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ASYMPTOTIC ANALYSIS OF REACTION-DIFFUSION WAVE FRONTS

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1. **Introduction.** The equations of reaction and diffusion in one space variable are

$$\underset{\sim}{u}_t = D\underset{\sim}{u}_{xx} + f(\underset{\sim}{u}), \quad \underset{\sim}{u} = (u_1, \cdots, u_n) \tag{1.1}$$

where D is a "diffusion" matrix. Such equations arise in chemical reaction and combustion problems, in population dynamics, and in other areas.

When the diffusion term $D\underset{\sim}{u}_{xx}$ (which may represent the random migration of the individuals of a population or chemical species in a mixture) is deleted from the system, we call the resulting set of ordinary differential equations the "corresponding kinetic equations".

Any stationary state of the kinetic equations is, of course, also a (constant) stationary state of the system (1.1). We consider the case when at least two such constant stationary states exist. Then conceptually, the possibility suggests itself of a solution of (1.1) which, for each t, approaches one state as $x \to -\infty$, and another as $x \to \infty$. In special cases [1, 2, 5, 7, 8, 9, 10, 13], such solutions are indeed known to exist, in the form of propagating wave fronts. Propagating fronts are solutions of the form $\underset{\sim}{u}(x, t) = \underset{\sim}{U}(x - ct)$, with $\underset{\sim}{U}(\pm\infty) = \underset{\sim}{u}_{\pm}$, the constant vectors $\underset{\sim}{u}_{\pm}$ being the two stationary states in question.

A very imprecise heuristic description of how and why wave fronts come about can be given as follows. Let us suppose that discontinuous initial data

$$\underset{\sim}{u}(x, 0) = \begin{cases} \underset{\sim}{u}_-, x < 0 \\ \underset{\sim}{u}_+, x > 0 \end{cases}$$

are imposed on the system (1.1). The solution of the resulting Cauchy problem is such that this discontinuity will be smoothed out by diffusion. The result is a time-dependent continuous distribution which approaches the two states $\underset{\sim}{u}_-$ and $\underset{\sim}{u}_+$ as $x \to \pm\infty$. Consider a value of x where $\underset{\sim}{u}$ is near $\underset{\sim}{u}_-$, but not equal to $\underset{\sim}{u}_-$. If $\underset{\sim}{u}_-$ is a stable stationary state of the kinetic system, then the reaction terms of the system (1.1) will tend to drive the solution $\underset{\sim}{u}(x, t)$ toward $\underset{\sim}{u}_-$ as t increases. This would have the effect of steepening the gradient in the transition

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region (unless a counteracting influence were also present on the other side). On the other hand, diffusion from regions to the right, where $\underset{\sim}{u}$ is even further from $\underset{\sim}{u}_-$, will have the opposite tendency, to drive the system away from $\underset{\sim}{u}_-$, thereby making the gradient less steep. In this way, there will be two opposing influences. One expects that for some gradient profile with the right amount of steepness, the two influences will balance each other out. If the gradient is too gradual, the reaction terms will make it steep; if it is too steep, the diffusion will make it less so.

In regions further to the right, where $\underset{\sim}{u}$ is near $\underset{\sim}{u}_+$, there will also be two opposing influences. There is no reason to believe, however, that the profile which balances the two tendencies on the left will also do so for the right. If that happens to be the case, a stationary transition solution results; but in most cases, the transition will move in one direction or the other. The stable moving profile will then be so steep that diffusion "wins out" over reaction on one side (in the direction ahead of the motion), but so flat that reaction wins out on the other side. Though both states $\underset{\sim}{u}_\pm$ are stable, the one behind the front could be thought of as the dominant one.

In applied contexts, our results are only suggestive. Nevertheless, front-type phenomena do occur in combustion theory [8, 9], and have been predicted for situations in genetics [7], chemical reaction theory [12, 13], and other population theories where the system in question has two steady states. In real populations, phenomena bearing some similarity to these are of course commonplace, when one species, genotype, or community takes over the territory of another. An interesting mathematical treatment of such a situation can be found in [11]. However in general, models based only on the system (1.1) are liable to be too idealized to describe real population dynamical phenomena with any sort of accuracy.

In this paper we are mainly concerned with an asymptotic analysis of wave fronts for systems with $n = 2$. At present the picture is quite complete in the case of a scalar equation ($n = 1$), but not much is known for systems ($n > 1$). The results given here are far from giving a complete picture of the possible wave fronts, even for $n = 2$, mainly because they require at least one small parameter to be present in a strategic place. This parameter makes possible the use of perturbation methods.

Moreover, we restrict attention entirely to the case when both $\underset{\sim}{u}_-$ and $\underset{\sim}{u}_+$ are stable solutions of the kinetic equations. This excludes, for example, the classical case studied by Fisher [7] and Kolmogorov, Petrovskiĭ, and Piscunov [10] which represents an idealized model of an unstable population state being taken over by a genotype (mutant or introduced) which has a selective advantage. ("Unstable" here is with respect to perturbations representing the introduction of advantageous genotypes, or their appearance by mutation.) Incidentally, an important practical difference between the case when one of the states $\underset{\sim}{u}_{\pm}$ is unstable and when they are both stable, is that in the former case there is a "hair trigger" effect [1], whereas in the latter there is a threshold effect. That is, for scalar equations at least, in the former case an arbitrarily small compact support perturbation away from the unstable state will lead to propagating wave phenomena [1, 2], and in the latter, the perturbation has to be large enough in some sense [1, 2, 6] for this to happen (also, the new state has to be the "dominant" one in the sense described above).

Section 2 is mainly a brief review of the known results in the scalar case $n = 1$. Sections 3–6 are concerned with the case $n = 2$, in which a far richer range of phenomena are possible. Section 2B develops a rather obvious perturbation (to $n = 2$) of the simplest scalar case. The fronts examined in Secs. 3–6 are more sophisticated; they involve a steep portion, with more gradually varying portions on the two sides. We construct, for them, a systematic formal asymptotic expansion in one or more small parameters. The existence of a steep portion suggests the use of two length scales in the asymptotic development. This, in fact, is done throughout Secs. 3–6. The problems in Sec. 6, in fact, often require a third length scale as well.

For purposes of developing the proper asymptotic techniques, the problems we are able to treat fall into three categories, as described in Sec. 3. In all cases, one of the components of $\underset{\sim}{u}$ suffers a jump across the front (in the lowest order approximation), whereas the other varies continuously. Categories (Types) I and II are characterized by the fact that the lowest order approximation for the continuous component is governed by a second order boundary value problem, whereas in Type III, it is a first order initial value problem. Type III problems are such that the system is very nearly uniform (constant) ahead of the wave front. A limiting case when the diffusion coefficient of one of the components is zero may be handled by the techniques of Type III.

An existence proof for fronts is not the object here. It appears likely that such proofs could be obtained, on the basis of topological methods developed by C. Conley and others (H. Kurland has recently done precisely this).

Wave fronts similar to these were introduced in [12, 13, 5], where the beginnings of an asymptotic development (lowest order) were described. The fronts discussed in [13]) were like those of our Type III. Our classification into three types corresponds to the classification of constant velocity wave fronts in [**5**, sec. 7] into types 1a, b, and c. However, the problems in [**5**] involve only one small parameter, whereas ours involve more. In [**5**], variable velocity wave fronts in bounded media were also treated; we do not discuss them here.

The stability of wave fronts is an important but different subject. Sattinger [**14**] has given results which affirm stability if certain properties of the spectrum of an associated linear operator are known. Probably the fronts we analyze here are quite stable. In fact, the results of [**6**] in the scalar case would indicate that they, as well as the constant stationary states, play a central role as possible limiting forms, as $t \to \infty$, for solutions of (1.1). A further heuristic argument pointing in the direction of stability in the case of fronts of the types treated here with $n = 2$ was given in [**5**].

2. **The scalar case. A. Summary of known results.** Here we consider the single equation

$$u_t = u_{xx} + f(u). \tag{2.1}$$

Stationary states $u_\pm$ are real numbers. For simplicity we assume $u_- = 0, u_+ = 1$. Thus

$$f(0) = f(1) = 0.$$

We assume they are both stable; in fact

$$f'(0) < 0, f'(1) < 0.$$

It follows that f must have at least one more, unstable, zero between these two. If there is only one such additional zero, a unique wave front $U(x - ct)$ with $U(-\infty) = 0$, $U(\infty) = 1$, exists [**1, 2**]. If there are still more zeros in (0, 1), a quite general sufficient condition for such a front to exist is given in [**6**]. In any case, the front's profile and velocity will be unique (unlike the case considered in [**10**]), and its direction of motion will be determined as follows:

$$c \gtreqless 0 \text{ according as } \int_0^1 f(u)\, du \lesseqgtr 0. \tag{2.2}$$

In the "upper" case, for $\int f\, du < 0$, we say the state $u = 0$ is "dominant," since the front moves to the right, increasing the "domain" of that state at the expense of the state $u = 1$. The other cases are when $u = 1$ is dominant, and when they are equally dominant (in this latter case, $c = 0$, so the front is stationary, representing the coexistence of the two states.)

Moreover, these wave fronts can be shown to be very stable [**6**].

To summarize, *when only two stable constant stationary states* *exist, there is a unique stable wave front representing a transition from one to the other. When there are more than two such stable stationary states, the important question is, which pairs of them may be connected by a stable wave front? Certain pairs may be so connected, but it is not necessarily true that every pair may be.*

I conjecture that these statements also hold, under mild additional assumptions, for the case $n > 1$.

B. A simple dimension-increasing perturbation of the scalar case. Consider the system of two equations

$$u_t = u_{xx} + f(u, v), \tag{2.3a}$$

$$v_t = v_{xx} + kg(u, v), \tag{2.3b}$$

where k is a large parameter. We suppose the corresponding kinetic equations have two rest states, which, for simplicity, we take to be $(0, 0)$ and $(1,1)$. Thus

$$f(0, 0) = g(0, 0) = f(1, 1) = g(1, 1).$$

We also assume that the relation

$$g(u, v) = 0$$

defines a function (Fig. 1)

$$v = H(u), \tag{2.4}$$

with $H(0) = 0, H(1) = 1$.

Finally, we assume that the two states $(0, 0)$ and $(1, 1)$ are stable for all large k. in fact, we assume that the matrix

$$\begin{bmatrix} f_u & f_v \\ kg_u & kg_v \end{bmatrix},$$

evaluated at either state, has both eigenvalues with negative real part, for large k. This is equivalent to the conditions that $g_v < 0$ and the determinant $D = f_u g_v - g_u f_v > 0$.

We shall look for wave front solutions

$$u = U(x - ct), v = V(x - ct)$$

of (2.3) satisfying $U(-\infty) = V(-\infty) = 0$, $U(\infty) = V(\infty) = 1$. Since such solutions may be translated at will, we normalize by requiring

$$U(0) = 1/2. \tag{2.5}$$

Setting $z = x - ct, \epsilon = k^{-1}$, we obtain the following equations for U and V:

$$cU_z + U_{zz} + f(U, V) = 0 \tag{2.6a}$$

$$\epsilon(cV_z + V_{zz}) + g(U, V) = 0. \tag{2.6b}$$

Of course c is to be determined as well as U and V.

As first approximation $(U_0(z), V_0(z), c_0)$, we set $\epsilon = 0$ and use (2.4) to obtain

$$U_0'' + c_0 U_0' + F(U_0) = 0 \tag{2.7}$$

where $F(u) = f(u, H(u))$.

Notice that $F'(u) = D/g_v < 0$ at $u = 0$ or 1, by the stability assumptions. Hence the lowest order approximation is reduced to the scalar case treated above in Sec. 2A. There will be a third zero of F in the interval $(0, 1)$; this represents a third stationary point for the kinetic system (the intermediate intersection point in Fig. 1). Let us suppose there are no others. Then according to the results stated in Sec. 2A, there exists a unique wave front solution $U_0(x - c_0 t)$ of (2.7), satisfying

$$U_0(-\infty) = 0, U_0(\infty) = 1, U_0(0) = 1/2.$$

This function, together with $V_0(z) = H(U_0(z))$, constitutes our first approximation.

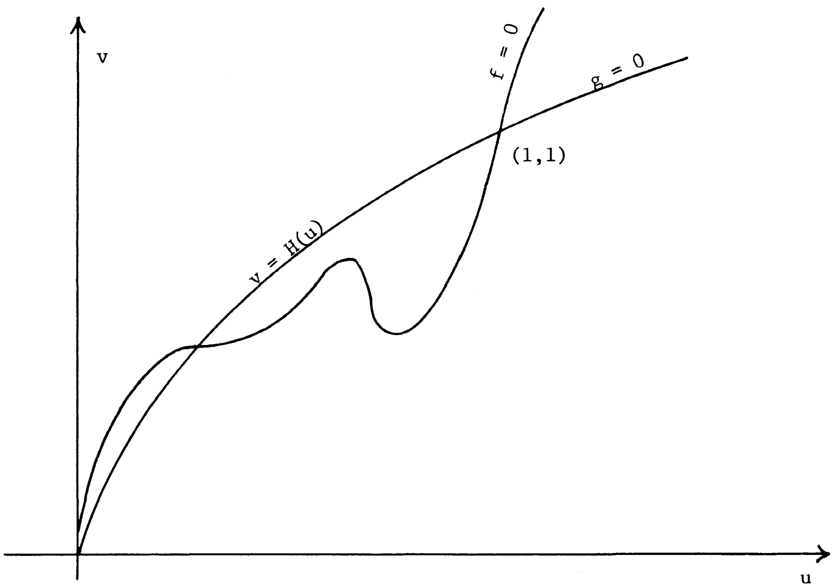

Figure 1.

To construct higher order approximations, we formally set

$$c = c_0 + \epsilon c_1 + \cdots,$$
$$U = U_0(z) + \epsilon U_1(z) + \cdots,$$

and

$$V = V_0(z) + \epsilon V_1(z) + \cdots.$$

Substituting these expressions into (2.6) and equating coefficients of ϵ, we obtain the following equations for U_1, V_1, and c_1:

$$U_1'' + c_0 U_1' + f_u U_1 + f_v V_1 = -c_1 U_0',$$
$$g_u U_1 + g_v V_1 = -c_0 V_0' - V_0''.$$

They can be decoupled and written as

$$LU_1 = -c_1 U_0' + r_1, \tag{2.8}$$

$$V_1 = -\frac{g_u}{g_v} U_1 + s_1, \tag{2.9}$$

where

$$LU \equiv U'' + c_0 U' + F'(U_0(z))U,$$

$$r_1(z) \equiv \frac{f_v}{g_v}(c_0 V_0' + V_0''),$$

$$s_1(z) \equiv -\frac{1}{g_v}(c_0 V_0' + V_0'').$$

In accordance with (2.5), we require $U_1(0) = 0$. We shall use the following property of the operator L:

LEMMA. *Let $q(z)$ be a continuous function on $\underset{\sim}{R}$ such that $\int_{-\infty}^{\infty} e^{c_0 z} q(z)^2\, dz < \infty$. Then there exists a bounded solution of*

$$Lp(z) = q(z) \tag{2.10}$$

on $\underset{\sim}{R}$ satisfying

$$\int_{-\infty}^{\infty} e^{c_0 z}[p''(z)^2 + p'(z)^2 + p(z)^2]\, dz < \infty, \tag{2.11}$$

if and only if

$$\int_{-\infty}^{\infty} e^{c_0 z} U_0'(z) q(z)\, dz = 0. \tag{2.12}$$

If this condition is fulfilled, there exists, in fact, a unique bounded solution of (2.10) *satisfying* (2.11) *and* $p(0) = 0$.

PROOF. All except the final statement is simply a particular case of Lemma 4.5 of [4]. Also by that lemma, every solution satisfying (2.11) is contained in the set $\{p_0(z) + aU_0'(z)\}$, where p_0 is some particular solution, and a is arbitrary. However $U_0'(0) \neq 0$ [6], so there is a unique value of a for which $p(0) = 0$.

We wish to apply the lemma to the equation (2.8). It is easily verified that the right hand side is square integrable with weight $e^{c_0 z}$. In fact, by linearization of (2.7) we can determine the exponential rate at which U_0 approaches its limits at $\pm\infty$. Since $F'(1) < 0$, for example, we know that $|U_0 - 1| \leqq Ce^{-\gamma z}$ $(z > 0)$, where $\gamma > c_0/2$. The same estimate holds for the derivatives of U_0. Hence

$$\int_0^\infty e^{c_0 z}(U_0'(z))^2\, dz \leqq C^2 \int_0^\infty \exp(-(2\gamma - c_0)z)\, dz < \infty.$$

Because $V_0 = H(U_0)$ and $H'(1) \neq 0$, $V_0 - 1$ and its derivatives, hence r_1, all approach 0 as $z \to \infty$ at the same rate as does U_0. A similar argument holds as $z \to -\infty$. Hence the right hand side of (2.8) satisfies the hypotheses of the lemma.

To guarantee a solution, we must apply the orthogonality condition (2.12). The result is

$$c_1 \int_{-\infty}^\infty \exp(c_0 z)(U_0'(z))^2\, dz = \int_{-\infty}^\infty \exp(c_0 z) r_1(z) U_0'(z)\, dz.$$

This determines c_1 and allows (2.8) to be solved for U_1. We can even require $U_1(0) = 0$.

The next approximation V_1 is now found from (2.9).

The higher order approximations are found in exactly the same manner.

3. **Fronts with two length scales: Formulation and classification.** As in Sec. 2B, we again consider problems with $n = 2$ and with two stable constant stationary states $(0,0)$ and $(1,1)$. And again, we suppose the equations depend on small parameters, but in a more general manner than before. We begin with a system

$$u_t = D_1 u_{xx} + f(u, v), \tag{3.1}$$

$$v_t = D_2 v_{xx} + g(u, v), \tag{3.2}$$

in which D_1, D_2, f and g may depend on several parameters with differing orders of magnitude.

Our first assumption (made for simplicity rather than necessity) is that the equations may be multiplied by appropriate constants to make f and g effectively independent of any of the parameters appearing in the problem. This, however, may introduce new parameters with the other terms. We next rescale time and space so that v_t will occur with coefficient unity, and u_{xx} with a small coefficient, which we denote by ϵ^2. In our asymptotic analysis, we shall always assume $\epsilon \ll 1$.

There remain only the coefficients of u_t and v_{xx}. We denote them by δ and γ, respectively. The orders of magnitude of various combinations of ϵ, δ, and γ will determine the type of asymptotic analysis we pursue.

The system we study, then, is

$$\delta u_t = \epsilon^2 u_{xx} + f(u, v) \tag{3.2a}$$

$$v_t = \gamma v_{xx} + g(u, v). \tag{3.26}$$

Our main assumptions concern the behavior of the functions f and g. Of course, they must vanish at the two stable rest points $(0, 0)$ and $(1, 1)$.

Assumptions A. The points $(0, 0)$ and $(1, 1)$ are rest points of the system, at which $f_u g_v - f_v g_u > 0$. The curve $f = 0$ consists of three monotone branches, as shown in Figure 2. There is only one additional intersection point with the curve $g = 0$ (in addition to the points $(0, 0)$ and $(1, 1)$); it is on the decreasing branch. The signs of f and g are as shown. Let the two ascending branches be represented by $u = h_{\pm}(v)$. Then

$$f_u(h_{\pm}(v), v) < 0 \tag{3.3}$$

for all v in the interior of the domains of $h_{\pm}$.

We shall seek travelling wave solutions of (3.2) with speed of the order $\epsilon\delta^{-1}$. That is, we want solutions which are functions only of $x - c\epsilon\delta^{-1}t$, where c (as well as u and v) may depend on ϵ, δ, and γ, but remains of order unity as the parameters of interest (see below) approach their asymptotic limits.

The procedure to be followed will depend on the orders of magnitude of certain combinations of the given parameters (in all cases, we assume $\epsilon \ll 1$). Accordingly, we define two new parameters

$$\rho = \epsilon\gamma^{-1/2}, \sigma = \epsilon\gamma^{-1/2}\delta^{-1}.$$

The problems we are able to treat follow naturally into three categories:

$$\text{Type I: } \rho \ll 1 \text{ and } \sigma \ll 1,$$

$$\text{Type II: } \delta \ll 1 \text{ and } \sigma = O(1),$$

$$\text{Type III: } \delta \ll 1 \text{ and } \sigma \gg 1.$$

We examine each type separately by giving the details for a particular example, when δ and γ are specific powers of ϵ, and then indicate how the general case may be handled.

When we use the moving coordinate $x - c\epsilon\delta^{-1}t$, the problem becomes one with a single independent variable. In all cases, we separate this problem into two parts: one on $\underset{\sim}{R}^+$ and one on $\underset{\sim}{R}^-$, each with prescribed (but for the time being, arbitrary) boundary values for v. The last step will be to patch the two parts together into a solution on all of $\underset{\sim}{R}$, by choosing the boundary values appropriately.

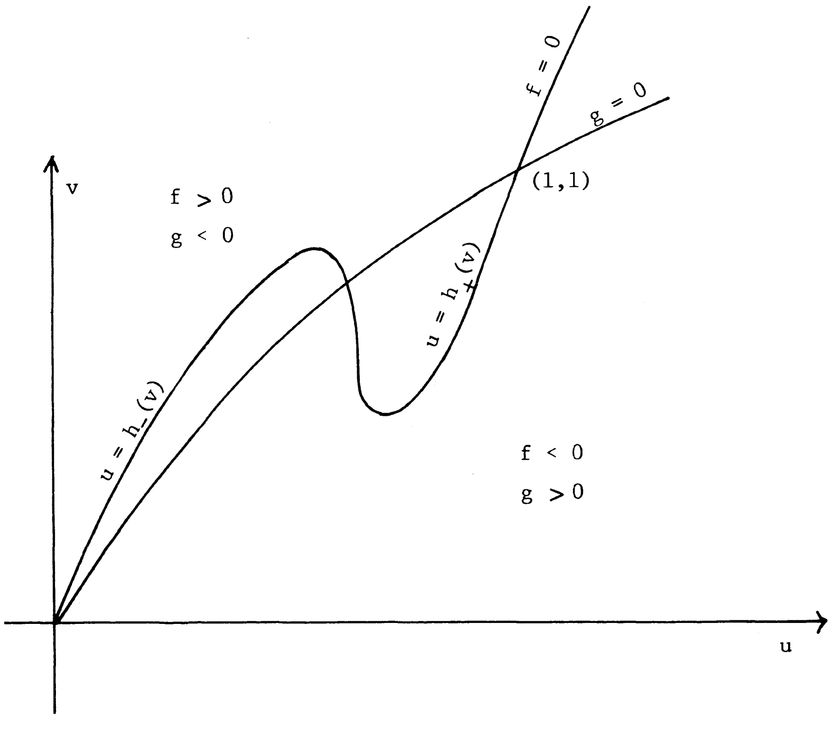

Figure 2

Incidentally, in connection with the notion of a solution on all of $\underset{\sim}{R}$, there is one technical clarification to be made. The equation in question will often be of the form $y'' + f(y) = 0$ or of a similar nature, with f discontinuous at one value of y. Then we speak of a function $y(x)$, strictly increasing in x, as being a solution on all of $\underset{\sim}{R}$ if it is in $C^1(\underset{\sim}{R})$ and satisfies the differential equation except where f is discontinuous.

In addition to Assumptions A, each type of problem will require an additional assumption of a geometric nature on the functions f and g.

4. **Type I.** We make the extra

ASSUMPTION B: *There is a value v^* in the domains of both h_+ and h_- such that*

$$\int_0^{v^*} g(h_-(v), v)\, dv + \int_{v^*}^1 g(h_+(v), v)\, dv = 0.$$

It can be seen from Figure 2 that the first integral here is negative, and the second positive. So this is a rather unrestrictive assumption.

First we treat the example $\delta = \gamma = 1$; extension to the general case of Type I will follow. There are two natural length scales in our example problem, and accordingly we define the two independent variables

$$z = x - c\epsilon t;\ \zeta = z/\epsilon = \epsilon^{-1}(x - c\epsilon t). \tag{4.1}$$

For functions of z, the system (3.2) ($\delta = \gamma = 1$) becomes

$$c\epsilon u_z + \epsilon^2 u_{zz} + f(u, v) = 0, \tag{4.2a}$$

$$c\epsilon v_z + v_{zz} + g(u, v) = 0. \tag{4.2b}$$

Of course any solution remains a solution when the variable z is translated at will. We therefore normalize by specifying the value of $u(z, \epsilon)$ at $z = 0$:

$$u(0, \epsilon) = a, \tag{4.3}$$

where $a \in (0, 1)$ is a constant which will be specified later.

Part of the problem will be to determine c as an asymptotic expansion in powers of ϵ.

A. **Problems on $\underset{\sim}{R}^+$.** As a first step, we look at the equations (4.2) for $z \geqq 0$ only, and in addition to (4.3), impose a further boundary condition at $z = 0$. Specifically, let $\{\omega_n\}$ be a set of numbers, considered arbitrary for the moment, and $\omega(\epsilon)$ the formal expansion $\omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \cdots$. Then we require

$$v(0, \epsilon) = \omega(\epsilon). \tag{4.5}$$

The sense of this is that v will be constructed as a formal expansion in powers of ϵ. When $z = 0$, the two expansions (4.5) should coincide.

Also in this first step, we think of the numbers $\{c_n\}$ as being given, and $c(\epsilon) = c_0 + \epsilon c_1 + \cdots$. Of course in a later step, the two sets $\{\omega_n\}$ and $\{c_n\}$ will be chosen in a unique manner, in order to complete the asymptotic analysis.

The solution will be sought in the form

$$
\begin{aligned}
u(z,\epsilon) &= U(z,\epsilon) + W(\zeta,\epsilon), \\
v(z,\epsilon) &= V(z,\epsilon) + Y(\zeta,\epsilon),
\end{aligned} \tag{4.6}
$$

where U, W, V, and Y are formal expansions in powers of ϵ: $U = U_0(z) + \epsilon U_1(z) + \cdots$, etc. The functions W and Y will serve as "boundary layer corrections". In accordance with this role, we impose conditions on them at ∞:

$$W_n(\infty) = Y_n(\infty) = 0, \text{ for all } n. \tag{4.7a}$$

To be consistent with (4.3), (4.5), we also impose the following boundary conditions at $z = 0$:

$$U_0(0) + W_0(0) = a,\ U_0(\infty) = 1, \tag{4.7b}$$

$$U_n(0) + W_n(0) = 0,\ U_n(\infty) = 0,\ n \geqq 1, \tag{4.7c}$$

$$V_0(0) + Y_0(0) = \omega_0,\ V_0(\infty) = 1, \tag{4.7d}$$

$$V_n(0) + Y_n(0) = \omega_n,\ V_n(\infty) = 0,\ n \geqq 1. \tag{4.7e}$$

For the time being, we also place individual boundary conditions on the terms V_n:

$$V_n(0) = \hat{\omega}_n, \tag{4.8}$$

where the $\hat{\omega}_n$ are related to the ω_n, and will be specified later.

The outer expansion. This expansion is defined to consist of U and V. To obtain its terms, we substitute $u = U(z,\epsilon)$, $v = V(z,\epsilon)$ into (4.2), obtaining the system

$$
\begin{aligned}
c\epsilon U_z + \epsilon^2 U_{zz} + f(U,V) &= 0, \\
c\epsilon V_z + V_{zz} + g(U,V) &= 0,
\end{aligned} \tag{4.9}
$$

which we rewrite in operator form as

$$N[U, V, c, \epsilon] = 0. \tag{4.10}$$

To obtain equations for the individual terms U_n, V_n, we think of U, V, and c as functions of ϵ, and require

(4.11) $$\left(\frac{\partial}{\partial \epsilon}\right)^k N[U(z,\epsilon), V(z,\epsilon), c(\epsilon), \epsilon]_{|\epsilon=0} = 0, k = 0, 1, \cdots.$$

[The logic behind this approach is as follows: for any practical purpose, the formal expansion would have to be truncated, leaving a polynomial in ϵ of some order m. Since it is too much to expect (4.10) to be satisfied exactly when the dependence of U, V, and c on ϵ is restricted to be polynomial, we require the next best thing, namely that the left side be as small as possible for small ϵ. This is accomplished by making as many as possible of its Taylor coefficients at $\epsilon = 0$ vanish. Hence (4.11).]

We write (4.11) for the various values of k separately:

$k = 0$:

$$f(U_0, V_0) = 0,$$
$$V_0'' + g(U_0, V_0) = 0.$$

Thus we may set

(4.12) $$U_0 = h_+(V_0);$$

this way the boundary conditions for U_0 and V_0 at $+\infty$ will be consistent. The problem for V_0 now becomes

(4.13) $$V_0'' + g(h_+(V_0), V_0) = 0, V_0(0) = \hat{\omega}_0, V_0(\infty) = 1.$$

This problem can be analyzed by standard methods, such as phase plane arguments. Using Assumptions A, it is seen to have a solution for any $\hat{\omega}_0 \leqq 1$ in the domain of h_+.

$k = 1$:

(4.14a) $$f_u(U_0, V_0)U_1 + f_v(U_0, V_0)V_1 + c_0U_0' = 0,$$
$$V_1'' + g_u(U_0, V_0)U_1 + g_v(\cdots)V_1 + c_0V_0' = 0;$$

(4.14b) $$V_1(0) = \hat{\omega}_1, \quad V_1(\infty) = 0.$$

Because of (3.3), we may solve the first equation for U_1, and substitute it into the second, to obtain

(4.15) $$V_1'' + H(z)V_1 = K_1(z), V_1(0) = \hat{\omega}_1, V_1(\infty) = 0,$$

where

$$H = g_v(U_0, V_0) - (f_v g_u / f_u),$$

and

$$K_1 = c_0((g_u U_0' / f_u) - V_0').$$

To show that (4.15) may be solved for V_1, we investigate the spectrum of the operator L_1, defined by

$$L_1 V \equiv V'' + HV,\ V(0) = 0,\ V \in \mathcal{L}_2(0, \infty],$$

considered as a self-adjoint operator in the space $\mathcal{L}_2(0, \infty)$. The continuous part of the spectrum [3, Thms. XIII.7.53–54] is confined to the portion of the real line below

$$\overline{\lim_{z\to\infty}} H(z) = H(\infty) = g_v(1, 1) - \frac{f_v(1, 1)g_u(1, 1)}{f_u(1, 1)}$$
$$= [f_u g_v - f_v g_u]/f_u(1, 1).$$

Here the denominator is negative and the numerator is positive, by Assumptions A. Hence $H(\infty) < 0$, and the continuous spectrum is bounded away from 0.

The point spectrum is also so bounded. In fact, let λ_0 be the largest eigenvalue, and $\psi(z) \geqq 0$ the corresponding (simple) eigenfunction:

$$L_1\psi = \psi'' + H\psi = \lambda_0\psi,\ \int_0^\infty \psi^2\,dz = 1,\ \psi(0) = 0.$$

Also let $\phi(z) = V_0'(z) > 0$. Differentiating (4.13) with respect to z, we obtain that $L_1\phi = 0$, and that ϕ decays exponentially. So now

$$0 = \int_0^\infty \psi L_1\phi\,dz = \int_0^\infty \phi L_1\psi\,dz + [\psi\phi' - \psi'\phi]_0^\infty$$

$$= \lambda_0 \int_0^\infty \phi\psi\,dz + \psi'(0)\phi(0).$$

But $\psi'(0) > 0$ (if $\psi'(0) = 0$, necessarily $\psi \equiv 0$), $\phi(0) = V_0'(0) > 0$ (by phase plane analysis of (4.13)), and $\int_0^\infty \phi\psi\,dz > 0$. Therefore $\lambda_0 < 0$, which shows the entire spectrum of L_1 is bounded away from 0. Since $K_1 \in \mathcal{L}_2(0, \infty)$, we may therefore solve (4.15) uniquely for $V_1(z)$.

We shall need the following representation of V_1. Let V_1^0 be the solution of (4.15) with $\hat{\omega}_1$ replaced by 0. Then we have

$$V_1(z) = V_1^0(z) + \frac{\hat{\omega}_1}{\phi(0)}\phi(z). \tag{4.16}$$

Clearly the right hand side satisfies (4.15), so by uniqueness it equals V_1.

Knowing V_1, we obtain U_1 from (4.14a).

Continuing in this way, all the terms U_n and V_n may be found.

The inner expansion. We have obtained a formal solution of the equations, but U and V do not satisfy the proper boundary conditions at $z = 0$. For this reason, we introduce the corrections W and Y, as indicated in (4.6). In (4.6) we replace z on the right by $\epsilon\zeta$ and substitute the resulting functions of ζ and ϵ into (4.2). In this, U and V are presumed known; one obtains a pair of equations for the unknown functions W and Y. These equations can be written, as before, in operator notation as

$$\hat{N}[W, Y, c, \epsilon] = 0, \tag{4.17}$$

where $\hat{N}$ is a nonlinear differential operator in the variable ζ. Of course, strictly speaking we cannot do this, since U and V at this stage are really only formal, divergent power series in ϵ. what we rather imagine is that we have obtained this expansion up to terms of the order ϵ^m, say, and put in this finite expansion for U and V. Then $\hat{N}$ really depends on m too. However, if we now expand $\hat{N}$ in powers of ϵ, we can see that the first m_0 terms of it are independent of m, so long as $m \geqq m_0$. So as long as we only look at these terms, the equations we get for the determination of the W_n and Y_n will be independent of m, and therefore well-defined.

So as before, we proceed by attempting to set

$$\left(\frac{\partial}{\partial\epsilon}\right)^k \hat{N}[W(\zeta,\epsilon), Y(\zeta,\epsilon), c(\epsilon), \epsilon]_{|\epsilon=0} = 0, k = 0, 1, \cdots. \tag{4.18}$$

But this will not immediately work this time, because $\hat{N}$ is formally of order ϵ^{-2} in its second component. In fact, this second component is

$$\epsilon^{-2}Y_{\zeta\zeta} + c(\epsilon)Y_\zeta + g(U + W, V + Y) - g(U, V)$$

or more explicitly,

$$\epsilon^{-2}Y_0'' + \epsilon^{-1}Y_1'' + Y_2'' + c_0Y_0' + g(U_0(0) + W_0, V_0(0) + Y_0)$$
$$- g(U_0(0), V_0(0)) + O(\epsilon).$$

Therefore, to annihilate the terms of orders ϵ^{-2} and ϵ^{-1}, we must require $Y_0'' = Y_1'' = 0$. From boundary conditions (4.7a) with $n = 0$, 1, we conclude that

$$Y_0 \equiv Y_1 \equiv 0. \tag{4.19}$$

With this, $\hat{N}$ is assured of being regular in ϵ at $\epsilon = 0$, so we may return to (4.18). We start with the first component of (4.18), and examine it for each k in order.

$k = 0$:

$$\text{(4.20)}\qquad \begin{aligned} &W_0'' + c_0 W_0' + f(U_0(0) + W_0, V_0(0)) = 0,\\ &W_0(0) = a - U_0(0) = a - h_+(\hat\omega_0);\ W_0(\infty) = 0. \end{aligned}$$

Let us defer this problem, assuming for the moment that it has a solution.

$k = 1$:

$$\text{(4.21a)}\qquad \begin{aligned} L_2 W_1 \equiv W_1'' + c_0 W_1' + f_u(h_+(\hat\omega_0) + W_0, \hat\omega_0) W_1 \\ = \tau_1(\zeta) - c_1 W_0'(\zeta), \end{aligned}$$

$$\text{(4.21b)}\qquad W_1(0) = -U_1(0),\ W_1(\infty) = 0,$$

where

$$\tau_1(\zeta) = -f_u(h_+(\hat\omega_0) + W_0(\zeta), \hat\omega_0)\zeta U_0'(0) - f_v \zeta V_0'(0).$$

We write this as

$$L_3 W_1 = e^{c_0\zeta}\tau_1 - c_1 e^{c_0\zeta} W_0',$$

where

$$L_3\phi \equiv \frac{d}{d\zeta}\left(e^{c_0\zeta}\frac{d\phi}{d\zeta}\right) + f_u(\cdots)e\quad \phi.$$

It can be shown, by the method used previously, that 0 is not an eigenvalue of L_3, and in fact its spectrum is bounded away from 0. Therefore L_2^{-1} exists on $\mathcal{L}_2^{(c_0)}(0, \infty)$, which denotes the $\mathcal{L}_2$-space weighted by the function $e^{c_0\zeta}$. But it can also be shown that the right hand side of (4.21a) is in this space. This is a routine calculation, based on the known rate of decay of $W_0(\zeta)$ and its derivatives as $\zeta \to \infty$, and on the identity obtained by differentiating the equation $f(h_+(v), v) = 0$ with respect to v.

Therefore (4.21) can be solved uniquely for W_1 (see [4, Sec. 4] for a similar argument). For future reference, we write the solution in the form

$$\text{(4.22)}\qquad W_1(\zeta) = W_1^0(\zeta) + c_1 Q(\zeta),$$

where

$$\text{(4.23)}\qquad \begin{aligned} &L_2 W_1^0 = \tau_1,\ W_1^0(0) = -U_1(0),\ W_1^0(\infty) = 0,\\ &L_2 Q = -W_0'(\zeta),\ Q(0) = Q(\infty) = 0. \end{aligned}$$

We shall need a certain property of Q later.

The remaining terms W_n for $n \geqq 2$ are found similarly. This completes the determination of W. We now take up the correction terms

Y_n, for $n \geqq 2$. Consider the second component of (4.18) for $k = 0$. Recalling that $Y_0 \equiv Y_1 \equiv 0$, we have

$$Y_2'' + g(h_+(\hat{\omega}_0) + W_0(\zeta), \hat{\omega}_0) - g(h_+(\hat{\omega}_0), \hat{\omega}_0) = 0; \tag{4.24}$$

and from (4.7e),

$$Y_2(0) = \omega_2 - V_2(0) = \omega_2 - \hat{\omega}_2.$$

But (4.24) has exactly one bounded solution. To see this, let us write (4.24) as

$$Y_2''(\zeta) = r(\zeta),$$

where r decays exponentially as $\zeta \to \infty$. Thus

$$Y_2(\zeta) = \int_\zeta^\infty \left(\int_{\zeta'}^\infty r(\zeta'')\, d\zeta'' \right) d\zeta'.$$

In particular, $Y_2(0)$ is determined uniquely. Hence $\hat{\omega}_2$ is also determined uniquely by

$$\hat{\omega}_2 = \omega_2 - Y_2(0).$$

In the same way, all the other terms Y_n, and as a matter of fact all the other $\hat{\omega}_n$, are determined (in particular we have $\hat{\omega}_0 = \omega_0$, $\hat{\omega}_1 = \omega_1$).

In summary, given arbitrary ω_n, c_n, the expansions U, W, V, Y are uniquely determined, subject only to the solvability of (4.20), to which we return in a moment.

B. **The problem on $\underset{\sim}{R}^-$ and the patching process.** With the same given parameters ω_n, c_n, we may set up and solve the analogous problems on $\underset{\sim}{R}^-$. To distinguish the two expansions, we use the superscripts "+" and "−". Thus, for example, it turns out that

$$U_0^{\pm}(z) = h_{\pm}(V_0^{\pm}(z)).$$

By construction, the two formal solutions (one defined for $z \leqq 0$, the other for $z \geqq 0$) coincide at $z = 0$:

$$U_n^-(0) + W_n^-(0) = U_n^+(0) + W_n^+(0),$$
$$V_n^-(0) + Y_n^-(0) = V_n^+(0) + Y_n^+(0).$$

If, in addition, we can match the derivatives:

$$W_0^{-\prime}(0) = W_0^{+\prime}(0), \tag{4.25}$$

$$U_n^{-\prime}(0) + W_{n+1}^{-\prime}(0) = U_n^{+\prime}(0) + W_{n+1}^{+\prime}(0), \tag{4.26}$$

$$V_n^{-\prime}(0) + Y_{n+1}^{-\prime}(0) = V_n^{+\prime}(0) + Y_{n+1}^{+\prime}(0), \tag{4.27}$$

then the patched expansion will be formal solution on all of $(-\infty, \infty)$. Our object, then is to arrange for (4.25–27) by an appropriate choice of the c_n and ω_n.

Determination of ω_0. We recall (4.13) the following equations satisfied by $V_0{}^{\pm}(z)$:

$$V_0{}^{\pm\prime\prime} + g(h_{\pm}(V_0{}^{\pm}), V_0{}^{\pm}) = 0, z \gtrless 0;$$
$$V_0{}^{+}(0) = V_0{}^{-}(0) = \omega_0; V_0{}^{-}(-\infty) = 0, V_0{}^{+}(\infty) = 1.$$

Since the function g appearing here is positive or negative according as the "+" or "−" sign is chosen, we see that the functions $V_0{}^{\pm}$ are monotone increasing in z. Therefore if we set

$$\bar{V}_0 \equiv \begin{cases} V_0{}^{-} & \text{for } z \leqq 0 \\ V_0{}^{+} & \text{for } z \geqq 0, \end{cases}$$

we have $\bar{V}_0 \gtrless \omega_0$ according as $z \gtrless 0$. The equations now become

$$\text{(4.28)} \qquad \begin{aligned} &\bar{V}_0{}'' + G(\bar{V}_0) = 0, z \neq 0, \\ &\bar{V}_0(-\infty) = 0, \bar{V}_0(\infty) = 1, \end{aligned}$$

where $G(v) = g(h_{\pm}(v), v)$ for $v \gtrless \omega_0$. We seek to determine ω_0 so that $\bar{V}_0{}'$ is continuous at $z = 0$. Despite the discontinuity in $G(v)$ at $v = \omega_0$, the problem (4.28) will have a solution with continuous derivative if $\int_0^1 G(v)\, dv = 0$. (This is easily seen by multiplying (4.28) by $V_0{}'$ and integrating with respect to z.) But our Assumption B states that this is indeed true for

$$\omega_0 = v^*.$$

With this choice, we have been able to patch V_0.

Having determined ω_0, we are in a position to specify the constant a appearing in (4.3). We choose it to be (1/2) $(h_{-}(\omega_0) + h_{+}(\omega_0))$.

Determination of c_0. Recall the equations for $W_0{}^{\pm}$:

$$W_0{}^{\pm\prime\prime} + c_0 W_0{}^{\pm\prime} + f(U_0{}^{\pm}(0) + W_0{}^{\pm}(\zeta), \omega_0) = 0,$$
$$W_0{}^{\pm}(0) = a - U_0{}^{\pm}(0), W_0{}^{-}(-\infty) = W_0{}^{+}(\infty) = 0.$$

Now define

$$\bar{W}(\zeta) = \begin{cases} U_0{}^{-}(0) + W_0{}^{-}(\zeta), \zeta \leqq 0 \\ U_0{}^{+}(0) + W_0{}^{+}(\zeta), \zeta \geqq 0. \end{cases}$$

Then

$$\text{(4.29a)} \qquad \overline{W}'' + c_0 \overline{W}' + f(\overline{W}, \omega_0) = 0, \zeta \neq 0,$$

$$\text{(4.29b)} \quad \overline{W}(-\infty) = U_0{}^{-}(0) = h_{-}(\omega_0); \overline{W}(\infty) = U_0{}^{+}(0) = h_{+}(\omega_0);$$

and $\overline{W}$ is continuous at $\zeta = 0$, assuming the value $(1/2)(h_-(\omega_0) + h_+(\omega_0))$ there.

But (4.29) (ignoring the stipulation $\zeta \neq 0$) is precisely the problem of determining a wave front solution of the scalar problem treated in Sec. 2 of this paper. In fact, $f(W, \omega_0)$ has two zeros, at $W = h_-(\omega_0)$ and $W = h_+(\omega_0)$; and $f_W < 0$ at each of them (there is, of course, a third unstable zero between them). By the well-known existence results stated there, there exists a unique c_0 for which a solution of (4.29) exists for all ζ, *including* $\zeta = 0$. This is the value of c_0 for which (4.25) holds. Incidentally, this settles the question of the existence of a solution of (4.20), which was left hanging.

Determination of c_1. Condition (4.26) with $n = 0$ will be satisfied with proper choice of c_1. To see this, we rewrite it, using (4.22), in the form

$$c_1[Q^{+\prime}(0) - Q^{-\prime}(0)] = U_0^{-\prime}(0) - U_0^{+\prime}(0) + W_1^{0-\prime}(0) - W_1^{0+\prime}(0).$$

So for the determination of c_1, we need only check that

$$Q^{+\prime}(0) \neq Q^{-\prime}(0).$$

Suppose this were not true, and the derivatives of $Q^{\pm}$ coincided at $\zeta = 0$. Then if we define a function

$$\bar{Q}(\zeta) = \begin{cases} Q^-(\zeta), & \zeta \leqq 0, \\ Q^+(\zeta), & \zeta \geqq 0, \end{cases}$$

it would have continuous derivatives, and we see from (4.23) that it would be a solution of

$$L_2\bar{Q} = -\overline{W}_0{}'(z)$$

on the whole line.

But we also see, from differentiating (4.20), that

$$L_2(\overline{W}_0{}') = 0, \overline{W}_0{}'(-\infty) = \overline{W}_0{}'(\infty) = 0.$$

Going over to the self-adjoint operator L_3, we now have

$$L_3\bar{Q} = -\overline{W}_0{}'e^{c_0\zeta}, L_3\overline{W}_0{}' = 0.$$

Therefore

$$0 = \int_{-\infty}^{\infty} \bar{Q}L_3\overline{W}_0{}'\, d\zeta = \int_{-\infty}^{\infty} \overline{W}_0{}'L_3\bar{Q}\, d\zeta = -\int_{-\infty}^{\infty} e^{c_0\zeta}(\overline{W}_0{}')^2\, d\zeta.$$

(It can be checked that $\bar{Q}$ and $\overline{W}_0{}'$ decay at ∞ fast enough for these integrals to exist.) This implies $W_0{}^{\pm\prime} \equiv 0$, hence $W_0{}^{\pm} \equiv 0$. This in turn implies that

$$a - h_-(\omega_0) = W_0^-(0) = 0 = W_0^+(0) = a - h_+(\omega_0),$$

so that $h_-(\omega_0) = h_+(\omega_0)$. This contradicts Assumptions A. Therefore c_1 is uniquely determined by condition (4.26), $n = 0$.

Determination of ω_1. This is done by using (4.27) with $n = 1$:

$$V_1^{+\prime}(0) - V_1^{-\prime}(0) = Y_2^{-\prime}(0) - Y_2^{+\prime}(0). \tag{4.30}$$

We rewrite (4.16) as

$$V_1^{\pm}(z) = V_1^{0\pm}(z) + \frac{\hat{\omega}_1}{V_0'(0)} V_0^{\pm\prime}(z).$$

Differentiating this and setting $\Delta V_1 = V_1^+ - V_1^-$, etc., we obtain

$$\Delta V_1'(0) = \Delta V_1^{0\prime}(0) + \frac{\hat{\omega}_1}{V_0'(0)} \Delta V_0''(0).$$

It is therefore clear that (4.30) can be satisfied with proper choice of ω_1, provided $\Delta V_0''(0) \neq 0$. By (4.13), this condition reduces to

$$g(h_+(\omega_0), \omega_0) \neq g(h_-(\omega_0), \omega_0).$$

But this inequality is true. In fact, Assumptions A imply the left side is positive and the right negative. In this way, ω_1 is determined.

The remaining parameters. The process of obtaining the c_n and ω_n for $n > 1$ involves no new ideas, and results in the satisfaction of all the conditions (4.25–27).

C. **Generalizations.** The foregoing has been a detailed analysis of a particular case: $\delta = \gamma = 1$. We shall now indicate how the same technique can be applied to the general problem of Type I.

In place of (4.1), we define

$$z = \gamma^{-1/2}(x - c\epsilon\delta^{-1}t); \; \zeta = z/\rho = \epsilon^{-1}(x - c\epsilon\delta^{-1}t).$$

As before, we split the problem into a part on $\underset{\sim}{R}^+$ and one on $\underset{\sim}{R}^-$. On $\underset{\sim}{R}^+$, for example, the solution will be sought in the form

$$u = U(z; \rho,\sigma) + W(\zeta; \rho, \sigma),$$
$$v = V(z; \rho,\sigma) + Y(\zeta; \rho, \sigma),$$

where U, W, V, and Y are formal series in both ρ and σ. Thus

$$U = U_{00}(z) + \rho U_{10}(z) + \sigma U_{01}(z) + \rho\sigma U_{11}(z) + \cdots.$$

The data $c(\rho, \sigma)$, $\omega(\rho, \sigma)$ for the problem on $\underset{\sim}{R}^+$ are given as similar expansions in the two parameters (but with constant coefficients).

The process of determining the various terms is as before: First the

U_{nm}, V_{nm} are determined; then the W_{nm} and Y_{nm}. The same is done for the problem on $\underset{\sim}{R}^-$. Finally, the c_{nm} and ω_{nm} are chosen for patching purposes.

5. **Type II.** Under Assumptions A and B, and if the three branches of the nullcurve $f = 0$ are (as in Fig. 2) all contained in the unit square $u \in [0, 1]$, $v \in [0, 1]$, then an asymptotic expansion of the solution can be achieved in this case also. Weaker assumptions will in fact be given later. The expansion will take the form

$$u = U(z, \delta) + W(\zeta, \delta),$$
$$v = V(z, \delta) + Y(\zeta, \delta),$$

where U, W, V, Y are power series in δ, and outer and inner variables are

$$z = \gamma^{-1/2}(x - c\epsilon\delta^{-1}t), \zeta = z/\delta.$$

As an example, we pursue the case $\delta = \epsilon$, $\gamma = 1$, so that ϵ is the only independent parameter. We take the same approach as in Sec. 4, first seeking a solution on $\underset{\sim}{R}^+$ in the form (4.6) satisfying (4.7), where $z = x - ct$, $\zeta = z/\epsilon$. In place of (4.9), we have a similar system with the coefficient ϵ in the second equation replaced by 1. As a result, (4.13) is replaced by

$$V_0'' + c_0V_0' + g(h_+(V_0)V_0) = 0, V_0(0) = \hat{\omega}_0, V_0(\infty) = 1.$$

The next change is in (4.14b), where the coefficient c_0 will be replaced by c_1. But it can still be solved.

Regarding the inner expansion, there are no essential changes needed.

We need to take a close look at the patching process, however. Equation (4.28) is replaced by

$$(5.1) \qquad \bar{V}_0'' + c_0\bar{V}_0' + G_{(\omega_0)}(\bar{V}_0) = 0; \bar{V}_0(-\infty) = 0, \bar{V}_0(\infty) = 1.$$

Let $I \subset (0, 1)$ be the interior of the intersection of the domains of h_+ and h_-. Then by the theory of Sec. 2, each $\omega_0 \in I$ determines a unique $c_0 = p(\omega_0)$ for which (5.1) has a solution, with $\bar{V}_0'$ continuous when $\bar{V}_0 = \omega_0$. The phase plane analysis of the problem reveals the effect on c_0 of varying G. In fact if G is not increased then c_0 is not decreased. But $G_{(\omega_0)}(V)$ is by its definition a nonincreasing function of ω_0. Hence the function p is nondecreasing. Furthermore by Assumption B, it assumes the value 0 for some $\omega_0 \in I$.

But the solvability condition for (4.29) provides a second relation between c_0 and ω_0. Again, for each $\omega_0 \in I$, a unique $c_0 = q(\omega_0)$ is determined. We shall show that the pair of equations

$$c_0 = p(\omega_0), c_0 = q(\omega_0)$$

has a solution (c_0, ω_0). Let $I = (a, b)(a \geqq 0$ is the least value of v for which $h_+(v)$ is defined, and $b \leqq 1$ is the greatest value for which h_- is defined). For ω_0 near a,

$$\int_{h_-(\omega_0)}^{h_+(\omega_0)} f(u, \omega_0)\, du > 0,$$

so that by (2.2), $c_0 = q(\omega_0) < 0$. On the other hand for ω_0 near b, $q(\omega_0) > 0$. Thus $q(a) \leqq 0 \leqq p(a)$, $q(b) \geqq 0 \geqq p(b)$. Hence there is an intermediate value ω_0 for which $q(\omega_0) = p(\omega_0)$. Hence there exists at least one solution (c_0, ω_0).

In determining c_0 and ω_0, the crucial assumptions were that the nullcurves are as depicted in Fig. 2, and Assumptions A and B. But in actuality, this is too restrictive. A sufficient condition would simply be that the equation $p(\omega) = q(\omega)$ has a solution in I, together with Assumptions A.

The determination of the c_n and ω_n for $n \geqq 1$ proceeds as before, with only minor changes.

6. **Type** III. This case is different in several important respects from the preceding two. An indication of this is seen by the following assumption, which is to replace Assumption B:

Assumption C: Either h_+ is defined for $v = 0$ and $\int_0^{h_+(0)} f(u, 0)\, du > 0$, or h_- is defined for $v = 1$, and $\int_{h_-(1)}^{1} f(u, 1)\, du < 0$.

For definiteness, we assume throughout that the first alternative holds. The changes to be made in the case of the second are rather clear. So we are supposing the null curves have the general appearance shown in Fig. 3. The wave fronts we construct will, to first approximation, be constant, with $u \approx v \approx 0$, to the left of the sharp front, but not constant to the right. The image in the $u - v$ plane will be shown in Fig. 3. The velocity will be negative, so that the front penetrates the uniform state.

The following parameters are relevant in problems of Type III:

$$\mu = \sigma^{-2} = \gamma\delta^2\epsilon^{-2} \ll 1; \nu = \delta\mu^{-1} = \gamma^{-1}\delta^{-1}\epsilon^2. \tag{6.1}$$

Type III can profitably be further broken down into the cases

(a) $\nu \ll 1$

and

(b) $\nu = O(1)$ or $\nu \gg 1$.

In case (a), there will be three length scales rather than the usual two.

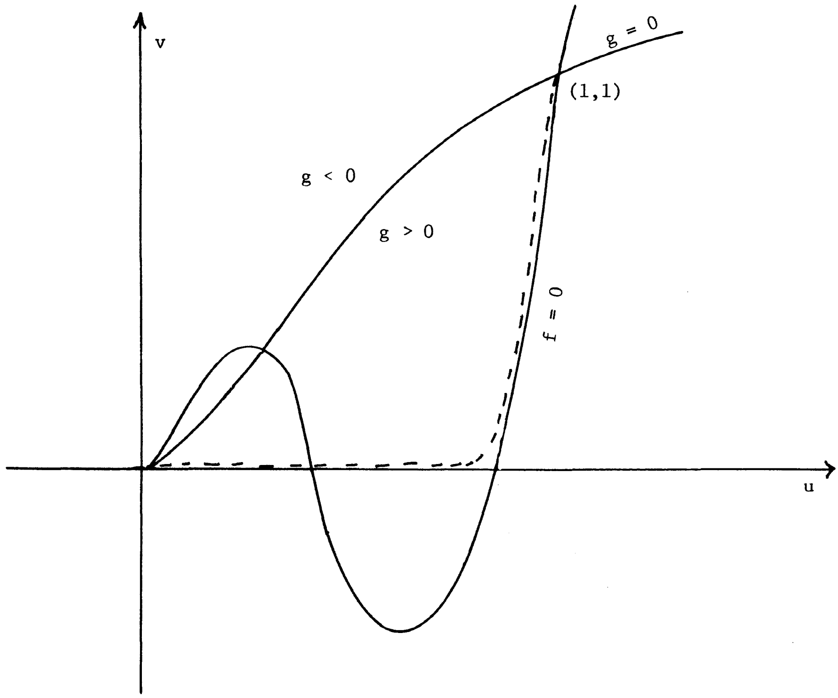

Figure 3
The dotted line shows the $u - v$ image of a wave front of Type III.

A. **Illustration.** By way of example, we take $\delta = \epsilon^2$, $\gamma = \epsilon^{-1}$, so that $\mu = \nu = \epsilon$. We define the three variables

$$z = \epsilon(x - c\epsilon^{-1}t) \qquad \text{(long range)}$$

$$\eta = z/\epsilon = x - c\epsilon^{-1}t, \qquad \text{(intermediate range)}$$

$$\zeta = z/\epsilon^2 = \eta/\epsilon. \qquad \text{(short range)}$$

On $\underset{\sim}{R}^+$, we seek a solution in the form

$$u = U(z, \epsilon) + Q(\eta, \epsilon) + W(\zeta, \epsilon),$$

$$v = V(z, \epsilon) + P(\eta, \epsilon) + Y(\zeta, \epsilon),$$

where the six functions are formal power series in ϵ. As before $c(\epsilon)$ and $\omega(\epsilon) = v(0, \epsilon)$ are at first prescribed.

The first step is to determine the outer expansion terms U_n and V_n formally as before, requiring $V_n(0) = \hat{\omega}_n$. For the lowest approximation, we obtain

$$f(U_0, V_0) = 0,$$
$$c_0 V_0{}' + g(U_0, V_0) = 0.$$

The main differences in Type III stem from the fact that the second equation is first order, rather than second order, as was the analogous equation in Types I and II. Hence $U_0(z) = h_+(V_0(z))$,

(6.2) $$c_0 V_0{}' + g(h_+(V_0), V_0) = 0, \;\; V_0(\infty) = 1, V_0(0) = \hat{\omega}_0.$$

Jumping ahead a bit, we note that we shall obtain an analogous equation for V_0^- on $\underset{\sim}{R}^-$:

$$c_0 V_0{}' + g(h_-(V_0), V_0) = 0, V_0(-\infty) = 0, V_0(0) = \hat{\omega}_0.$$

But notice that $g(h_+(v), v) \geqq 0$ for $v \leqq 1$, whereas $g(h_-(v), v) \leqq 0$ for $v \geqq 0$. So if V_0 takes on values in the interval $[0, 1]$ only, $V_0{}'$ must change sign as z passes from negative to positive, *unless* $V_0 \equiv 0$ on the left or $V_0 \equiv 1$ on the right. In fact, this will be the only way that V_0 can satisfy the boundary conditions at $z = \pm\infty$, even if we allow V_0 to take values outside the interval $[0, 1]$. So V_0 is constant on $\underset{\sim}{R}^-$ or $\underset{\sim}{R}^+$.

According to Assumption C, $h_+(0)$ is defined, and we accordingly set

$$U_0^-(z) \equiv V_0^-(z) \equiv 0, z \leqq 0.$$

Hence $\hat{\omega}_0 = 0$. For $z \geqq 0$, we require (6.2) with $\hat{\omega}_0 = 0$.

Notice that for $V_0{}'$ to be positive (as it must be) on $\underset{\sim}{R}^+$, the coefficient c_0 in (6.2) must be negative. So from this point on, we require that

$$c_0 < 0.$$

Later, when c_0 is actually determined, its negativity will be verified.

This determines V_0. Proceeding in an orderly fashion, we encounter no difficulties in obtaining all the other terms $U_n^\pm$, $V_n^\pm$. It turns out that $U_n^- \equiv V_n^- \equiv 0$, so that only boundary layer corrections appear for $z < 0$.

Next, we construct the intermediate expansion by setting $u \sim U(\epsilon\eta, \epsilon) + Q(\eta, \epsilon)$, $v \sim V(\epsilon\eta, \epsilon) + P(\eta, \epsilon)$, with $Q(\infty, \epsilon) = P(\infty, \epsilon) = 0$. To lowest order, we have

(6.3a) $$f(U_0(0) + Q_0(\eta), V_0(0) + P_0(\eta)) = 0,$$

(6.3b) $$c_0 P_0{}' + P_0{}'' = 0.$$

Since $c_0 < 0$, this latter equation has no bounded solution on $\underset{\sim}{R}^+$

except constants, and since $P_0(\infty) = 0$, we conclude $P_0{}^+ \equiv 0$. Hence (6.3a) becomes

$$f(h_+(0) + Q_0(\eta), 0) = 0, Q_0(\infty) = 0.$$

The continuity of $Q_0{}^+(\eta)$ then implies $Q_0{}^+ \equiv 0$.

By similar reasoning, we conclude that all of the P_n and Q_n vanish identically on $\underset{\sim}{R}^+$. The situation is quite different on $\underset{\sim}{R}^-$. There do exist nontrivial solutions satisfying $P_0{}^-(-\infty) = 0$, and in fact we may prescribe $P_n{}^-(0) = \beta_n \geqq 0$ arbitrarily. The terms $Q_n{}^-(\eta)$ are then determined uniquely.

Up to this point, we have found the terms of the outer and intermediate expansions, with arbitrary boundary values $V_n{}^+(0) = \hat{\omega}_n$ and $P_n{}^-(0) = \beta_n$.

Next, we consider the inner expansion, by setting $u \sim U(\epsilon^2\zeta, \epsilon) + Q(\epsilon\zeta, \epsilon) + W(\zeta, \epsilon)$; $v \sim V(\epsilon^2\zeta, \epsilon) + P(\epsilon\zeta, \epsilon) + Y(\zeta, \epsilon)$. To lowest order for $W_0{}^+$, we have

$$c_0 W_0{}' + W_0{}'' + f(h_+(0) + W_0, 0) = 0, \zeta > 0.$$

The terms $W_n{}^+$, $Y_n{}^+$ are found exactly as in Sec. 4. The boundary conditions imposed on $W_n{}^+$ are (since $Q_n{}^+ \equiv 0$)

$$W_n{}^+(0) + U_n{}^+(0) = \begin{cases} a \equiv \dfrac{1}{2}h_+(0), & n = 0 \\ 0 \quad , & n > 0. \end{cases}$$

On $\underset{\sim}{R}^-$, the terms $W_n{}^-$, $Y_n{}^-$ are found in a straightforward manner. The boundary conditions are (since $U_n{}^- \equiv 0$)

$$W_n{}^-(0) + Q_n{}^-(0) = \begin{cases} a, & n = 0 \\ 0, & n > 0. \end{cases}$$

Of course, boundary conditions may not be imposed on the $Y_n{}^\pm$, for the same reason they could not for problems of Type I. So the $\hat{\omega}_n$, β_n are determined by (recall $P_n{}^+ = 0$, $V_n{}^- = 0$)

(6.4a) $$V_n{}^+(0) + Y_n{}^+(0) = \hat{\omega}_n + Y_n{}^+(0) = \omega_n,$$

(6.4b) $$P_n{}^-(0) + Y_n{}^-(0) = \beta_n + Y_n{}^-(0) = \omega_n.$$

Incidentally, the possibility of choosing β_n so these latter equations are satisfied is one reason for introducing the intermediate expansion in the first place. If there were no $P_n{}^-$'s, hence no β_n's, it would in general be impossible to patch at $\zeta = 0$.

We remark that $Y_0{}^\pm = 0$. Since we have seen that $\omega_0 = 0$, (6.4b) with $n = 0$ implies that $\beta_0 = P_0{}^- = Q_0{}^- = 0$. But in general $P_n{}^- \neq 0$ for $n > 0$.

The last step is to choose the terms c_n, ω_n, so that the derivatives match. Let

$$\overline{W}_0(\zeta) = \begin{cases} U_0{}^+(0) + W_0{}^+(\zeta), & \zeta > 0 \\ W_0{}^-(\zeta) & , \ \zeta < 0 \end{cases}$$

Then

$$c_0\overline{W}_0{}' + \overline{W}_0{}'' + f(\overline{W}_0, 0) = 0, \ \overline{W}_0(-\infty) = 0,$$

$$\overline{W}_0(\infty) = h_+(0), \overline{W}_0(0) = a = \frac{1}{2} h_+(0).$$

By Sec. 2, this has a unique solution, with $\overline{W}_0{}'$ continuous at $\zeta = 0$ with a unique value for c_0. By Assumption C and (2.2), we have $c_0 < 0$, which is in accord with our previous requirement (this, in fact, is the origin of Assumption C).

The other constants ω_n, c_n are determined in much the same way as before. We shall not give the details.

B. **Generalizations.** In case (a), when $\nu \ll 1$, we use the following variables:

$$z = \delta\epsilon^{-1}(x - c\epsilon\delta^{-1}t),$$

$$\eta = z/\mu,$$

$$\zeta = z/\delta = \eta/\nu.$$

We seek a solution in the form

$$u = U(z; \mu, \nu) + Q(\eta; \mu, \nu) + W(\zeta; \mu, \nu),$$

$$v = V(z; \mu, \nu) + P(\eta; \mu, \nu) + Y(\zeta; \mu, \nu),$$

where the six functions on the right are series in powers of the two parameters μ and ν. They are determined as in the above example: first on $\underset{\sim}{R}^+$, then on $\underset{\sim}{R}^-$, then patched.

In case (b), $\nu \geqq O(1)$. The intermediate variable η may be dispensed with. If $\nu \gg 1$, we seek a solution in the form

$$u = U\left(z; \delta, \frac{1}{\nu}\right) + W\left(\zeta; \delta, \frac{1}{\nu}\right),$$

$$v = V\left(z; \delta, \frac{1}{\nu}\right) + Y\left(\zeta; \delta, \frac{1}{\nu}\right),$$

expanding in powers of δ and $1/\nu$. Alternately, the expansion may be made in powers of δ alone, the coefficients then depending on $1/\nu$. This latter expansion then holds for $\nu = O(1)$ as well.

C. **The case $\gamma = 0$.** This case, in which v does not diffuse at all, can be considered the limit, in case (b), when $1/\nu \to 0$. Since the solutions are expressed in powers of $1/\nu$ and δ, there is no difficulty in carrying out this limit. Other limiting cases, such as $\delta \to 0$, may also be handled.

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