aligned} |\bar{F}_i(a, x) - \bar{F}_i(b, y)| &\leqq |\bar{F}_i(a, x) - \bar{F}_i(b, x)| \\ &\quad + |\bar{F}_i(b, x) - \bar{F}_i(b, y)| \\ &\leqq \|H_i(\,\cdot, a, b)\|_1 \|f(\,\cdot, (-1)^i x(\,\cdot\,))\|_\infty \\ &\quad + \|\vartheta(b, x) - \vartheta(b, y)\|_\infty, \end{aligned}$$

where

$$\begin{aligned} H_i(s, a, b) = {} & G_t(a_i, s; a_i, a_{i+1})\chi_{(a_i, a_{i+1})}(s) \\ & - G_t(b_i, s; b_i, b_{i+1})\chi_{(b_i, b_{i+1})}(s). \end{aligned}$$

It is easily shown that $H_i(s, a, b) \to 0$ $s - a.e.$ on $[0, 1]$ as $b \to a$. Further, $|H_i(s, a, b)| \leqq 1$. By Lemma 6.2 and Lebesgue's bounded convergence Theorem, the right side of the preceding inequality tends to zero as $(b, y) \to (a, x)$. The boundedness follows from Lemma 6.1.

7. **Complete Continuity of B and F : (t, x, x') Dependence of f.** The introduction of x' dependence produces additional complications. The first problem is to find a function space E in which derivatives may be taken in some sense, the obvious choices being $C^1([0, 1])$ and $\mathrm{Lip}_1([0, 1])$. Unfortunately, the operators fail to be compact when defined on these spaces. To overcome the difficulty we proceed as follows.

Let $E = \{x \in AC([0, 1]) : x' \in L_p([0, 1])\}$ where $p > 1$ and define $\|x\| = \|x\|_C + \|x'\|_p$. The space E is a Banach space. Let $K = \{x \in E : x(t) \geqq 0,\ 0 \leqq t \leqq 1\}$. Throughout this section the topology on K will be that induced by E and $\bar{T} \times K$ will have the product topology.

Concerning assumptions of f, it will be assumed that the function $(y, z) \to f(\,\cdot, y(\,\cdot\,), z(\,\cdot\,))$ is continuous from $E \times L_p([0, 1])$ into $L_q([0, 1])$, $1 < q \leqq p$. Further, this mapping shall take bounded sets to bounded sets. Finally, $yf(s, y, z) \geqq 0$ for all $(s, y, z) \in [0, 1] \times R^2$.

Define $\vartheta : T \times K \to L_q$ by

$$\vartheta(a, x) = \sum_{i=0}^{n} (-1)^i f(\,\cdot, (-1)^i x(\,\cdot\,), (-1)^i x'(\,\cdot\,))\chi_{[a_i, a_{i+1})}\,.$$

In the following lemmas $\|\cdot\|_C$ will denote the norm in E and $\|\cdot\|_p$ the norm in L_p, $1 \leqq p \leqq \infty$.

LEMMA 6.5. *$B : \bar{T} \times K \to K$ is continuous.*

PROOF. The definition of B and the sign hypothesis on f make it clear that the range of B lies in K. Now suppose that $|a^m - a^0| + \|x_m - x_0\|_C \to 0$ as $m \to \infty$ where $(a^m, x_m) \in \bar{T} \times K$ for $m = 0, 1, 2, \cdots$. Then

$$\|B(a^m, x_m) - G(a^0, x_0)\|_C$$

$$\leqq \left\| \int_0^1 [G(\cdot, s; a^m) - \mathcal{G}(\cdot, s; a^0)]\, \mathcal{F}(a^0, x_0)(s)\, ds \right\|_C$$

$$+ \left\| \int_0^1 \mathcal{G}(\cdot, s; a^m)[\mathcal{F}(a^m, x_m) - \mathcal{F}(a^0, x_0)](s)\, ds \right\|_C$$

$$\leqq \|\mathcal{F}(a^0, x_0)\|_\infty \left\| \int_0^1 [\mathcal{G}(\cdot, s; a^m) - \mathcal{G}(\cdot, s; a^0)]\, ds \right\|_C$$

$$+ \frac{1}{4}[\|\mathcal{F}(a^m, x_m) - \mathcal{F}(a^m, x_0)\|_\infty + \|\mathcal{F}(a^m, x_0) - \mathcal{F}(a^0, x_0)\|_1],$$

because $|\mathcal{G}(t, s; a^m)| \leqq 1/4$. By Lemmas 6.2, 6.3 and 6.4 this term tends to zero as $m \to \infty$.

LEMMA 6.6. *$B : \bar{T} \times K \to K$ is completely continuous.*

PROOF. It remains to show B is compact. To this end, let $A \times H \subseteq \bar{T} \times K$ be bounded and let $(a, x) \in A \times H$ where $a = (a_1, \cdots, a_n)$.

$$\|B(a, x)\|_C = \left\| \int_0^1 \mathcal{G}(\cdot, s; a)\, \mathcal{F}(a, x)(s)\, ds \right\|_C \leqq \|\mathcal{F}(a, x)\|_\infty/4,$$

and Lemma 6.1 shows that $B(A, H)$ is a uniformly bounded family in K.

Further, $B(a, x)'(t)$ exists for each $t \notin \{a_i\}_{i=1}^n$. In fact, $B(a, x)'(t) = (-1)^i \int_{a_i}^{a_{i+1}} G_t(t, s; a_i, a_{i+1}) f(s, (-1)^i x(s))\, ds$ for all $t \in (a_i, a_{i+1})$, $0 \leqq i \leqq n$. Hence, $D^\pm B(a, x)$ exists on $(0, 1)$ and $|D^\pm B(a, x)(t)| \leqq \sup\{|G_t(t, s; a_i, a_{i+1})| : 0 \leqq i \leqq n \text{ and } t \notin \{a_i\}_{i+1}^n\}$. $\|\mathcal{F}(a, x)\|_\infty \leqq \|\mathcal{F}(a, x)\|_\infty$. By Lemma 6.1, the family of functions $B(A, H)$ is a uniformly Lipschitz, hence equicontinuous, family in K. The Arzela-Ascoli Theorem (see Royden [17]) applies to complete the proof.

LEMMA 6.7. *$F : \bar{T} \times K \to R^n$ is continuous and maps bounded sets to bounded sets.*

6. Complete Continuity of B and F: (t, x) – Dependence of f. In order to apply the degree computations of preceding sections it is necessary to verify that the *nodal operator* B and the *matching operator* F are completely continuous.

Considered in this section is the special case where f does not depend on x'.

Throughout, $F = C([0, 1])$, equipped with the topology of uniform convergence, $K = \{x \in E : x(t) \geqq 0\}$, the cone of nonnegative functions in E with the relative topology. The sets R^n and T will have the topology induced by the norm $|a| = \max\{a_i : 1 \leqq i \leqq n\}$, where $a = (a_1, \cdots, a_n) \in R^n$. The product space $\bar{T} \times K$ will have the product topology. If $a \in \bar{T}$, we will number its components a_i, $1 \leqq i \leqq n$, $a_0 \equiv 0$ and $a_{n+1} \equiv 1$.

It is assumed that the mapping $x \to f(\cdot, x(\cdot))$ is continuous from E into E, and further, this mapping takes bounded sets to bounded sets. Finally, $zf(t, z) \geqq 0$ for almost all $t \in [0, 1]$, $z \in R$.

The *nodal operator* B will be written in the form

$$B(a, x)(t) = \int_0^1 G(t, s; a)\,\vartheta(a, x)(s)\,ds$$

for all $(z, x) \in \bar{T} \times K$, where

$$G(t, s; a) = \begin{cases} \displaystyle\sum_{i=0}^{n} G(t, s; a_i, a_{i+1})\chi_{(a_i,a_{i+1})\times(a_i,a_{i+1})}(t, s), \\ \qquad\qquad \text{for } s, t \notin \{a_i\}_{i=0}^{n+1} \\ 0, \text{ for } s \text{ or } t \in \{a_i\}_{i=0}^{n+1} \end{cases}$$

and

$$\vartheta(a, x)(s) = \sum_{i=0}^{n} (-1)^i f(s, (-1)^i x(s))\chi_{[a_i,a_{i+1})}(s) \text{ for } s \in [0, 1).$$

LEMMA 6.1. *$\vartheta : \bar{T} \times K \to L_\infty([0, 1])$ maps bounded sets to bounded sets.*

LEMMA 6.2. *$\vartheta(a, \cdot) : K \to L_\infty([0, 1])$ is continuous, uniformly in $a \in \bar{T}$.*

LEMMA 6.3. *$\vartheta(\cdot, x) : \bar{T} \to L_1([0, 1])$ is continuous, uniformly for x in bounded subsets of K.*

LEMMA 6.4. *If $|a^m - a^0| \to 0$ as $m \to \infty$, then $\|G(t, \cdot; a^m) - G(t, \cdot; a^0)\|_C \to 0$ uniformly in $t \in [0, 1]$.*

degree $d(F(\cdot, x(\cdot)), T, 0)$ is defined for any continuous mapping $x : \bar{T} \to K$ for which $F(a, x(a)) \neq 0$, $a \in \partial T$.

The purpose of this section is to give some geometric results which insure that $d(F(\cdot, x(\cdot)), T, 0) = 1$ for a certain class of functions x.

DEFINITION 5.1. A function $N : \partial S \to R^n$ is called an *outer normal* to the bounded open set $S \subseteq R^n$ provided for some $a^* \in S$

$$N(a) \cdot (a - a^*) > 0 \text{ for } a \in \partial S \tag{5.1}$$

$$\bar{S} \subseteq \{b \in R^n : N(a) \cdot (b - a) \leqq 0\} \text{ for each } a \in \partial S. \tag{5.2}$$

Define an outer normal to T in the following way. Let $\{e_i\}_{i=1}^n$ denote the standard unit vectors in R^n and let $e_{n+1} = 0$. Put

$$S_0 = \{a \in \partial T : 0 = a_1 = \cdots = a_j < a_{j+1} \text{ for some } j, 1 \leqq j \leqq n\},$$

and define faces $S_1, \cdots, S_n$ of ∂T inductively by

$$S_i = \{a \in \partial T : a_{i-1} < a_i = a_{i+1}\} \setminus \bigcup_{k=0}^{i-1} S_k, \ 1 \leqq i \leqq n.$$

Then ∂T is the disjoint union of the faces $S_0, \cdots, S_n$. Define

$$N(a) = e_k - e_{k+1} \text{ for } a \in S_k, 1 \leqq k \leqq n, \tag{5.3}$$

$$N(a) = -e_j \text{ for } a \in S_0 \text{ satisfying } 0 = a_1 = \cdots = a_j < a_{j+1}. \tag{5.4}$$

LEMMA 5.2. *The function $N : \partial T \to R^n$ given by* (5.3)–(5.4) *is an outer normal to the convex open set T.*

LEMMA 5.3. *Suppose $x : \bar{T} \to K$ is continuous and that $x(a)(t) = 0$ if and only if $t \in \{a_i : 0 \leqq i \leqq n + 1\}$, then the outer normal N given by* (5.3)–(5.4) *and the operator F of section* 3 *satisfy*

$$N(a) \cdot F(a, x(a)) < 0. \tag{5.5}$$

LEMMA 5.4. *Suppose $x : \bar{T} \to K$ is continuous and that $x(a)(t) = 0$ if and only if $t \in \{a_i : 0 \leqq i \leqq n + 1\}$, then the Brouwer degree*

$$d(F(\cdot, x(\cdot)), T, 0) = 1.$$

PROOF. For $a^* \in T$ the homotopy given by

$$H(a, \lambda) = (1 - \lambda)(a - a^*) - \lambda F(a, x(a))$$

does not vanish for $a \in \partial T$, $0 \leqq \lambda \leqq 1$. To see this consider $N(a) \cdot H(a, \lambda)$ and apply Lemmas 5.2 and 5.3.

PROOF OF THEOREM 4.1. In views of Lemmas 4.7 and 4.8 we may again use the homotopy invariance property of Leray-Schauder degree, this time using ϵ as the homotopy parameter, $0 \leqq \epsilon \leqq \epsilon_1$. This gives

$$\begin{aligned} d(I - A, T \times (B_R \backslash \bar{B}_r), 0) &= d(I - H_0(\cdot, \cdot, 0), T \times (B_R \backslash \bar{B}_r), 0) \\ &= d(I - H_{\epsilon_1}(\cdot, \cdot, 0), T \times (B_R \backslash \bar{B}_r), 0) \\ &= - d(-F(\cdot, k(\cdot)), T, 0). \end{aligned}$$

The proof of Theorem 4.2 follows in a similar manner from the following lemmas, which are stated without proof.

LEMMA 4.9. *Suppose that the nonsingularity condition and the compression conditions* (b) *or* (d) *holds.*

If (4.4) *holds, then*

$$d(I - H_\epsilon(\cdot, \cdot, 0), T \times B_r, 0) = 0.$$

LEMMA 4.10. *Suppose that the nonsingularity condition and* (4.3) *hold, then for each $\epsilon > 0$ sufficiently small $L_\epsilon(\cdot, \cdot, \mathcal{T})$ is fixed-point free on $\partial(T \times B_R)$, $0 \leqq \mathcal{T} \leqq 1$. Therefore,*

$$\begin{aligned} d(I - L_\epsilon(\cdot, \cdot, 0), T \times B_R, 0) &= d(I - L_\epsilon(\cdot, \cdot, 1), T \times B_R, 0) \\ &= d(I - H_\epsilon(\cdot, \cdot, 0), T \times B_r, 0), \end{aligned}$$

for each $\epsilon > 0$ sufficiently small.

LEMMA 4.11. *Let the nonsingularity condition, the smallness condition on k and the compression conditions* (b) *or* (d) *be in force.*

If (4.3) *and* (4.4) *are satisfied, then for some $\epsilon_2 > 0$ and $0 < \epsilon \leqq \epsilon_2$,*

$$d(I - H_\epsilon(\cdot, \cdot, 0), T \times (B_R \backslash B_r), 0) = d(-F(\cdot, k(\cdot)), T, 0).$$

LEMMA 4.12. *Suppose that the nonsingularity condition holds, together with* (4.3) *and* (4.4), *then A is fixed-point free on $\partial(T \times (B_R \backslash \bar{B}_r))$.*

PROOF OF THEOREM 4.2. In view of lemmas 4.11 and 4.12 we may use the homotopy invariance of degree with parameter $\epsilon \in [0, \epsilon_2]$. This gives

$$\begin{aligned} d(I - A, T \times (B_R \backslash \bar{B}_r), 0) &= d(I - H_0(\cdot, \cdot, 0), T \times (B_R \backslash \bar{B}_r), 0) \\ &= d(I - H_{\epsilon_2}(\cdot, \cdot, 0), T \times (B_R \backslash \bar{B}_r), 0) \\ &= d(-F(\cdot\, k(\cdot)), T, 0). \end{aligned}$$

5. **Geometric Properties of F.** It will be shown in sections 6 and 7 *infra*, that $F : \bar{T} \times K \to R^n$ is continuous. It follows that the Brouwer

$$V(a, x, t) = (a + F^*(a, tk(a) + (1 - t)x), k(a)),$$
$$W(a, x, s) = (a + F(a, k(a)), sk(a)).$$

If $V(\cdot, \cdot, t)$ has a fixed point $(a, x) \in \partial(T \times B_r)$, then $x = k(a) \in K$ and $a \in \partial T$ by (iii). Further $x = k(a)$ gives $0 = F(a, tk(a) + (1 - t)x) = F(a, k(a))$, which contradicts *the nonsingularity condition.*

If $W(\cdot, \cdot, s)$ has a fixed point $(a, x) \in \partial(T \times B_r)$, then $\|x\| = s\|k(a)\| < r$, by (iii), so $a \in \partial T$. Now $F(a, k(a)) = 0$ again contradicts the nonsingularity condition.

By homotopy invariance of degree, we obtain the following:

$$\begin{aligned} d(I - L_\epsilon(\cdot, \cdot, 0), T \times B_r, 0) &= d(I - V(\cdot, \cdot, 0), T \times B_r, 0) \\ &= d(I - V(\cdot, \cdot, 1), T \times B_r, 0) \\ &= d(I - W(\cdot, \cdot, 1), T \times B_r, 0) \\ &= d(I - W(\cdot, \cdot, 0), T \times B_r, 0) \\ &= d((-F(\cdot, k(\cdot)), I), T \times B_r, 0) \\ &= d(-F(\cdot, k(\cdot)), T, 0) \cdot d(I, B_r, 0) \\ &= d(-F(\cdot, k(\cdot)), T, 0). \end{aligned}$$

The preceding also establishes the following result:

LEMMA 4.6. *B_r may be replaced by B_R in the preceding lemma.*

LEMMA 4.7. *Assume the nonsingularity condition, the smallness condition on k and the compression conditions* (a) *or* (c). *If* (4.1) *and* (4.2) *hold, then for some $\epsilon_1 > 0$ and $0 < \epsilon \leqq \epsilon_1$,*

$$d(I - H_\epsilon(\cdot, \cdot, 0), T \times (B_R \backslash \bar{B}_r), 0) = -d(-F(\cdot, k(\cdot)), T, 0).$$

PROOF. The result follows immediately from Lemmas 4.3, 4.4 and 4.5 once one has observed that

$$\begin{aligned} d(I - H_\epsilon(\cdot, \cdot, 0), T \times (B_R \backslash \bar{B}_r), 0) = {} & d(I - H_\epsilon(\cdot, \cdot, 0), T \times B_R, 0) \\ & - d(I - H_\epsilon(\cdot, \cdot, 0), T \times B_r, 0), \end{aligned}$$

by virtue of the excision property.

LEMMA 4.8. *Suppose that the nonsingularity condition holds, together with* (4.1) *and* (4.2), *then A is fixed-point free on $\partial(T \times (B_R \backslash \bar{B}_r))$.*

PROOF. Suppose $A(a, x) = (a, x)$, then $x = B^*(a, x) \in K$ and we have $\|x\| \geqq r > 0$, $x = B(a, x)$, $F(a, x) = 0$, and so $a \in T$ by (ii). But $a \in T$ implies $\|x\| \neq r$ by (4.2) with $\lambda = 1$ and $\|x\| \neq R$ by (4.1) with $\lambda = 0$.

which contradicts *the compression condition* (a). If the norm is not monotone on K, then $\|x - B(a, x)\| = (1 + \epsilon)\|h(a)\| > \|h(a)\|$, contradicting *the compression condition* (c).

For each $\epsilon > 0$, define the completely continuous map $L_\epsilon : \bar{T} \times E \times [0, 1] \to R^n \times K$ by

$$L_\epsilon(a, x, \tau) = (a + F^*(a, x), \tau B^*(a, x) + (1 - \tau)k(a) + \tau\epsilon h(a)).$$

Lemma 4.4. *Suppose that the nonsingularity condition holds together with* (4.2), *then for each* $\epsilon > 0$ *sufficiently small the operator* $L_\epsilon(\,\cdot\,, \cdot\,, \tau)$ *is fixed-point free on* $\partial(T \times B_r)$, *for* $0 \leqq \tau \leqq 1$. *Hence,*

$$\begin{aligned} d(I - L_\epsilon(\,\cdot\,, \cdot\,, 0), T \times B_r, 0) &= d(I - L_\epsilon(\,\cdot\,, \cdot\,, 1), T \times B_r, 0) \\ &= d(I - H_\epsilon(\,\cdot\,, \cdot\,, 0), T \times B_r, 0). \end{aligned}$$

Proof. If $L_\epsilon(a, x, \mathcal{T}) = (a, x)$ for some $a \in \partial T$, then since $B^*(a, x) \in K$ we have $x \geqq (1 - \mathcal{T})k(a) + \mathcal{T}\epsilon h(a)$ and $F(a, x) = 0$. This contradicts (ii).

Now suppose there exist sequences $\{\mathcal{T}_n\} \subseteq [0, 1]$, $\{a_n\} \subseteq T$, $\{x_n\} \subseteq \partial B_r$ and $\{\epsilon_n\} \subset R^+$, $\epsilon_n \to 0$ as $n \to \infty$, satisfying $L_{\epsilon n}(a_n, x_n, \mathcal{T}_n) = (a_n, x_n)$, $n = 1, 2, \cdots$. Again, $x_n \in K$, $x_n = \mathcal{T}_n B(a_n, x_n) + (1 - \mathcal{T}_n)k(a_n) + \mathcal{T}_n\epsilon_n h(a_n)$ and $F(a_n, x_n) = 0$, $n = 1, 2, \cdots$. By compactness of $\bar{T}$, $[0, 1]$, and the operator B, subsequences can be selected (let us use the same numbering), $\{x_n\}$, $\{a_n\}$ and $\{\mathcal{T}_n\}$, so that $\mathcal{T}_n \to \mathcal{T} \in [0, 1]$, $a_n \to a \in \bar{T}$ and $B(a_n, x_n) \to y \in K$ as $n \to \infty$. Hence, $x_n \to \mathcal{T}_y + (1 - \mathcal{T})k(a) \equiv x \in K$ as $n \to \infty$, by continuity of k and h. Furthermore, by continuity of B and F,

$$x = \mathcal{T} B(a, x) + (1 - \mathcal{T})k(a), \ F(a, x) = 0.$$

Since $\{x_n\} \in \partial B_r$, $\|x\| = r$. Now if $a \in \partial T$ both $\mathcal{T} < 1$ and $\mathcal{T} = 1$ lead to contradictions of *the nonsingularity condition*, whereas if $a \in T$, (4.2) is violated. The invariance of degree under homotopy and the fact that $L_\epsilon(\,\cdot\,, \cdot\,, 1) \equiv H_\epsilon(\,\cdot\,, \cdot\,, 0)$ now gives, for each $\epsilon > 0$ sufficiently small,

$$\begin{aligned} d(I - L_\epsilon(\,\cdot\,, \cdot\,, 0), T \times B_r, 0) &= d(I - L_\epsilon(\,\cdot\,, \cdot\,, 1), T \times B_r, 0) \\ &= d(I - H_\epsilon(\,\cdot\,, \cdot\,, 0), T \times B_r, 0). \end{aligned}$$

Lemma 4.5. *Suppose that the nonsingularity condition and the smallness condition on k holds, then*

$$d(I - L_\epsilon(\,\cdot\,, \cdot\,, 0)\ T \times B_r, 0) = d(-F(\,\cdot\,, k(\,\cdot\,)), T, 0).$$

Proof. Define the completely continuous maps V, $W : \bar{T} \times B_r \times [0, 1] \to R^n \times K$ by

Then the following Leray-Schrauder degree identity holds:

$$d(I - A, T \times (B_R \backslash B_r), 0) = -d(-F(\cdot, k(\cdot)), T, 0).$$

THEOREM 4.2. *Suppose that the complete continuity condition, the nonsingularity condition and the smallness condition on k are satisfied. Suppose that the compression conditions* (b) *or* (d) *are satisfied. Assume*:

(4.3) *If* $(a, y) \in T \times M$, $y = \lambda B(a, y) + (1 - \lambda)k(a), F(a, y) = 0$ *for some* $\lambda \in [0, 1]$, *then* $\|y\| \neq R$.

(4.4) *If* $(a, y) \in T \times M$, $y = B(a, y) + \lambda h(a)$, $F(a, y) = 0$ *for some* $\lambda \geqq 0$ *then* $\|y\| \neq r$.

Then the following Leray-Schauder degree identity holds:

$$d(I - A, T \times (B_R \backslash \bar{B}_r), 0) = d(-F(\cdot, k(\cdot)), T, 0).$$

The proofs of these theorems will follow from a sequence of lemmas. For each $\epsilon > 0$ define $H_\epsilon : \bar{T} \times E \times [0, 1] \to R^n \times K$ by

$$H_\epsilon(a, x, \lambda) = (a + F^*(a, x), B^*(a, x) + (\lambda + \epsilon)h(a)).$$

Under the *complete continuity condition*, H_ϵ is completely continuous. We shall assume that this condition holds throughout the remainder of this section.

LEMMA 4.3. *Suppose that the nonsingularity condition and the compression conditions* (a) *or* (c) *are satisfied. If* (4.1) *holds, then the operator* $H_\epsilon(\cdot, \cdot, \lambda)$ *is fixed point free on* $\partial(T \times B_R)$ *for each* $\lambda \in [0, 1]$. *Therefore*, $d(I - H_\epsilon(\cdot, \cdot, 0), T \times B_R, 0) = d(I - H_\epsilon(\cdot, \cdot, 1), T \times B_R, 0) = 0$.

PROOF. Suppose $H_\epsilon(a, x, \lambda) = (a, x)$, then $x = B^*(a, x) + (\lambda + \epsilon)h(a) \in K$. If $a \in \partial T$, then $x \geqq (\lambda + \epsilon)h(a)$, and $F^*(a, x) = F(a, x) = 0$ contradicts (ii). If $a \in T$ and $\|x\| = R$, then $x = B(a, x) + (\lambda + \epsilon)h(a)$ and $F(a, x) = 0$. This implies that $\lambda + \epsilon > 1$ by (4.1). Thus if the norm is monotone, then $\|x\| \geqq (\lambda + \epsilon)\|h(a)\| > \|h(a)\| \geqq R$, by *the compression condition* (a), a contradiction. If the norm is not monotone, then $\|x - B(a, x)\| = (\lambda + \epsilon)\|h(a)\| > \|h(a)\|$ contradicting *the compression condition* (c). This shows that $H_\epsilon(\cdot, \cdot, \lambda)$ is fixed-point free on $\partial(T \times B_R)$. Invariance of degree under homotopy gives $d(I - H_\epsilon(\cdot, \cdot, 0), T \times B_R, 0) = d(I - H_\epsilon(\cdot, \cdot, 1), T \times B_R, 0)$. Suppose, now that $d(I - H_\epsilon(\cdot, \cdot, 1), T \times B_R, 0) \neq 0$, then for some $(a, x) \in T \times B_R$, $x = B^*(a, x) + (1 + \epsilon)h(a) \in K$ and $F^*(a, x) = F(a, x) = 0$. If the norm is monotone on K then $R > \|x\| \geqq (1 + \epsilon)\|h(a)\| > \|h(a)\|$, Let E be a real Banach space, K a cone in E, $P : E \to K$ any continuous extension of the identity on K, which maps E onto K. Let $M = \{x \in K : r \leqq \|x\| \leqq R\}$ where $0 < r < R$, and let T be a bounded open set in R^n.

Suppose the following maps are given:

$$B : \bar{T} \times K \to K, F : \bar{T} \times K \to R^n, k : \bar{T} \to K, h : \bar{T} \to K.$$

Define:

$$\begin{array}{ll} B^* : \bar{T} \times E \to K & \text{by } B^*(a, x) = B(a, Px), \\ F^* : \bar{T} \times E \to R^n & \text{by } F^*(a, x) = F(a, Px), \\ A : \bar{T} \times E \to R^n \times K & \text{by } A(a, x) = (a + F^*(a, x), B^*(a, x)). \end{array}$$

B_δ will denote the open ball of radius δ in E about 0.

Further, assume that the following conditions are satisfied:

(i) The *complete continuity condition*: B and F are completely continuous, h and k are continuous.

(ii) The nonsingularity condition: If $x = B(a, x) \neq 0$, or $x \geqq k(a)$, or $x \geqq \tau h(a)$ for some $\tau > 0$, $(a, x) \in \partial T \times K$, then $F(z, x) \neq 0$.

(iii) *The smallness condition on k*: For $a \in T$, $0 < \|k(a)\| < r$.

(iv) *The compression conditions*: If $\|\cdot\|$ is monotone on K, then either

(a) $\|h(a)\| \geqq R$, $a \in T$, or

(b) $\|h(a)\| \geqq r$, $a \in T$.

If $\|\cdot\|$ is not monotone on K, then either

(c) $\sup\{\|x - B(a, x)\| : x \in K, \|x\| \leqq R, F(a, x) = 0\} < \|h(a)\|$ for $a \in T$

or

(d) $\sup\{\|x - B(a, x)\| : x \in K, \|x\| \leqq r, F(a, x) = 0\} < \|h(a)\|$ for $a \in T$.

The following theorems are true:

THEOREM 4.1. *Suppose that the complete continuity condition, the nonsingularity condition and the smallness condition on k are satisfied. Suppose that the compression conditions* (a) *or* (c) *are satisfied.*

Assume:

(4.1) *If $(a, y) \in T \times M$, $y = B(a, y) + \lambda h(a)$, $F(a, y) = 0$ for some $\lambda \in [0, 1]$, then $\|y\| \neq R$;*

(4.2) *If $(a, y) \in T \times M$, $y = \lambda B(a, y) + (1 - \lambda)k(a)$, $F(a, y) = 0$ for some $\lambda \in [0, 1]$, then $\|y\| \neq r$.*

$$x(t) = \operatorname{sign}(y'(0)) \sum_{i=0}^{n} (-1)^i y(t) \chi_{[a_i, a_{i+1}]}(t).$$

PROOF. Suppose $(a, x) \in T \times K$ is a solution to (3.3), then $B(a, x) = x$, and for $t \in [a_i, a_{i+1}]$,

$$x(t) = (-1)^i \int_{a_i}^{t} \frac{(a_{i+1} - t)(s - a_i)}{a_{i+1} - a_i} f(s, (-1)^i x(s), (-1)^i x'(s))\, ds$$

$$+ (-1)^i \int_{t}^{a_{i+1}} \frac{(a_{i+1} - s)(t - a_i)}{a_{i+1} - a_i} f(s, (-1)^i x(s), (-1)^i x'(s))\, ds,$$

$0 \leqq i \leqq n$. Since the integrand of each integral belongs to $L_q \subset L_1$, $x(t)$ is differentiable almost everywhere on $a_i < t < a_{i+1}$, and its derivative agrees almost everywhere with a continuous function $u(t)$. Replacing $x'(t)$ by $u(t)$ in the preceding equations shows x is twice continuously differentiable on $a_i < t < a_{i+1}$ and

$$x'(t) = (-1)^i f(t, (-1)^i x(t), (-1)^i x'(t)), a_i < t < a_{i+1}.$$

This shows that y satisfies (1.1) on each subinterval (a_i, a_{i+1}), $0 \leqq i \leqq n$. Since $x(a_i) = x(a_{i+1}) = 0$ and $x(t) > 0$ on (a_i, a_{i+1}), it follows that y satisfies the nodal conditions (1.2). It might first appear that we have demonstrated that y solves (1.1), (1.2). This is not the case, however. It is conceivable that the right and left-hand derivatives of y do not "match up" at the nodal points. The *matching operator* comes to the rescue at this stage. The following is true:

$$\lim_{h \to 0} - \left[\frac{y(a_i + h) - y(a_i)}{h} - \frac{y(a_i) - y(a_i - h)}{h} \right]$$

$$= (-1)^i \lim_{h \to 0} - \left[\frac{x(a_i + h) - x(a_i)}{h} + \frac{x(a_i) - x(a_i - h)}{h} \right]$$

$$= (-1)^{i-1} F_i(a, x) = 0, 1 \leqq i \leqq n.$$

This completes the proof of (i). The proof of (ii) is similar.

4. **Degree Computation.** This section deals with abstract operator equations in Banach spaces. The main results show that the Leray-Schauder degrees of certain mappings relative to zero and appropriate open sets are computable in terms of Brouwer degree. The computation enters as an essential step in the convergence proof for a projection method of Galerkin type; see section 9 *infra*.

The function $\vartheta(a, x)$ is defined $s-a.e.$ on $[0,1]$.

LEMMA 7.1. *The mapping ϑ is continuous and maps bounded sets to bounded sets.*

Define $B: \bar{T} \times K \to K$ by $B(a, x)(t) = \int_0^1 \mathcal{G}(t, s; a) \vartheta(a, x)(s)\, ds$ for each $a = (a_1, \cdots, a_n) \in \bar{T}$ and $x \in K$, where

$$\mathcal{G}(t, s; a) = \sum_{i=0}^{n} \mathcal{G}(t, s; a_i, a_{i+1}) \chi_{[a_i, a_{i+1})^2}(t, s).$$

Both $\mathcal{G}(\cdot, \cdot; a)$ and $\mathcal{G}_t(\cdot, \cdot; a)$ lie in $L_\infty([0,1]^2)$, their norms being 1/4 and 1, respectively. One can easily see that B maps $\bar{T} \times K$ into K.

LEMMA 7.2. *If $|a^m - a| + \|x_m - x\|_C + \|x_m' - x'\|_p \to 0$ as $m \to \infty$, then $\|B(a^m, x_m) - B(a, x)\|_C \to 0$ as $m \to \infty$, where $(a^m, x^m), (a, x) \in \bar{T} \times K$ for $m = 1, 2, \cdots$.*

LEMMA 7.3. *$B: \bar{T} \times K \to K$ is continuous.*

PROOF. In view of the previous lemma it suffices to show that $\|B(a^m, x_m)' - B(a^0, x)'\|_p \to 0$, given $|a^m - a^0| + \|x_m - x\|_c + \|x_m' - x'\|_p \to 0$ as $m \to \infty$. By Minkowski's inequality,

$$\begin{aligned}
&\|B(a^m, x_m)' - B(a^0, x)'\|_p \\
&\leqq \left\| \int_0^1 \mathcal{G}_t(\cdot, s; a^m) - \mathcal{G}_t(\cdot, s; a^0)] \vartheta(a^0, x)(s)\, \mathrm{ds} \right\|_p \\
&\quad + \left\| \int_0^1 \mathcal{G}_t(\cdot, s; a^m)[\vartheta(a^m, x_m) - \vartheta(a^0, x)]\, ds \right\|_p \\
&\leqq \|\vartheta(a^0, x)\|_q \left\| \left[\int_0^1 |\mathcal{G}_t(\cdot, s; a_m) \right.\right. \\
&\quad \left.\left. - \mathcal{G}_t(\cdot, s; a^0)|^{q'}\, ds \right]^{1/q'} \right\|_p \\
&\quad + \|\vartheta(a^m, x_m) - \vartheta(a^0, x)\|_q.
\end{aligned}$$

Again, the second term approaches zero as $m \to \infty$. Now if $t \notin \{a_i{}^m\}_{i=1, m=0}^{n, \infty}$, $J_m(t) = \int_0^1 |\mathcal{G}_t(t, s; a^m) - \mathcal{G}_t(t, s; a^0)|^{q'}\, ds$ exists, and furthermore, $J_m(t) \to 0$ as $m \to \infty$ by Lebesgue's dominated convergence theorem. Clearly, $J_m(t) \leqq 1$ *a.e.* and so again by Lebesgue's dominated convergence theorem $\|J_m\|_p \to 0$ as $m \to \infty$.

The lemmas which follow will establish the compactness of B.

LEMMA 7.4. *$B: \bar{T} \times K \to C([0,1])$ is completely continuous.*

PROOF. Continuity has been established, so it remains to prove compactness. Let $A \times H \subset \bar{T} \times K$ be bounded and let $(a, x) \in A \times H$, then

$$\|B(a, x)\|_C = \left\| \int_0^1 G(\cdot, s; a)\vartheta(a, x)(s)\, ds \right\|_C$$

$$\leqq \frac{1}{4}\|\vartheta(a, x)\|_q.$$

The right side of this inequality is uniformly bounded by some number $M > 0$, by virtue of Lemma 7.1.

If $(a, x) \in A \times H$, $a = (a_1, \cdots, a_n)$, then

$$B(a, x)'(t) = (-1)^i$$

$$\int_{a_i}^{a_{i+1}} G_t(t, s; a_i, a_{i+1}) f(s, (-1)^i x(s), (-1)^i x'(s))\, ds$$

for almost all $t \in (a_i, a_{i+1})$ for $0 \leqq i \leqq n$. The fact that the integrand is in $L_1([a_i, a_{i+1}])$ implies that $B(a, x) \in AC^2([a_i, a_{i+1}])$ and that $B(a, x)' \in C([a_i, a_{i+1}])$, $0 \leqq i \leqq n$. Furthermore, $D^{\pm}B(a, x)$ exists on $(0, 1)$ and $|D^{\pm}B(a, x)(t)| \leqq \|\vartheta(a, x)\|_1 \leqq \|\vartheta(a, x)\|_q \leqq 4M$ for all $(a, x) \in A \times H$. Hence, $B(a, x)$ is Lipschitz with Lipschitz constant $4M$ for all $(a, x) \in A \times H$. Since $B(A, H)$ is a uniformly bounded equicontinuous family in $C([0, 1])$, it is precompact by the Arzela-Ascoli Theorem.

LEMMA 7.5. *$B : \bar{T} \times K \to K$ is completely continuous.*

PROOF. Lemma 7.3 has established the continuity of B so it suffices to show compactness.

Suppose $\{(a^m, x_m)\}_{m=1}^{\infty}$ is a bounded sequence in $\bar{T} \times K$. By the previous lemma we may assume that a subsequence has been selected so that $\{B(a^m, x_m)\}_{m=1}^{\infty}$ converges to some $g \in C([0, 1])$. We may also assume that $\{a^m\}_{m=1}^{\infty}$ converges to some $a = (a_1, \cdots, a_n) \in \bar{T}$.

Let $I = \{i : 0 \leqq i \leqq n \text{ and } a_{i+1} - a_i > 0\}$, $3\delta = \min\{a_{i+1} - a_i : i \in I\}$ and $C_k = \bigcup_{i \in I} [a_i + \delta/k, a_{i+1} - \delta/k]$ for $k = 1, 2, \cdots$. Fix k and choose N_k such that $a_i^m \notin C_k$ for $0 \leqq i \leqq n + 1$ for all $m \geqq N_k$. It was shown in the proof of the previous lemma that $\{B(a^m, x^m)'\}_{m \geqq N_k}$ is a uniformly bounded sequence in $C(C_k)$. Also, for $u, v \in C_k$, if $|u - v| < 2\delta/k$, then $u, v \in (a_i^m, a_{i+1}^m)$ for some $i \in I$ and for all $m \geqq N_k$. Therefore,

$$\begin{aligned}|B(a^m, x_m)'(u) - B(a^m, x_m)'(v)| &= \left| \int_v^u B(a^m, x_m)''(s)\, ds \right| \\ &= \left| \int_v^u \mathcal{G}(a^m, x_m)(s)\, ds \right| \\ &\leqq |u - v|^{1/q'} \| \mathcal{G}(a^m, x_m) \|_q.\end{aligned}$$

Hence, $\{B(a^m, x_m)'\}_{m \geqq N_k}$ is a uniformly bounded equicontinuous family in $C(C_k)$ and has a uniformly convergent subsequence by the Arzela-Ascoli Theorem. We construct a subsequence as follows: let $S_k = \{B(a^m, x_m)'\}_{m \geqq N_k}$ for $k = 1, 2, \cdots$, and let T_1 be a subsequence of S_1 which converges uniformly on C_1. Proceeding inductively, suppose that the subsequences T_i have been chosen with T_i converging uniformly on C_i for $i = 1, \cdots, j$; choose $T_{j+1} \subseteq T_j \cap S_{j+1}$ to be a subsequence which converges uniformly on C_{j+1}.

Let us write $T_i = \{B(a^{m,i}, x_{m,i})'\}_{m=1}^{\infty}$ for each $i = 1, 2, \cdots$, and consider the diagonal subsequence $\{B(a^{m,m}, x_{m,m})'\}_{m=1}^{\infty}$. It is clear from the construction that this subsequence converges uniformly on compact subsets of $(0,1)\backslash\{a_i\}_{i=1}^{n}$ to a function $g_1 \in C((0,1)\backslash\{a_i\}_{i=1}^{n})$. A further consequence of this convergence is that $g_1(t) = g'(t)$ for $t \in (0,1)\backslash\{a_i\}_{i=1}^{n}$ and hence, $\{B(a^{m,m}, x_{m,m})\}_{m=1}^{\infty}$ converges to g in E.

We will now turn our attention to the matching operator F.

Define $F : \bar{T} \times K \to R^n$ by $F = (F_1, \cdots, F_n)$ where

$$\begin{aligned}F_i(a, x) = &\int_{a_{i-1}}^{a_i} G_t(a_i, s; a_{i-1}, a_i)\, \mathcal{G}(a, x)(s)\, ds \\ &+ \int_{a_i}^{a_{i+1}} G_t(a_i, s; a_i, a_{i+1})\, \mathcal{G}(a, x)(s)\, ds\end{aligned}$$

for $x \in K$, $a = (a_i, \cdots, a_n) \in \bar{T}$.

LEMMA 7.6. *F is continuous and maps bounded sets to bounded sets.*

PROOF. The proof is similar to that of Lemma 6.7 and will be omitted. The essential difference between the proofs is that $\mathcal{G}(a, x) \notin L_\infty$ in general, however, the fact that $\mathcal{G}(a, x) \in L_q$ allows Hölder's inequality to be used in conjunction with Lemma 7.1.

8. **The Auxiliary Functions h and k.** The theorems of section 4 made use of auxiliary functions h and k. Below, these functions are constructed and some of their properties are given. In particular, the construction makes it clear that h and k depend only upon the linear differential operator and the nodal conditions.

Let $a = (a_1, \cdots, a_n) \in \bar{T}$ and let c be a fixed positive parameter, to be determined later.

DEFINITION 8.1. For $0 \leqq t \leqq 1$, define

$$h(t; a, c) = 2c \sum_{i=0}^{n} m_1(c_i, d_i)(t)\chi_{[c_i,d_i]}(t)$$

where

$$m_1(a, b)(t) = \begin{cases} \min\{t - a, b - t\} /(b - a)^2, & \text{if } t \in (a, b) \\ 0, & \text{otherwise} \end{cases}$$

and $c_i = a_i + (a_{i+1} - a_i)/3$, $d_i = a_i + 2(a_{i+1} - a_i)/3$.

The notation $\Delta a_i \equiv a_{i+1} - a_i$ will be used. It is verified that $h(\,\cdot\,; a, c) \in C([0, 1])$ with norm $3c/m$, $m \equiv \min\{\Delta a_i : 0 \leqq i \leqq n, \Delta a_i > 0\}$. Further, $\int_0^1 h(t; a, c)\, dt = c(n + 1)$.

DEFINITION 8.2. For each $c > 0$, define the operator $h : T \rightarrow K$ by $h(a)(t) = \int_0^1 \mathcal{G}(t, s; a)h(s; a, c)\, ds$, $0 \leqq t \leqq 1$; where the kernel $\mathcal{G}$ is defined in section 6.

Explicitly,

$$(8.1) \quad h(a)(t) = \begin{cases} c(t - a_i)/4, \ a_i \leqq t \leqq c_i \\ c(t - a_i)/4 - 3c(t - c_i)/\Delta a_i^2, \ c_i \leqq t \leqq (a_i + a_{i+1})/2 \\ d(a_{i+1} - t)/4 - 3c(d_i - t)/\Delta a_i^2, \ (a_i + a_{i+1})/2 \leqq t \leqq d_i \\ c(a_{i+1} - t)/4, \ d_i \leqq t \leqq a_{i+1}, \end{cases}$$

for $a_i \leqq t \leqq a_{i+1}$, $0 \leqq i \leqq n$.

In view of (8.1) and the definition of h some of the following lemmas do not require proof.

LEMMA 8.3. *The function $h(a)$ is continuous on $[0, 1]$ for each $a \in \bar{T}$ and $h(a)(t) > 0$ for $t \in [0, 1] \setminus \{a_i\}_{i=0}^{n+1}$.*

LEMMA 8.4. *The function $h(a) \in C^2([a_i, a_{i+1}])$ whenever $a_i < a_{i+1}$.*

LEMMA 8.5. *For each $a \in \bar{T}$, $\|h(a)\|_C \leqq c/9$.*

PROOF. Definitions 8.1, 8.2 and the representation (8.1) show that $h(a)$ is concave on each of the intervals $[a_i, a_{i+1}]$, $0 \leqq i \leqq n$, and that the maximum occurs at $(a_i - a_{i+1})/2$ for some i, $0 \leqq i \leqq n$. Now (8.1) gives

$$h(a)((a_i - a_{i+1})/2) = c\Delta a_i/9 \leqq c/9.$$

LEMMA 8.6. *If* $a \in \bar{T}$ *and* $t \notin \{a_i\}_{i=1}^n$, $|h(a)'(t)| \leqq c/4$.

A consequence of Lemma 8.6 is that $h(\bar{T}) \subseteq K$ for each choice of the Banach space E in sections 6, 7. Further, $\|h(a)\| \leqq 13\, c/36$ and $\|h(a)'\|_p \leqq c/4$.

Let D^+ and D^- represent the right and left-hand Dini derivative operators, respectively.

LEMMA 8.7. *If* $a \in T$, *then* $D^+ h(a)(a_i) + D^- h(a)(a_i) = 0$, $1 \leqq i \leqq n$.

In order to show continuity of $h : T \to K$ a representation is given which is more readily manipulated. If $a = (a_1, \cdots, a_n) \in T$ and $a_i < t$, $s < a_{i+1}$, substitute $\mu = (t - a_i)/\Delta a_i$, $\lambda = (a - a_i)/\Delta a_i$ into definition 8.2. This gives

$$h(a)(t) = 6c\Delta a_i \int_{1/3}^{2/3} g(\mu, \lambda)\phi(\lambda)\, d\lambda, \; a_i \leqq t \leqq a_{i+1},$$

(8.2) where $g(\mu, \lambda) = \min\{\mu(1 - \lambda), \lambda(1 - \mu)\}$ and

$$\phi(\lambda) = \min\{3\lambda - 1, 2 - 3\lambda\}.$$

REMARK 8.8. For $\lambda \in [1/3, 2/3]$, $0 \leqq \phi(\lambda) \leqq 1/2$ and $0 \leqq g(\mu, \lambda) \leqq \psi(\mu) \equiv \min\{\mu, 1 - \mu\}$, $0 \leqq \mu \leqq 1$. It follows that (8.2) can be used even when $a_i = a_{i+1}$.

LEMMA 8.9. *The function g satisfies* $|g(\mu, \lambda) - g(\mu_1, \lambda)| \leqq 2|\mu - \mu_1|$ *whenever* $\mu, \mu_1, \lambda \in [0, 1]$.

LEMMA 8.10. *If* $\delta > 0$ *and* $a - \delta < a' < a + \delta \leqq t \leqq b - \delta < b' < b + \delta$, *then*

$$|(t - a)/(b - a) - (t - a')/(b' - a')| < 3\delta \min\{(b - a)^{-1}, (b' - a')^{-1}\}.$$

Furthermore, if $\delta < (b - a)/6$, *then*

$$2(b - a) < 3(b' - a') < 4(b - a).$$

LEMMA 8.11. *The function* $h : \bar{T} \to C([0, 1])$ *is continuous.*

PROOF. Using the notation developed above, suppose that $|a - b| < \delta$ and $6\delta < \min\{\Delta a_i : 0 \leqq i \leqq n$ and $\Delta a_i > 0\}$ where $b = (b_1, \cdots, b_n) \in \bar{T}$. Put $\Delta b_j = b_{j+1} - b_j$ and $\mu_1 = (t - b_j)/\Delta b_j$ whenever $b_j \leqq t \leqq b_{j+1}$. Let $t \in [0, 1]$ be fixed. Then $t \in [a_i, a_{i+1}] \cap [b_j, b_{j+1}]$ for some $0 \leqq i, j \leqq n$ and by (8.2)

$$|h(a)(t) - h(b)(t)| \leqq 6c[|\Delta a_i - \Delta b_j| \int_{1/3}^{2/3} g(\mu, \lambda)\phi(\lambda)\, d\lambda$$

$$+ \Delta b_j \int_{1/3}^{2/3} |g(\mu, \lambda) - g(\mu_1, \lambda)|\phi(\lambda)\, d\lambda]$$

$$\leqq c|\Delta a_i - \Delta b_j|\mu + c\Delta b_j \min\{\psi(\mu) + \psi(\mu_1), 2|\mu - \mu_1|\},$$

by Remark 8.8 and Lemma 8.9. Consider the following possibilities:

Case 1. *If* $t \in [a_i, a_i + \delta)$, then $\psi(\mu) \leqq \mu \leqq \delta/\Delta a_i$ and $\psi(\mu_1) \leqq 2\delta/\Delta b_j$. Hence

$$|h(a)(t) - h(b)(t)| \leqq c[\delta(1 - \Delta b_j/\Delta a_i) + \delta(\Delta b_j/\Delta a_i + 2)] \leqq 6c\delta.$$

If $t \in (a_{i+1} - \delta, a_{i+1}]$, then the same estimate holds.

Case 2. If $t \in [a_i + \delta, a_{i+1} - \delta]$, then $t \in (b_i, b_{i+1})$, that is, $j = i$. Lemma 8.10 gives

$$|h(a)(t) - h(b)(t)| \leqq c[|\Delta_{a_i} - \Delta b_i| + \Delta b_i 2|\mu - \mu_1|] \leqq 8c.$$

It follows that $\|h(a) - h(b)\|_C \to 0$ as $\delta \to 0$.

LEMMA 8.12. *The function* $h : \bar{T} \to K$ *is continuous.*

PROOF. It has been shown in previous lemmas that h maps $\bar{T}$ into K and that $\|h(a) - h(b)\|_C \to 0$ as $|a - b| \to 0$. This is sufficient if $E = C([0, 1])$ as in section 6 but if $E = L_p{}^1([0, 1])$ as in section 7, then it remains to show that $\|h(a)' - h(b)'\|_p \to 0$ as $b \to a$ in $\bar{T}$. The notation of the previous lemma will be used. Suppose that $a_i < t < a_{i+1}$, then

$$h(a)'(t) = \frac{d\mu}{dt}\frac{d}{d\mu}(h(a)(t)) = 6c\frac{d}{d\mu}\left(\int_{1/3}^{2/3} g(\mu, \lambda)\phi(\lambda)\, d\lambda\right)$$

$$= 6c\int_{1/3}^{2/3} g_\mu(\mu, \lambda)\phi(\lambda)\, d\lambda,$$

where

$$g_\mu(\mu, \lambda) = \begin{cases} 1 - \lambda, & \mu < \lambda \\ -\lambda, & \mu > \lambda. \end{cases}$$

Thus, if $|a - b| < \delta$ and $t \in (a_i, a_{i+1}) \cap (b_j, b_{j+1})$, then

$$|h(a)'(t) - h(b)'(t)| = 6c\left|\int_{1/3}^{2/3} [g_\mu(\mu, \lambda) - g_\mu(\mu_1, \lambda)]\phi(\lambda)\, d\lambda\right|$$

$$\leqq 6c \begin{cases} \int_{\mu}^{\mu^1} (1 - \lambda - (-\lambda))\phi(\lambda)\, d\lambda, \ \mu < \mu_1, \\ \int_{\mu_1}^{\mu} | - \lambda - (1 - \lambda)|\phi(\lambda)\, d\lambda, \ \mu_1 < \mu \end{cases}$$

$$\leqq 3c|\mu - \mu_1|.$$

Case 1. If $t \in [a_i, a_i + \delta) \cup (a_{i+1} - \delta, a_{i+1}]$, then

$$|h(a)'(t) - h(b)'(t)| \leqq 3c|\mu - \mu_1| \leqq 3c.$$

Case 2. If $t \in [a_i + \delta, a_{i+1} - \delta]$, then $t \in (b_i, b_{i+1})$ and $|h(a)'(t) - h(b)'(t)| \leqq 9c\delta/\Delta a_i$ by Lemma 8.10.

Hence,

$$\|h(a)' - h(b)'\|_p^p = \int_0^1 |h(a)'(t) - h(b)'(t)|^p\, dt$$
$$\leqq 2(n + 1)\delta(3c)^p + (9c\delta)^p m^{1-p},$$

where $m = \min\{\Delta a_i : 0 \leqq i \leqq n, \Delta a_i\} > 0$. Therefore $\|h(a)' - h(b)'\|_p \to 0$ as $\delta \to 0$.

The function k will be the same as the function h but with a different value for the parameter c, again, to be determined later. To avoid confusion, whenever we are dealing with the function k we will call this parameter d.

9. **Projection Methods.** The idea behind a projection method is to approximate the equation to be solved by another, simpler equation, which usually reduces to a finite system of scalar equations.

Let E and F be real Banach spaces and $N : D(N) \subseteq E \to R(N) \subseteq F$ a nonlinear operator. The *projection method* for the nonlinear equation $Nx = 0$ is as follows. Let $\{E_n\}$, $\{F_n\}$ be two sequences of subspaces of E and F, respectively, $E_n \subseteq D(N) \subseteq E$, $F_n \subseteq F$ and let P_n be a linear projection of F onto F_n, i.e.,

$$P_n{}^2 = P_n, P_nF = F_n.$$

The projection method replaces $Nx = 0$ by the *approximate equation*

$$P_nNx_n = 0, x_n \in E_n. \tag{9.1}$$

In contrast to these usual projection methods, the one used in this work is motivated by the pair of coupled operator equations

$$\begin{cases} x = B(a, x), \\ 0 = F(a, x), \end{cases} \tag{9.2}$$

the second of which is already finite-dimensional. The idea is to replace the first equation by a finite-dimensional approximate equation. Unfortunately, this leads us to define projections parameterized by the variable a, hence it does not precisely fit the usual context of a projection method. However, it does seem reasonable to call it a projection method, in view of the properties of the parameterized projection operators.

The projection method used in this work considers a sequence $\{E_m(a)\}_{m=1}^{\infty}$ of subspaces of E and a sequence of projections

$$P_m(a) : E \to E_m(a),$$

defined for each $a \in \bar{T}$. These projections will be constructed so as to map the cone K of nonnegative functions back into itself.

The nonlinear system (9.2) will be replaced by the *system of approximate equations*

$$\begin{cases} x_m = P_m(a)B(a, x_m), \; x_m \in E_m(a), \\ 0 = F(a, x_m), \; a \in \bar{T}. \end{cases} \tag{9.3}$$

This system is finite-dimensional.

In contrast to standard methods, it seems that (9.3) cannot be viewed as a projection scheme of the form (9.1). This is due primarily to the intermixing of the variable a in the projections themselves.

10. **The Projection Operators.** A family of projection operators shall be constructed, tailored to the problem at hand. Each of the projections will map E onto a finite-dimensional subspace where the corresponding approximate equations may be solved using numerical methods. It is shown that the identity is uniformly approximated by these projections, and that the approximate equations (9.3) suitably approximate (9.2) in the case of a nodal problem (3.3).

Let $E = \{x \in AC([0, 1]) : x' \in L_p([0, 1])\}$ with norm $\|x\| = \|x\|_C + \|x'\|_p$.

For each $a = (a_1, \cdots, a_n) \in \bar{T}$ and $m \geqq 2$. Let $\Pi_m(a) = \{s_{i,j} = a_j + (i - 1)h_j : 1 \leqq i \leqq m, \quad 0 \leqq j \leqq n\}$ where $h_j = (a_{j+1} - a_j)/(m - 1)$. Notice that $s_{1,j} = a_j = s_{m,j-1}$ and that $s_{i,j} = a_j$ if $a_{j+1} = a_j$.

Define the projection $P_m(a) : E \to E$ by letting $P_m(a)x$ be the continuous function on $[0, 1]$ which agrees with x at each point of $\Pi_m(a)$ and which is linear on $I_{i,j} \equiv [s_{i,j}, s_{i+1,j})$, $1 \leqq i \leqq m - 1$, $0 \leqq j \leqq n$. We may write

$$P_m(a)x = \sum_{i=1}^{m-1} \sum_{j=0}^{n} x(s_{i,j})e_{i,j} + x(s_{m,n})e_{m,n}$$

whose $e_{i,j}$ is the continuous function which satisfies: $e_{i,j}(s_{k,\ell}) = \delta_{i,k}\delta_{m,\ell}$ and $e_{i,j}$ is linear on each $I_{k,\ell}$, $1 \leqq i, k \leqq m$, $0 \leqq m$, $0 \leqq m$, $0 \leqq j, \ell \leqq n$. The operator $P_m(a)$ is a projection from E onto the subspace $E_m(a) = \text{span}\{e_{i,j} : 1 \leqq i \leqq m, 0 \leqq j \leqq n\}$. The dimension of $E_m(a)$ is $(m-1)(n+1)+1$ is $a \in T$; it is strictly less if $a \in \partial T$. The topology on $E_m(a)$ is the topology induced by E.

LEMMA 10.1. *$P_m(a)$ is a bounded linear operator from E onto $E_m(a)$ such that $\|P_m(a)\| = 1$ for all $a \in \bar{T}$, $m \geqq 2$.*

LEMMA 10.2. *If $x \in E$ is Lipschitz with constant M then $\|P_m(a)x - x\|_C \leqq M/2(m-1)$.*

PROOF. Let $t \in [0,1]$ be arbitrary. If $t = 1$, $P_m(a)x(t) = x(t)$; otherwise $t \in I_{i,j}$ for some $1 \leqq i \leqq m-1$, $0 \leqq j \leqq n$, and

$$\begin{aligned}|P_m(a)x(t) - x(t)| &= |x(s_{i,j})[s_{i+1,j} - t]h_j^{-1} \\ &\quad + x(s_{i+1,j})[t - s_{i,j}]h_j^{-1} - x(t)| \\ &\leqq |x(s_{i,j}) - x(t)|[s_{i+1,j} - t]h_j^{-1} \\ &\quad + |x(s_{i+1,j}) - x(t)|[t - s_{i,j}]h_j^{-1} \\ &\leqq 2M(s_{i+1,j} - t)(t - s_{i,j})h_j^{-1} \\ &\leqq Mh_j/2 \leqq M/2(m-1).\end{aligned}$$

LEMMA 10.3. *If $x \in E$ is such that $x' \in AC(I_{i,j})$ for $1 \leqq i \leqq m-1$, $0 \leqq j \leqq n$ and $x'' \in L_q([0,1])$ for some $1 < q \leqq p$, then $\|(P_m(a)x)' - x'\|_p \leqq \|x''\|_q(m-1)^{-1/q'}$, where $1/q + 1/q' = 1$.*

PROOF. Let x by as above, then

$$\|(P_m(a)x)' - x'\|_p$$

$$= \left[\int_0^1 \left|\sum_{i=1}^{m-1}\sum_{j=0}^{n} (x(s_{i+1,j}) - x(s_{i,j}))h_j^{-1}\chi_{I_{i,j}}(t) - x(t)\right|^p dt\right]^{1/p}$$

$$\leqq \left[\sum_{i=1}^{m-1}\sum_{j=0}^{n}\int_{I_{i,j}}\left(\int_{I_{i,j}}|x'(s) - x'(t)|h_j^{-1}\,ds\right)^p dt\right]^{1/p}$$

$$\leqq \left[\sum_{i=1}^{m-1}\sum_{j=0}^{n}\int_{I_{i,j}}\left(\int_{I_{i,j}}h_j^{-1}\left|\int_t^s \right| x''(u)|\,du|\,ds\right)^p dt\right]^{1/p}$$

$$\leqq \left[\sum_{i=1}^{m-1}\sum_{j=0}^{n}\int_{I_{i,j}}\left(\int_{I_{i,j}}h_j^{-1}\left|\int_t^s 1^{q'}\,du\right|^{1/q'}\|x''\|_q\,ds\right)^p dt\right]^{1/p}$$

$$\leqq \|x''\|_q \left[\sum_{i=1}^{m-1} \sum_{j=0}^{n} \int_{I_{i,j}} \left(\int_{I_{i,j}} h_j^{1/q'-1}\, ds \right)^p dt \right]^{1/p}$$

$$= \|x''\|_q \left[\sum_{i=1}^{m-1} \sum_{j=0}^{n} \int_{I_{i,j}} h_j^{p/q'}\, dt \right]^{1/p}$$

$$= \|x''\|_q h_j^{1/q'} \leqq \|x''\|_q/(m-1)^{1/q'}.$$

LEMMA 10.4. *Let y be Lipschitz on $[0, 1]$ with constant M, and suppose $[a, b], [c, d] \subset [0, 1]$.*

Let $\ell(t) = y(b)\lambda(t) + y(a)(1 - \lambda(t)) - y(d)\mu(t) - y(c)(1 - \mu(t))$, *where* $\lambda(t) = (t - a)/(b - a)$ *and* $\mu(t) = (t - c)/(d - c)$, *then the following estimates hold for*

$$\Lambda \equiv \max\{|\ell(t)| : t \in [a, b] \cap [c, d]\}:$$

(i) $\Lambda \leqq \max\{\min\{|b - c|, |c - a|\}, \min\{|b - d|, |d - a|\}\}2M$ if $[c, d] \subset [a, b]$,

(ii) $\Lambda \leqq \max\{\min\{|d - a|, |c - a|\}, \min\{|b - d|, |b - c|\}\}2M$ if $[a, b] \subset [c, d]$,

(iii) $\Lambda \leqq \max\{\min\{|b - c|, |c - a|\}, \min\{|b - d|, |b - c|\}\}2M$ if $a \leqq c \leqq b \leqq d$,

(iv) $\Lambda \leqq \max\{\min\{|d - a|, |c - a|\}, \min\{|b - d|, |d - a|\}\}2M$ if $c \leqq a \leqq d \leqq b$.

It will be shown below that if $y \in E$ is Lipschitz with constant M and if $a^k \to a^0$ in $\bar{T}$ as $k \to \infty$ then $\|P_m(a^k)y - P_m(a^0)y\| \to 0$ as $k \to \infty$ for each fixed $m \geqq 2$. Let $\{J_i\}_{i=1}^{\alpha}$ be an enumeration of the $I_{i,j}$'s corresponding to $\Pi_m(s)$ and $\{K_i\}_{i=1}^{\beta}$ and enumeration of those corresponding to $\Pi_m(a^k)$. We have suppressed the dependence of K_i on k. It is clear that $\alpha, \beta \leqq (n + 1)(m - 1)$.

LEMMA 10.5. *Let a^0, $a^k \in \bar{T}$ and let* $\delta = \min\{b - a : a < b \in \Pi_m(a^0)\}$, *where $m \geqq 2$ is fixed. Suppose that $|a^k - a^0| < \delta/4$ and that $[a, b) \cap [c, d) \neq \emptyset$ where a, b are adjacent points in $\Pi_m(a^0)$ and c, d are adjacent points in $\Pi_m(a^k)$, then*

$$\begin{aligned} &\max\{\min\{|b - c|, |d - a|, |c - a|\}, \\ &\min\{|b - c|, |d - a|, |b - d|\}\} \leqq |a^k - a^0|. \end{aligned} \tag{10.1}$$

PROOF. Since $\delta_k \equiv |a^k - a^0| < \delta/4$, the definition of δ implies that the δ_k-neighborhoods of the points in $\Pi_m(a^0)$ are disjoint. Further, each of these neighborhoods contains a point of $\Pi_m(a^k)$. In addition, each of the points of $\Pi_m(a^k)$ lies in one of these neighborhoods.

The proof is completed by examining each of five distinct cases arising from eight possible locations of a, b, c, d. One deduces the following:

$$\text{if } |d-a| > \delta_k, \text{ then } |b-c| \leqq \delta_k \text{ or } |c-a| \leqq \delta_k;$$

$$\text{if } |b-c| > \delta_k, \text{ then } |d-b| \leqq \delta_k \text{ or } |d-a| \leqq \delta_k.$$

LEMMA 10.6. *Suppose that $y \in E$ is Lipschitz with constant M and that a^k, $a^0 \in T$ with $|a^k - a^0| < \delta/4$, δ being defined in the previous lemma, then*

$$\|P_m(a^k)y - P_m(a^0)y\|_C \leqq 2M|a^k - a^0|, \text{ for each } m \geqq 2.$$

PROOF. Let $t \in [0,1]$, then either $t = 1$, in which case $P_m(a^k)y(t) = P_m(a^0)y(t)$, or $t \in J_i \cap K_j$ for some $1 \leqq i \leqq \alpha$, $1 \leqq j \leqq \beta$. Write $J_i = [a,b)$ and $K_j = [c,d)$, then $|P_m(a^k)y(t) - P_m(a^0)y(t)| = |\ell(t)| \leqq \Lambda$, for $t \in J_i \cap K_j$, where we are using the notation of Lemma 10.4. This lemma gives estimates for Λ with the four possible locations of a, b, c and d under the restriction that $[a,b] \cap [c,d] \neq \emptyset$. We will combine these estimates with the result of Lemma 10.5 to reach our conclusion.

If $|b-c|$, $|d-a| > \delta_k \equiv |a^k - a^0|$, then (10.1) implies that $|c-a|$, $|b-d| \leqq \delta_k$ and so $\Lambda \leqq 2M\,\delta_k$ in all cases. If $|d-a| \leqq \delta_k$, then only cases (i) and (iv) can occur; in case (i) this condition implies that $|c-a| \leqq \delta_k$ and so $\Lambda \leqq 2M\,\delta_k$ in all possible cases. If $|b-c| \leqq \delta_k$, then only cases (i) and (iii) can occur; in case (i) this condition implies that $|b-d| \leqq \delta_k$ and so $\Lambda \leqq 2M\,\delta_k$ in all possible cases.

LEMMA 10.7. *Under the assumptions of Lemma* 10.6,

$$\|(P_m(a^k)y - P_m(a^0)y)'\|_p \leqq 4M\,\alpha\beta|a^k - a^0|^{1/p},$$

where $\alpha, \beta \leqq (n+1)(m-1)$.

PROOF. The notation is as above. Then for $t \in [a,b] \cap [c,d]$

$$|\ell'(t)| \leqq \min\{L_1, L_2\}$$

where

$$L_1 \equiv (|y(b) - y(d)| + |y(c) - y(a)|)/|b-a| + |y(d) - y(c)|\,|(b-a)^{-1} - (d-c)^{-1}|,$$

$$L_2 \equiv |y(b) - y(a)|/|b-a| + |y(d) - y(c)|/|d-c|.$$

Therefore,

$$|\ell'(t)| \leqq 2M \cdot \min\{(|b-d| + |c-a|)/|b-a|, 1\}.$$

Each set $J_i \cap K_j$ is of the form $[a, b) \cap [c, d)$, furthermore, one of the following must occur: (i) $[a, b] \cap [c, d]$ has length less than $|a^k - a^0|$; (ii) $|a - c|, |b - d| < |a^k - a^0|$.

Considering cases (i) and (ii) leads to the estimate

$$\int_{[a,b]\cap[c,d]} |\ell'(t)|^p \, dt \leqq (4M)^p |a^k - a^0|.$$

This inequality implies the conclusion.

Let B be as defined in section 3. If P is a continuous projection of E onto K (for example $Px = |x|$) define $B^*(a, x) \equiv B(a, Px)$.

LEMMA 10.8. The function $(a, x) \to P_m(a)B^*(a, x)$ is continuous from $\bar{T} \times E$ into K for each $m \geqq 2$.

PROOF. Suppose $(a^k, x^k) \to (a^0, x^0)$ in $\bar{T} \times E$ as $k \to \infty$. Then by continuity of B, $y^k \equiv B^*(a^k, x^k) \to y^0 \equiv B^*(a^0, x^0)$, as $k \to \infty$. We have

$$\begin{aligned}\|P_m(a^k)y^k - P_m(a^0)y^0\| &\leqq \|P_m(a^k)(y^k - y^0)\| \\ &\quad + \|P_m(a^k)y^0 - P_m(a^0)y^0\| \\ &\leqq \|y^k - y^0\| + 2M|a^k - a^0| + 4M\alpha\beta|a^k - a^0|\end{aligned}$$

by Lemmas 10.6 and 10.7. Here, M is a Lipschitz constant for y^0, constructed as in the proof of Lemma 7.4.

LEMMA 10.9. The function $(a, x) \to P_m(a)B^*(a, x)$ is completely continuous from $\bar{T} \times E$ into K for each $m \geqq 2$.

PROOF. This follows from compactness of B and Lemma 10.8.

LEMMA 10.10. *Let $\mathcal{B}$ be a subset of $T \times E$ such that there exists a uniform Lipschitz constant M for $B^*(a, x)$ and a uniform bound M_1 for $\|B^*(a, x)''\|_q$, $(a, x) \in \mathcal{B}$. Then*

$$\|P_m(a)B^*(a, x) - B^*(a, x)\| \leqq (M + M_1)/(m - 1)^{1/q'},$$

where $q > 1$ and $1/q + 1/q' = 1$.

In particular, $P_m(a)B^*(a, x)$ uniformly approximates $B^*(a, x)$ on $\mathcal{B}$.

PROOF. Apply Lemmas 10.2 and 10.3 and the definition of the norm.

11. **Convergence.** This section uses the results of sections 10, 11 to obtain existence of solutions to the finite-dimensional approximate equations (9.3). Furthermore, the approximating operators are shown to be collectively compact. The solutions of the approximate equations are shown to cluster to the solution of the nodal problem, in the presence of uniqueness (theorem 11.4).

Recall $A : \bar{T} \times E \to R^n \times K$ is given by $A(a, x) = (a + F^*(a, x), B^*(a, x))$. Let us define $A_m : \bar{T} \times E \to R^n \times K$ by $A_m(a, x) = (a + F^*(a, x), P_m(a)B^*(a, x))$ for each $m \geqq 2$. Let $\Omega \subseteq \bar{\Omega} \subseteq T \times (B_R \backslash \bar{B}_r)$ be open.

THEOREM 11.1. *If the Leray-Schauder degree $d(I - A, \Omega, 0)$ exists, then, for m sufficiently large, so does $d(I - A_m, \Omega, 0)$ and the two degrees are equal. In particular, $d(I - A, \Omega, 0) \neq 0$ implies $d(I - A_m, \Omega, 0) \neq 0$ for all large m.*

PROOF. The operator A_m is completely continuous by virtue of Lemma 10.9. Due to the computations of section 7, the hypotheses of Lemma 10.10 are satisfied for any bounded subset $\mathcal{B}$, therefore A_m uniformly approximates A on $\bar{\Omega}$ as $m \to \infty$. The result follows from basic theorems of Leray-Schauder degree.

COROLLARY 11.2. *Suppose that the fixed points of A are isolated and that $d(I - A, T \times (B_R \backslash \bar{B}_r), 0) \neq 0$. Then there exists a sequence $\{\phi_{m_j}\}_{j=1}^{\infty} \subseteq T \times (B_R \backslash \bar{B}_r)$ converging to a fixed point of A and such that $A_{m_j} \phi_{m_j} = \phi_{m_j}, j = 1, 2, \cdots$.*

LEMMA 11.3. *The family of operators $\{A_m\}_{m \geqq 2}$ is collectively compact.*

PROOF. Let $\{(a^k, x^k)\}_{k=1}^{\infty}$ be a bounded sequence in $\bar{T} \times E$. By taking subsequences, if necessary, assume that $a^k \to a^0 \in \bar{T}$, $y_k \equiv B^*(a^k, x^k) \to y_0 \in E$, and $b_k \equiv a^k + F^*(a^k, x^k) \to b_0 \in R^n$ as $k \to \infty$, this being possible by the results in Section 7.

Let $\{m_k\}_{k=1}^{\infty} \subseteq N$ and consider the sequence $\{A_{m_k}(a^k, x^k)\}_{k=1}^{\infty}$. It suffices to show that $S \equiv \{P_{m_k}(a^k)B^*(a^k, x^k)\}_{k=1}^{\infty}$ has a convergent subsequence. If $\{m_k\}_{k=1}^{\infty}$ is bounded, then S is bounded in a finite dimensional space and possesses a convergent subsequence. If $\{m_k\}_{k=1}^{\infty}$ is not bounded, then by taking a subsequence, if necessary, we may assume that $m_k \to \infty$ as $k \to \infty$. In this case

$$\begin{aligned} \|P_{m_k}(a^k)y_k - y_0\| &\leqq \|P_{m_k}(a^k)y_k - P_{m_k}(a^k)y_0\| \\ &\quad + \|P_{m_k}(a^k)y_0 - y_0\| \\ &\leqq \|y_k - y_0\| + \|P_{m_k}(a^k)y_0 - y_0\|. \end{aligned}$$

Lemmas 10.2 and 10.3 complete the proof.

An immediate consequence of Lemma 11.3 is the following:

THEOREM 11.4. *If Ω is a bounded set in $T \times E$ then $p \equiv \{\phi \in \Omega : A_k\phi = \phi$ for some $k\}$ is precompact. In particular, if $\phi_0 \in \Omega$ is a fixed point of A, unique in $\bar{\Omega}$, then all sequences $\{\phi_k\}_{k=1}^{\infty} \subseteq P$ satisfying $\phi_k = A_k\phi_k$ converge to ϕ_0.*

11.5. REMARK. The uniqueness in $\bar{\Omega}$ of a fixed point of A does not imply the uniqueness of fixed points of the operators $\{A_k\}$. However, the fixed-point sets $K_p = \{\phi : \phi = A_p\phi\}$ are compact sets geometrically close to the singleton $\{\phi_0\}$ in the Hausdorff metric. In this sense, any solution $\phi \in K_p$ is a "good" approximation to ϕ_0, provided p is sufficiently large.

A-priori estimates for the rate of convergence need to be developed. This should be possible in the sublinear case, due to the more likely possibility of uniqueness (i.e., K_p is a singleton). Our progress on these problems is presently too incomplete to report.

12. **A-priori Bounds: The Superlinear Case.** This section contains several technical lemmas which give a-priori bounds for the solutions to certain types of differential equations satisfying nodal properties. These bounds are used to define open sets on which degree computations can be made.

Consider the differential equation

$$y''(t) + g(t, y(t), y'(t)) = 0, 0 < t < 1. \tag{12.1}$$

Suppose that g has the form

$$g(t, y, z) = f(t, y, z) + \operatorname{sign}(y)q(t), \tag{12.2}$$

where $f: [0,1] \times R^2 \to R$ and $q: [0,1] \to [0, \infty)$ are continuous. Further, assume that f satisfies

$$f(t, y, z)/y \geqq p(t)|y|^{\alpha}, \tag{12.3}$$

$t \in [0,1]$, $z \in R$, $|y| \geqq Y$, for some fixed α, $Y > 0$, and $p: [0,1] \to [0, \infty)$ is continuous and not identically zero on any interval of length $3^{-1}(n+1)^{-1}$. In addition, $f(t, y, z)y \geqq 0$ for all $(t, y, z) \in [0,1] \times R^2$.

If x is a continuous function on $[a, b]$ let $\|x\|_{a,b} \equiv \max\{|x(t)| : a \leqq t \leqq b\}$. Define $m(a, b)(t) = \min\{t - a, b - t\}/(b - a)$. An immediate consequence of the concavity of positive solutions of (12.1) is the following:

LEMMA 12.1. *Suppose that y is a solution of* (12.1) *having $a < b$ as adjacent zeros, then*

$$|y(t)| \geqq \|y\|_{a,b} m(a, b)(t), a \leqq t \leqq b,$$

and

$$\min\{|y'(a)|, |y'(b)|\} \geqq \|y\|_{a,b}(b - a)^{-1}.$$

LEMMA 12.2. *The function p satisfies*

$$p^* \equiv \inf \left\{ \int_a^b m(a,b)^{2+\alpha}(t)p(t)\,dt : b-a \geqq 3^{-1}(n+1)^{-1}, 0 \leqq a < b \leqq 1 \right\} > 0.$$

REMARK 12.3. If $p(t) \geqq p > 0$ for $0 \leqq t \leqq 1$, then $p^* \geqq p[3(n+1) \cdot (3+\alpha) \cdot 2^{2+\alpha}]^{-1}$.

LEMMA 12.4. *Let y be a solution of* (12.1) *having $a < b$ as consecutive zeros. Let $c = a + (b-a)/3$, $d = b - (b-a)/3$. If λ_0 is the minimum eigenvalue of*

$$\begin{cases} u''(t) + \lambda p(t)m(a,b)^{\alpha}(t)u(t) = 0 \\ u(c) = u(d) = 0 \end{cases} \tag{12.4}$$

then $\|y\|_{a,b} \leqq \max\{3Y, \lambda_0^{1/2}\}$.

PROOF. Suppose that $y(t) \geqq 0$ for $a \leqq t \leqq b$. If $\|y\|_{a,b} > 3Y$ then by Lemma 12.1, $y(t) \geqq Y$ for $c \leqq t \leqq d$. Furthermore, if $\|y\|_{a,b}^{\alpha} > \lambda_0$, then conditions (12.2) and (12.3) and Lemma 12.1 show that (12.4), with $\lambda = \lambda_0$, is strictly majorized by (12.1) on (c, d). This contradicts y not vanishing on (c, d), by Sturm's comparison theorem.

For computational purposes the following is given:

LEMMA 12.5. *Using the notation of Lemma* 12.4, *if* $(b-a) \geqq (n+1)^{-1}$, *then*

$$\lambda_0 \leqq 3(n+1)/p^* \equiv (R_0)^{\alpha}.$$

Given $\rho > 0$ ($\rho = 1$, usually), define $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$\phi(r,t) = \max\{|f(s,y,z)| : 0 \leqq s \leqq 1, 3|y| \leqq |z| \leqq r + \rho, |y| \leqq t + \rho\}.$$

Let $\Phi_t : [0,\infty) \to [0,\infty)$ be defined by

$$\Phi_t(u) = \int_0^u r[\phi(r,t)]^{-1}\,dr.$$

Hereafter, we assume a Nagumo Condition: $t < \int_{3t}^{\infty} r[\phi(r,t)]^{-1}\,dr$, $t \geqq 0$. This allows us to define $\Psi : [0,\infty) \to [0,\infty)$ by

$$\Psi(t) = \Phi_t^{-1}(t + \Phi_t(3t)).$$

LEMMA 12.6. *If $t \geqq 0$, then $\Psi \circ \Psi(t) \geqq \Psi(t) \geqq t$.*

LEMMA 12.7. *If y is a solution of* (12.1) *having $b > c$ as adjacent zeros in* $[0, 1]$, *and if* $q(t) = 0$ *for* $t \in [b, b + (c - b)/3] \cap [c - (c - b)/3, c]$, *then* $u = \|y\|_{b,c}$ *satisfies*

$$\max\{|y'(b)|, |y'(c)|\} \leqq \Phi_u^{-1}(u + \Phi_u(3u(c - b)^{-1})). \tag{12.5}$$

PROOF. The inequality for $y'(b)$ will be proved, the argument for $y'(c)$ is similar. Without loss of generality assume that $y'(b) > 0$. If $y'(b) \geqq 3\|y\|_{b,c}(c - b)^{-1}$, then (12.5) follows by the monotonicity of Φ.

If $y'(b) < 3\|y\|_{b,c}(c - b)^{-1}$, then there exists a first value $d \in (b, b + (c - b)/3]$ such that $y'(d) = 3\|y\|_{b,c}(c - b)^{-1}$ (otherwise, $y(t) = \int_b^t y'(s)\, ds > 3\|y\|_{b,c} \cdot (t - b)/(c - b)$, which fails at $t = b + (c - b)/3$). The following is valid:

$$\|y\|_{b,c} \geqq y(d) = \int_b^d y'(s)\, ds \geqq -\int_b^d y'(s) \frac{y''(s)}{\phi(y'(s), u)}\, ds;$$

the last inequality is a consequence of the definition of ϕ and the inequality $y'(s) \geqq 3\|y\|_{b,c}$, $s \in [b, d]$. Performing the change of variables $t = y'(s)$ and integrating leads to

$$\|y\|_{b,c} \geqq \Phi_u(y'(b)) - \Phi_u(y'(d)).$$

Solving for $y'(b)$ gives the desired result.

REMARK 12.7. Since Φ is increasing the right hand side of (12.5) may be replaced by $\Psi(\|y\|_{b,c}(c - b)^{-1})$.

Define $R_1 = (n + 1)R_0$, R_0 being given by Lemma 12.5, and set $R_{i+1} = \Psi(R_i)$ for $i \geqq 1$.

THEOREM 12.8. *Suppose that y is a solution of* (12.1) *having exactly $n + 2$ zeros at $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$. If $q(t) = 0$ for $t \in \bigcup_{i=0}^{n} [a_i, a_i + (a_{i+1} - a_i)/3] \cup [a_{i+1} - (a_{i+1} - a_i)/3, a_{i+1}$, then*

$$\|y\|_C \leqq R_{n+1} \text{ and } \|y'\|_C \leqq R_{n+2}.$$

PROOF. For brevity put $N_i = \|y\|_{a_i,a_{i+1}} (a_{i+1} - a_i)^{-1}$, $i = 0, 1, \cdots, n$. By Lemma 12.1,

$$N_i \leqq |y'(a_i)|, |y'(a_{i+1})| \leqq \Psi(N_i). \tag{12.6}$$

for $i = 0, 1, \cdots, n$.

Now for some j, $0 \leqq j \leqq n$, $a_{j+1} - a_j \geqq (n + 1)^{-1}$. It follows from Lemmas 12.4, 12.5 and 12.6 that

$$N_j \leqq (n + 1)R_0 = R_1 \leqq R_n,$$
$$\|y'\|_{a_j,a_{j+1}} = \max\{|y'(a_j)|, |y'(a_{j+1})|\} \leqq \Psi(N_j) \leqq R_2.$$

Equation (12.6) implies that

$$N_{j-1} \leqq |y'(a_j)| \leqq R_2 \text{ and } N_{j+1} \leqq |y'(a_{j+1})| \leqq R_2,$$

so

$$N_{j-2} \leqq |y'(a_{j-1})| \leqq \Psi(N_{j-1}) \leqq \Psi(R_2) = R_3$$

and

$$N_{j+2} \leqq |y'(a_{j+2})| \leqq \Psi(N_{j+1}) \leqq \Psi(R_2) = R_3.$$

Continuing by induction (when permissible) one gets

$$N_{j-k} \leqq |y'(a_{j-k+1})| \leqq R_{k+1}$$

and

$$N_{j+i} \leqq |y'(a_{j+i})| \leqq R_{i+1}$$

for $k \leqq j$ and $i \leqq n - j$. The conclusion follows from the observations that $\|y\|_{a_i,a_{i+1}} \leqq N_i$, $0 \leqq i \leqq n$, and $\|y'\|_C = \max\{|y'(a_i)| : 0 \leqq i \leqq n + 1\}$.

Using the notation of section 7, $\|y\|_E = \|y\|_c + \|y'\|_p \leqq R_{n+1} + R_{n+2}$.

To find a lower bound for the norm of a nontrivial solution of (12.1), the following definition is made.

Definition 12.9. Define $\eta : (0, \infty) \to (0, \infty)$ by

$$(12.7)\eta(r) \equiv \sup\{f(t, y, z)/y : t \in [0, 1], |z| \leqq R_{n+2}, 0 < |y| \leqq \eta\}.$$

Theorem 12.10. *Suppose that y is a solution of* (12.1), *where f satisfies* (12.2), (12.3). *If $\eta(r) < 8/5$ and*

$$\int_0^1 |q(t)|\, dt < r(8 - 5\eta(r))/10$$

for some $r < R_{n+1}$, then $\|y\| \equiv \|y\|_C + \|y'\|_p \neq r$.

Proof. Let $a < b$ be consecutive zeros of y in $[0, 1]$ such that for some $t_0 \in [a, b]$, $|y(t_0)| = \|y\|_C$. It can be assumed that $y \geqq 0$ on $[a, b]$. By Theorem 12.8 and definition 12.9, if $\|y\| = r$, then

$$\|y\|_C = y(t_0) = \int_a^b G(t_0, s; a, b)[f(s, y(s), y'(s)) + q(s)]\, ds$$

$$\leqq \int_a^b G(t_0, s; a, b)[\eta(r) + q(s)]\, ds$$

$$\leqq r\eta(r)/8 + \int_0^1 q(s)\, ds/4.$$

Similarly, $\|y'\|_p \leqq \|y'\|_C \leqq r\eta(r)/2 + \int_0^1 q(s)\,ds$. These inequalities contradict $\|y\| = r$ and the hypothesis.

REMARK 12.11. In the previous theorem the requirement that $\int_0^1 q(t)\,dt < r(8 - 5\eta(r))/10$ may be replaced by $\|q\|_C < r(8 - 5\eta(r))/5$.

13. **A-priori Bounds: The Sublinear Case.** Bounds for solutions of (12.1) in the C^1-norm will be obtained in the setting where f is sublinear in y near ∞ and satisfies certain other growth conditions. The specific requirements are as follows:

Suppose that there exist $Q > 0$, $Y_1 \geqq 0$ and $r > 0$ such that

(13.1) $$|f(t, y, z)| \leqq Q(1 + z^2)$$

for $|y| \leqq Y_1$, $z \in R$, $t \in [0, 1]$,

(13.2) $$\gamma \equiv \sup\{f(t, y, z)/y : |y| > Y_1, z \in R, t \in [0, 1]\} < 8,$$

and

(13.3) $$p(t)\psi(y) \leqq f(t, y, z)/y$$

for $0 < |y| \leqq r$, $z \in R$, $t \in [0, 1]$, where $p : [0, 1] \to [0, \infty)$ is continuous and positive off a set of measure zero, $\psi : [-r, r]\backslash\{0\} \to [0, \infty)$ is continuous and $\psi(y) \to +\infty$ as $y \to 0$.

LEMMA 13.1. *If y is a solution of (12.1) having $a < b$ as consecutive zeros and if $q(t) = 0$ on $I \equiv [a, a + (b - a)/3] \cup [b - (b - a)/3, b]$ and $[a, a_1] \cup [b_1, b] \subseteq \{t \in I : |y(t)| \leqq Y_1, |y'(t)| > 0\}$, then*

$$1 + (y'(a))^2 \leqq (Y_1^2/(a - a_1)^2 + 1)e^{2QY_1}$$

and

$$1 + (y'(b))^2 \leqq (Y_1^2/(b - b_1)^2 + 1)e^{2QY_1}.$$

PROOF. Suppose that $y(t) \geqq 0$ on $[a, b]$. The first inequality will be proved; all other cases are proved in a similar manner.

For $a \leqq t \leqq a_1$, $|y(t)| \leqq Y_1$ and by (13.1),

$$y'(t) \geqq y'(t)\frac{f(t, y(t), y'(t))}{Q(1 + (y'(t))^2)} = \frac{-y'(t)y''(t)}{Q(1 + (y'(t))^2)}.$$

By integrating with a change of variables it follows that

$$Y_1 \geqq \int_a^{a_1} y'(t)\,dt \geqq -\int_{y'(a)}^{y'(a_1)} \frac{s\,ds}{Q(1 + s^2)} = \frac{1}{2Q}\ln\frac{1 + (y'(a))^2}{1 + (y'(a_1))^2},$$

and

$$1 + (y'(a))^2 \leqq (1 + (y'(a_1))^2)e^{2QY_1}.$$

Finally, the concavity of y implies that

$$0 \leqq y'(a_1) \leqq y(a_1)/(a_1 - a) \leqq Y_1/(a_1 - a).$$

THEOREM 13.2. *If y is a solution of* (12.1), *with f replaced by λf, $\lambda \in [0, 1]$, having $a < b$ as consecutive zeros in* $[0, 1]$, *and if $q(t) = 0$ on $I \equiv [a, a + (b - a)/3] \cup [b - (b - a)/3, b]$, then $\|y\|_{a,b} \leqq R'$, where*

$$R' \equiv \max\{3Y_1, (8 - \gamma)^{-1}(8(9Y_1^2 + 1)Qe^{2QY_1} + 5\|q\|_C)/9\}$$

PROOF. Again, assume that $y(t) \geqq 0$ for $a \leqq t \leqq b$. Let $t_0 \in [a, b]$ be such that $y(t_0) = \|y\|_{a,b}$ and suppose that $\|y\|_{a,b} \geqq 3Y_1$, then by Lemma 12.1 there exist $a_1 < b_1 \in [a, b]$ such that $y(a_1) = y(b_1) = Y_1$ and $y' > 0$ on $[a, a_1]$, $y' < 0$ on $[b_1, b]$, and furthermore $J \equiv [a, a_1] \cup [b_1, b] \subseteq I$. It follows from Lemma 13.1 and condition (13.1) and (13.2) that

$$\begin{aligned}
\|y\|_{a,b} = y(t_0) = &\int_a^b G(t_0, s; a, b)(\lambda f(s, y(s), y'(s)) + q(s))\, ds \\
\leqq &\int_J G(t_0, s; a, b) Q(1 + (y'(s))^2)\, ds \\
&+ \int_{a_1}^{b_1} G(t_0, s; a, b)\gamma \|y\|_{a,b}\, ds \\
&+ \|q\|_C \int_{a+(b-a)/3}^{b-(b-a)/3} G(t_0, s; a, b)\, ds \\
\leqq &\, Q[Y_1/(a - a_1)^2 + 1]\, e^{2QY_1} \int_a^{a_1} G(t_0, s; a, b)\, ds \\
&+ Q[Y_1/(b - b_1)^2 + 1]\, e^{2QY_1} \int_{b_1}^{b} G(t_0, s; a, b)\, ds \\
&+ \gamma \|y\|_{a,b}/8 + 5\|q\|_C/72.
\end{aligned}$$

The integrals $\int_a^{a_1} G(t_0, s; a, b)\, ds$ and $\int_{b_1}^{b} G(t_0, s; a, b)\, ds$ are bounded by $(a - a_1)^2/2$ and $b - b_1)^2/2$, respectively, therefore

$$\begin{aligned}
\|y\|_{a,b}(1 - \gamma/8) \leqq &\, qY_1^2 e^{2QY_1} + 5\|q\|_C/72 \\
&+ Qe^{2QY_1}((a - a_1)^2 + (b - b_1)^2)/2.
\end{aligned}$$

Recall that $(a_1 - a)$, $(b - b_1) \leqq (b - a)/3 \leqq 1/3$, hence, $\|y\|_{a,b} \leqq R'$.

It should be noted that the above proof does not depend on the number of nodes of y in $[0, 1]$, nor does the bound depend upon $[a, b]$, except that it be contained in $[0, 1]$. Hence the following:

COROLLARY 13.3. *If y is a solution of* (12.1), *with f replaced by λf, $0 \leqq \lambda \leqq 1$, with $y(0) = y(1) = 0$ and if* support(q) *lies in the union of the middle thirds of the intervals between the nodes of y, then*

$$\|y\|_C \leqq R'.$$

COROLLARY 13.4. *If y and q are as above and if y has no more than n distinct zeros in* (0, 1), *then*

$$\|y'\|_C \leqq R'_{n+2}$$

where R'_{n+2} is $(n + 1)$-fold composition of Ψ applied to $(n + 1)R'$, Ψ being defined in Section 12 (see (13.1), (13.2)).

PROOF. Proceeding as in section 12, Lemma 12.7 is still valid in this case. Furthermore, Theorem 12.8 relied only upon $\|y\|_{a,b}(b - a)^{-1}$ having a bound independent of $(b - a)$, for $a < b$ some pair of consecutive zeros of y. In this case, that bound is $R'(n + 1)$.

LEMMA 13.5. *Consider the problem*

$$\begin{cases} u'' + \lambda p(t)u = 0, t \in (a, b) \\ u(a) = u(b) = 0. \end{cases} \tag{13.4}$$

If $\lambda_{a,b}$ is the minimum eigenvalue of (13.4) *for $a < b$* in (0, 1) *with $b - a \geqq (n + 1)^{-1}$, then $\lambda_{a,b} < (n + 1)\tilde{p}^{-1}$ where $\tilde{p} =$* $\inf\{\int_a^b m(a, b)^2(s)p(s)\, ds : b - a \geqq (n + 1)^{-1}\} > 0$.

With the above lemma it is easy to obtain a lower bound for the norm of nontrivial solutions of (12.1).

THEOREM 13.6. *Suppose that y is a nontrivial solution of* (12.1) *with $y(0) = y(1) = 0$ and having exactly n distinct zeros in* (0, 1). *Let $r \leqq r$ be such that $\psi(s) \geqq (n + 1)/\tilde{p}$ for all $0 < s \leqq \tilde{r}$, then*

$$\|y\|_C < \tilde{r}.$$

PROOF. Suppose that $\|y\|_C \leqq \tilde{r}$. Because of the nodal assumption on y there exist $a < b$ consecutive zeros of y such that $b - a \geqq (n + 1)^{-1}$. By conditions (12.3), (13.3)

$$u''(t) + \frac{g(t, y(t), y'(t))}{y(t)} u(t) = 0$$

majorizes

$$u''(t) + p(t)\psi(y(t))u(t) = 0$$

which, in turn, majorizes

$$u''(t) + p(t)(n + 1)\tilde{p}^{-1}u(t) = 0$$

on (a, b), by the choice of $\tilde{r}$. The previous lemma together with Sturm's comparison theorem implies that y must vanish in (a, b), a contradiction.

14. **The Main Results for the Superlinear Case.** This section shows how certain superlinear ordinary differential equations with nodal conditions can be handled with the results in earlier sections. Consider the differential equation with nodal condition

(14.1) $$y''(t) + f(t, y(t), y'(t)) = 0, 0 < t < 1,$$

(14.2) $y(0) = y(1) = 0$, y has exactly n distinct zeros in $(0, 1)$.

The assumptions on f shall be as follows:

(14.3) $f: [0, 1] \times R \times R \to R$ is continuous;

(14.4) $yf(t, y, z) \geqq 0$ for all $t \in [0, 1], y, z \in R$;

(14.5) The function $(y, z) \to f(\cdot, y(\cdot), z(\cdot))$ is continuous from $L_p^1([0, 1]) \times L_p([0, 1])$ into $L_q([0, 1])$ for some q, $1 < q \leqq p$, and bounded sets are taken to bounded sets;

(14.6) $f(t, y, z) = 0$ if and only if $y = 0$, $t - a.e.$ in $[0, 1]$;

(14.7) for some fixed α, $Y > 0$ and continuous function $p: [0, 1] \to [0, \infty)$, $f(t, y, z)/y \geqq p(t)|y|^\alpha$ for all $t \in [0, 1]$, $z \in R$, $|y| \geqq Y$. Further, it is assumed that p does not vanish on any interval of length $(3n + 3)^{-1}$. Finally, f satisfies a Nagumo Condition (see section 12).

Consider the finite-dimensional nonlinear system

(14.8) $$\begin{cases} F(a, x) = 0, \\ P_m(a)B(a, x) = x, \end{cases}$$

where $P_m(a)$ is the projection defined in section 10.

THEOREM 14.1. *Assume* (14.3)–(14.7). *Then for sufficiently large* m, *system* (14.8) *has a solution* $(a^m, x^m) \in T \times B_R \backslash \bar{B}_r$. *The sequence* $\{(a^m, x^m)\}$ *converges in the Hausdorff metric to the solution set of* (14.1), (14.2).

In order to prove Theorem 14.1, we prepare by verifying the hypotheses of Theorem 4.1.

The complete continuity condition holds by virtue of Sections 6, 7, 8.

The smallness condition on k holds because of Lemma 8.6 and our freedom to choose the parameter $d > 0$ as small as we please, (recall that k is, in fact, h with a different choice of the parameter, $c = d$). The number $r > 0$ will be given later and will not depend upon d.

If $f(t, y, z) = f(t, y)$, then the space $E = C([0, 1])$ and $\|\cdot\|$ is monotone on the cone K. Furthermore, if $a \in T$ $\Delta a_i \geqq (n+1)^{-1}$ for some i, $0 \leqq i \leqq n$, and so by Lemma 8.5, $\|h(a)\| = \|h(a)\|_C \geqq c/(9n+9)$. Hence, in this case *the compression condition* (*a*) will hold by choosing c sufficiently large. The number R will be given later and will not depend upon c.

If f does depend upon z, its third argument, then the space $E = L_p^1([0, 1])$ and $\|\cdot\|$ is not monotone on the cone K. This causes no problem, however, because $\sup\{\|x - B(a, x)\| : x \in K, \ \|x\| \leqq R, F(a, x)\} < N$, for some number N since B is completely continuous. Furthermore, $\|h(a)\| = \|h(a)\|_C + \|h(a)'\|_p \geqq c/(9n+9) + c/4$ by Lemma 8.5 and 8.6, hence *the compression* condition (*c*) will hold in this case too, by choosing c sufficiently large.

LEMMA 14.2. *The nonsingularity condition holds.*

PROOF. If $a \in \partial T$ and $x \in K$ are such that $x \geqq \tau k(a)$ or $x \geqq \tau h(a)$, (these are equivalent for our definitions of h and k), for some $\tau > 0$, then $x(a)(t) \equiv x(t)$ is zero precisely when $t \in \{a_i : 0 \leqq i \leqq n+1\}$. Lemma 5.3 shows that $F(a, x) \neq 0$.

If, on the other hand, $x = B(a, x) \neq 0$, then there exists an interval (a_j, a_{j+1}) which is nonempty and on which $x(t) > 0$. Suppose that $F(a, x) = 0$. Now

$$0 = F_j(a, x) = (-1)^{j-1}$$

$$\int_{a_{j-1}}^{a_j} G_t(a_j, s; a_{j-1}, a_j) f(s, (-1)^{j-1}x(s), (-1)^{j-1}x'(s))\, ds$$

$$+ (-1)^j \int_{a_j}^{a_{j+1}} G_t(a_j, s; a_j, a_{j+1}) f(s, (-1)^j x(s), (-1)^j x'(s))\, ds,$$

and since the latter integral is nonzero, neither is the former. Hence, $a_j \neq a_{j-1}$ and $x(t) > 0$ on (a_{j-1}, a_j) by concavity. Continuing by induction, it is clear that if $F_i(a, x) = 0$ for all $1 \leqq i \leqq j$, then $a_i \neq a_{i-1}$. We may obtain the same conclusion for all $j \leqq i \leqq n + 1$ using induction, started by supposing that $F_{j+1}(a, x) = 0$. This implies that $a \in T$, a contradiction.

Let $R = R_{n+1} + R_{n+2} + 1$ and $r > 0$ be such that $\eta(r) < 8/5$, $r < R_{n+1}$, where R_{n+1}, R_{n+2} and $\eta(r)$ are given in section 12. Let $M = \{x \in K : r \leqq \|x\| \leqq R\}$.

Lemma 14.3. *If $(a, y) \in T \times M$, $y = B(a, y) + \lambda h(a)$, $F(a, y) = 0$, for some $\lambda \in [0, 1]$, then $\|y\| \neq R$, that is, condition* (4.1) *holds.*

Proof. By definition of B, F and by virtue of Lemma 8.6 if $y = B(a, y) + \lambda h(a)$, $F(a, y) = 0$, then the function $u(t) \equiv \sum_{i=0}^{n} (-1)^i y(t) \chi_{[a_i, a_{i+1}]}(t)$ satisfies equation (12.1) where $q(t)$ in (12.2) is given by $q(t) = \lambda h(t; a, c)$. Furthermore, u has exactly $n + 2$ simple zeros at $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$. Theorem 12.8 shows that $\|y\| = \|u\| < R$.

Lemma 14.4. *If $(a, y) \in T \times M$, $y = \lambda B(a, y) + (1 - \lambda)k(a)$, $F(a, y) = 0$, for some $\lambda \in [0, 1]$, then $\|y\| \neq r$, that is, condition* (4.2) *holds.*

Proof. It shall be assumed that $d < r(8 - 5\eta(r))/(10n + 10)$ so that $(1 - \lambda)\int_0^1 k(t; a, d)\, dt \leqq d(n + 1) < r(8 - 5\eta(r))/10$. Now suppose that $y = \lambda B(a, y) + (1 - \lambda)k(a)$, $F(a, y) = 0$, then the function $u(t)$, defined as above, satisfies an equation of the type (12.1) where f is replaced by λf and q by $(1 - \lambda)k(t; a, d)$ in (12.2). Theorem 12.10, with a slight modification to take into account the λ coefficient of f, gives the result.

Proof of Theorem 14.1. The results in section five, together with Theorem 4.1 show that $d(I - A, T \times B_R \backslash \bar{B}_r, 0) \neq 0$. By Theorem 11.1, the approximating operator A_m has a fixed point in $T \times B_R \backslash \bar{B}_r$ for all large m. This means that system (14.8) has a solution in $T \times B_R \backslash \bar{B}_r$ for all large m.

The proof is completed by appeal to section 11.

Remark 14.5. The finite dimensional nonlinear system (14.8) can be written down explicitly, inasmuch as the operators involved are directly obtainable from (14.1), (14.2).

In the presence of (local) uniqueness of solutions to (14.1), (14.2), any sequence $\{(a^m, x_m)\}$ converges by virtue of Lemma 11.4 to the unique solution of (14.1), (14.2).

15. The Main Results for the Sublinear Case. Consider the nodal problem

(15.1) $$y''(t) + f(t, y(t), y'(t)) = 0, 0 < t < 1,$$

(15.2) $y(0) = y(1) = 0$, y has exactly n distinct zeros in $(0, 1)$.

The assumptions on f are (14.3)–(14.6) renumbered as

(15.3) $$f : [0, 1] \times R \times R \to R \text{ is continuous;}$$

(15.4) $$yf(t, y, z) \geqq 0 \text{ for all } t \in [0, 1], y, z \in R;$$

(15.5) The function $(y, z) \to f(\cdot, y(\cdot), z(\cdot))$ is continuous from $L_p{}^1([0, 1]) \times L_p([0, 1])$ into $L_q([0, 1])$ for some q, $1 < q \leqq p$, and bounded sets are taken to bounded sets;

(15.6) $f(t, y, z) = 0$ if and only if $y = 0$, $t -$ *a.e.* on $[0, 1]$;

The sublinear conditions on f contained in section 13 are also assumed:

(15.7) For some $Q > 0$, $Y_1 \geqq 0$ and $r > 0$ $|f(t, y, z)| \leqq Q(1 + z^2)$ for $|y| \leqq Y_1$, $z \in R$, $t \in [0, 1]$;

(15.8) $\gamma \equiv \sup\{f(t, y, z)/y : |y| > Y_1, z \in R, t \in [0, 1]\} < 8$;

(15.9) there exists $p : [0, 1] \to [0, \infty)$ and $\psi : [-r, r]\backslash\{0\} \to [0, \infty)$ continuous with $\psi(y) \to \infty$ as $y \to 0$, such that $p(t)\psi(y) \leqq f(t, y, z)/y$ for $0 < |y| \leqq r$, $z \in R$, $t \in [0, 1]$. Further, it is assumed that p is positive off a set of measure zero.

Let $\tilde{R} = R' + R'_{n+2} + 1$, R' *and* R'_{n+2} being defined in section 13. The R and r of Theorem 4.2 will be this $\tilde{R}$ and the $\tilde{r}$ of Theorem 13.6, respectively and $M = \{x \in K : \tilde{r} \leqq \|x\| \leqq R\}$.

Consider the finite-dimensional nonlinear system

(15.10) $$\begin{cases} F(a, x) = 0 \\ P_m(a)B(a, x) = x. \end{cases}$$

THEOREM 15.1. *Assume* (15.3)–(15.9). *Then for sufficiently large m, system* (15.10) *has a solution* $(a^m, x^m) \in T \times B_{\tilde{R}}\backslash\bar{B}_{\tilde{r}}$. *The sequence* $\{(a^m, x^m)\}$ *converges in the Hausdorff metric to the solution set of* (15.1), (15.2).

To prove Theorem 15.1, we first satisfy the hypotheses of Theorem 4.2. The remainder of the proof is similar to that of Theorem 14.1. In section 14 it was shown that *the complete continuity condition, the smallness condition on* k, and *the nonsingularity condition* were all satisfied (for the parameter d sufficiently small). The proofs did not rely upon the growth of f and so these conditions are satisfied in this situation also. Furthermore, *the compression conditions* (b) or (d) may be satisfied as before, by choosing c sufficiently large.

LEMMA 15.2. *If* $(a, y) \in T \times M$, $y = \lambda B(a, y) + (1 - \lambda)k(a)$, $F(a, y) = 0$ *for some* $\lambda \in [0, 1]$, *then* $\|y\| \neq R$, *that is, condition* (4.3) *holds.*

PROOF. If $(a, y) \in T \times M$, $y = \lambda B(a, y) + (1 - \lambda)k(a)$, $F(a, y) = 0$ for some $\lambda \in [0, 1]$, then the function $u(t) \equiv \sum_{i=0}^{n} (-1)^i y(t) \chi_{[a_i, a_{i+1}]}(t)$ satisfies equations (12.1), with f replaced by λf and q by $(1 - \lambda)k(t; a, d)$ in (12.2). Furthermore, $u(0) = u(1) = 0$ and u has exactly n distinct zeros in $(0, 1)$. Corollaries 13.3 and 13.4 show that $\|y\| = \|u\| < \tilde{R}$.

LEMMA 15.3. *If* $(a, y) \in T \times M$, $y = B(a, y) + \lambda h(a)$, $F(a, y) = 0$ *for some* $\lambda \geqq 0$ *then* $\|y\| \neq \tilde{r}$, *that is, condition* (4.4) *holds.*

PROOF. Apply Theorem 13.6 where q in (12.2) is replaced by $\lambda h(t; a, c)$.

REMARK 15.4. The comments made at the end of section 14 concerning the existence of solutions to the "approximate" equations (14.9) and their convergence to solutions of the nodal problem, also apply here.

BIBLIOGRAPHY

1. M. G. Crandall and P. Rabinowitz, *Nonlinear Sturm-Liouville Eigenvalue Problems and Topological Degree,* J. Math. Mech. 19 (1970), 1083-1102.

2. G. B. Gustafson, *Fixed point methods for nodal problems in differential equations* in *Fixed Point Theory and its applications* (editor, S. Swaminathan), Academic Press, New York (1976), 69-77.

3. G. B. Gustafson and Klaus Schmitt, *Nonlinear Analysis Methods in Differential Equations,* The University of Utah: 1973.

4. P. Hartman, *Ordinary Differential Equations,* John Wiley and Sons, Inc., New York: 1964.

5. Yasuhiko Ikebe, *The Galerkin Method for the Numerical Solution of Fredholm Integral Equations of the 2nd Kind,* Siam Review 14 (1972), 467-491.

6. L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces,* The MacMillan Company, New York: 1964.

7. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations,* The MacMillan Company, New York: 1964.

8. ——, *Positive Solutions of Operator Equations,* P. Noordhoff Ltd., Groningen, The Netherlands: 1964.

9. M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii, and V. Ya. Stetsenko, *Approximate Solution of Operator Equations*, Wolters-Noordhoff Pub., Groningen, The Netherlands: 1972.

10. J. LeRay and J. Schauder, *Topologie Et Equations Fonctionnelles*, Ann. Sci. Ecole Norm. Sup. 3, **51** (1934), 45–78.

11. Z. Nehari, *On a Class of Nonlinear Second Order Differential Equations*, Trans. Amer. Math. Soc. **95** (1960), 101–123.

12. ——, *Characteristic Values Associated with a Class of Nonlinear Second Order Differential Equations*, Acta. Math. **105** (1961), 144–175.

13. G. Pimbley, *A Superlinear Sturm-Liouville Problem*, Trans. Amer. Math. Soc. **103** (1962), 229–248.

14. ——, *A Sublinear Sturm-Liouville Problem*, J. Math. Mech. **11** (1962), 121–138.

15. P. Rabinowitz, *Nonlinear Sturm-Liouville Problems for Second Order Ordinary Differential Equations*, Comm. Pure Appl. Math. **23** (1970), 939–961.

16. ——, *A Nonlinear Sturm-Liouville Problem*, Japan-United States Seminar on Ordinary Differential and Functional Equations, Lecture Notes in Mathematics, No. 243, Springer Verlag, New York, New York: 1971.

17. H. L. Royden, *Real Analysis*, The MacMillan Company, New York: 1963.

18. J. T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach, New York: 1969.

19. Frank Stenger, *On the Convergence of the Bubnov-Galerkin Method*, Springer Verlag Lecture Notes No. 362 (1974): 434–450.

20. R. E. L. Turner, *Nonlinear Sturm-Liouville Problems*, J. Diff. Equations **10** (1971), 141–146.

21. R. E. L. Turner, *Superlinear Sturm-Liouville Problems*, J. Diff. Equations **13** (1973), 157–171.

UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112