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ALMOST AUTOMORPHIC FUNCTIONS AND TOTALLY BOUNDED GROUPS

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Introduction. S. Bochner introduced the concept of an almost automorphic function on the real numbers in [4] and the first extensive study of such functions on a topological group G was made by Veech [23, 24, 25]. More recently, Terras [21, 22] and Reich [19], among others, have also made contributions. One of the most interesting theorems about almost automorphic functions asserts that they are just the bounded Bohr continuous functions. Since one can dispense with some of the continuity hypotheses involved in the term *topological group* and still make sense out of the terms *almost periodic function* and hence *Bohr topology,* one might expect that the theory of almost automorphic functions could be developed in a more general setting, that of semitopological groups. (One can not hope to dispense with the operation of inversion in the group, which *can* be done without affecting the development of the theory of the almost periodic functions, since the definition of an almost automorphic function involves inverse elements.) Terras obviously agreed that the theory of almost automorphic functions should be workable in some setting more general than that of topological groups; in [21], he proved the theorem mentioned above for semitopological groups with continuous inversion. In this paper, we go all the way and prove this theorem for general semitopological groups. In fact, in this setting we establish most of the known theory of almost automorphic functions and prove some results that are new even for topological groups.

The chief theorem in part I, which contains mainly preliminaries, is a theorem about representing continuous functions on topological groups as functions on metrizable quotient groups. This theorem is essential in proving some of the characterization and approximation theorems in part II, where, for example, we determine that a continuous function f on a semitopological group G is almost automorphic if and only if it satisfies the following condition: given any sequence $n' = \{n_i'\} \subset G$, $\exists$ a subsequence $n = \{n_i\}$ of $n' \ni$ the joint limit $\lim_{i,j} f(n_i n_j^{-1} t) = f(t)$, $\forall t \in G$. (We believe this result is new even for topological groups.) Veech's approximation theorem for almost automorphic functions [24, Theorem 7.1] also follows with

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little difficulty, as does the assertion that a continuous function f is almost automorphic if and only if it satisfies the condition: given $\epsilon > 0$ and $t \in G$, $\exists$ a left relatively dense subset $E \subset G \ni E^{-1}E \subset \{s \in G \mid |f(ts) - f(t)| < \epsilon\}$. This last condition is simpler (and weaker) than the one used by Reich [19].

In part III we discuss a number of topics relating to almost automorphic functions. In § 1 we discuss at length the existence of almost automorphic functions that are not almost periodic, making full use of the results of Comfort and Ross [5] concerning pseudocompact groups. Among other things, we prove that such functions fail to exist if and only if the canonical image of the group in its almost periodic compactification is pseudo-compact. And, if this is the case, every Bohr continuous function is also almost periodic, i.e., there are no unbounded Bohr continuous functions.

In III, § 2, we give examples of groups with subgroups on which there exist almost automorphic functions that cannot be extended to functions almost automorphic on the whole group. Perhaps the most startling example is of a locally compact group G with a closed, normal, abelian subgroup H such that the only almost automorphic functions on H that extend to functions almost automorphic on G are the constant functions.

The final section of the paper is centered around the question, whose answer we do not know: Do the almost automorphic functions always admit an invariant mean?

Part I.

1. **Almost automorphic functions.** A topological space G that is also a group is called a *semitopological group* if the maps $s \to st$ and $s \to ts$ from G into G are continuous $\forall t \in G$, i.e., multiplication is separately continuous. G is called a *topological group* if the map $(s, t) \to st^{-1}$ from $G \times G$ into G is continuous. Specific reference to the topology of a topological or semitopological group is generally omitted. Later we will want to discuss more than one topology on a given group space and will use the notation $(G, \mathcal{T})$ to indicate that the topology $\mathcal{T}$ is the one being discussed. We let $C(G)$ denote the C^*-algebra of all continuous, bounded, complex-valued functions on G and indicate the supremum norm by $\| \ \|$. The *left* (*right*) *translate* $f_s(f^s)$ of $f \in C(G)$ by $s \in G$ is defined at $t \in G$ by $f_s(t) = f(st)$ $f^s(t) = f(ts))$ and a subspace X of $C(S)$ is called *translation invariant* if f_s, $f^s \in X$ whenever $f \in X$, $s \in G$. We often use the following compact notation which was first used by Bochner [4]: a net (sequence) in G is written as a single symbol with subscripts

used to designate members of the net (sequence), for example, $\alpha = \{\alpha_i\}_{i \in I}$ is a net ($n = \{n_i\}_{i \in N}$ is a sequence); also, if f is a function defined on G and α is a net in G such that the net $\{f_{\alpha_i}\}$ of left translates of f converges pointwise on G to a function g, we indicate this fact by writing $T_\alpha f = g$. It follows from Tychonoff's theorem that if f is a bounded function on G and α' is a net in G, then $\exists$ a subnet α of α' and a function g on $G \ni T_\alpha f = g$. The *right uniformly continuous* subspace $\mathrm{RUC}(G) \subset C(G)$ is defined to be $\{f \in C(G) \mid \|f^{\alpha_i} - f^s\| \to 0$ whenever α is a net in G converging to $s \in G\}$. (The *left uniformly continuous* subspace is defined analogously using left translates.) In case G is a topological group, the assertion that $f \in \mathrm{RUC}(G)$ is equivalent to saying $f \in C(G)$ and $\forall \epsilon > 0$, $\exists$ a neighbourhood $V = V(\epsilon)$ of the identity $e \in G \ni |f(s) - f(t)| < \epsilon$ whenever $s^{-1}t \in V$.

Definition. (Bochner, Veech). A continuous complex-valued function f on a semitopological group G is called *left almost automorphic* if every net $\alpha' \subset G$ has a subnet $\alpha \ni T_\alpha f = g$ and $T_{\alpha^{-1}} g = f$. (Here $\alpha^{-1} = \{\alpha_i^{-1}\}$.) Let $\mathrm{LAA}(G)$ denote the family of all left almost automorphic functions on G. f is called *right almost automorphic* if it satisfies the analogous condition involving right translates.

A function $f \in C(G)$ is called *almost periodic* if $\{f_s \mid s \in G\}$ is relatively compact in $C(G)$, and the family of such functions is denoted by *AP(G)*. It follows from the fact that the functions in $\mathrm{AP}(G)$ can be represented on the almost periodic compactification G_a of G, which is a compact topological group (see [6, Theorem 4.6] and [18, Corollary 2], for example), that $\mathrm{AP}(G) \subset \mathrm{LAA}(G)$. Also, whenever the pointwise limit $T_\alpha f$ exists for $\alpha \subset G$ and $f \in \mathrm{AP}(G)$, it actually exists uniformly on G.

The following assertions are easy to prove directly from the definition above. Proofs have been given by Veech [23].

Theorem 1. *$\mathrm{LAA}(G)$ is a translation invariant C^*-subalgebra of $C(G)$.*

Lemma 2. *If $f \in \mathrm{LAA}(G)$ and α is a net in $G \ni T_\alpha f = g$, then $T_{\alpha^{-1}} g = f$.*

The next lemma also has an easy direct proof (as in [22, Proposition 2.9]).

Lemma 3. *If G_1 and G_2 are semitopological groups and ψ is a continuous homomorphism of G_1 into G_2, then the transpose map ψ^* maps $\mathrm{LAA}(G_2)$ into $\mathrm{LAA}(G_1)$.*

REMARK. Example 4 of III, §2, shows how far ψ^* can be from mapping $\mathrm{LAA}(G_2)$ *onto* $\mathrm{LAA}(G_1)$, even if ψ is also a homeomorphism. (If $H = G_1$, $G = G_2$ and ψ is the identity injection, then $\psi^*(\mathrm{LAA}(G_2))$ consists only of constant functions.)

Terras [22, p. 760] has remarked that, for $f \in \mathrm{LAA}(G)$, the pointwise limit $T_\alpha f = g$ can fail to be continuous. If G is a Hausdorff topological group that is complete in a left invariant metric or locally compact and $f \in C(G)$, it follows from Theorem 3.1 of [15] and §5(d) and Theorem 7 of [16] that $\exists$ a net $\alpha \subset G$ for which $T_\alpha f$ is discontinuous if and only if $f \notin \mathrm{RUC}(G)$. Hence $\mathrm{LAA}(G)/\mathrm{RUC}(G)$ is not always void. In fact many of the interesting examples of left almost automorphic functions, namely the ones that are not almost periodic, are also not uniformly continuous (see the example following Theorem 17 and III, § 1, ahead).

2. Totally bounded topological groups.

DEFINITION. A topological group G is called *totally bounded* if, given any non-void neighbourhood $V \subset G$, $\exists$ a finite number of elements $\{s_k\}_{k=1}^N \subset G \ni G = \bigcup_1^N s_k V$.

Terras [21, Lemma 3.2.2] and Landstad [12, Lemma 1] have proved that the left and right uniformities of a totally bounded topological group are equal. This can also be proved using the facts that every continuous homorphism of topological groups is left (and right) uniformly continuous and that the left and right uniformities of a compact Hausdorff topological group are equal, and the following two theorems which we will need again later.

THEOREM 4 (WEIL [26]). *Every Hausdorff totally bounded topological group G is isomorphic and homeomorphic to a dense subgroup of a compact topological group $\bar{G}$. $\bar{G}$ may be regarded as the completion of G with respect to the left (or right) uniformity.*

THEOREM 5 (LOOMIS [13, p. 112]). *A topological group G has a smallest closed normal subgroup, and hence a largest quotient group that is a Hausdorff space.*

We note that Loomis's proof of Theorem 5 can be applied in the following more general setting to prove a weaker result (see also Remark (e) following Corollary 15).

THEOREM 6. *A semitopological group G with continuous inversion has a smallest closed normal subgroup, and hence a largest quotient group that is a T_1-space.*

An example of a semitopological group with continuous inversion that is a T_1-space but not a T_2-space, i.e., not a Hausdorff space, is the additive reals with the topology for which complements of finite subsets are the only open sets. We pose the

QUESTION. Is Theorem 6 true for general semitopological groups?

We have one more result about totally bounded topological groups, which provides an interesting example concerning open mappings. We note first that any continuous homomorphism ψ of one compact Hausdorff topological group G_1 onto another such group G_2 is open; this is because ψ factors through the quotient group $G/\{s \in G_1 \mid \psi(s) = e \in G_2\}$, and because the quotient map is open and a 1-1 continuous map of one compact Hausdorff space onto another is a homeomorphism. (For another setting in which homomorphisms are open, see Theorem 5.29 of [9].) In the next theorem we require G to be a locally compact abelian group under each of two topologies τ_1 and τ_2 such that τ_1 is strictly weaker than τ_2; then G has characters that are continuous with respect to τ_2, but are not continuous with respect to τ_1. Let $G_a{}^i$ be the almost periodic compactification of (G, τ_i) and let $a_i : G \to G_a{}^i$ be the canonical continuous homomorphism, $i = 1, 2$. Since the identity map of (G, τ_2) onto (G, τ_1) is continuous, the universal mapping property of the almost periodic compactification [6; Theorem 6.2] gives a continuous homomorphism ψ of $G_a{}^2$ onto $G_a{}^1$ such that $\psi \circ a_2(s) = a_1(s)$ for all $s \in G$.

THEOREM 7. *Let G, τ_1, τ_2 and $\psi : G_a{}^2 \to G_a{}^1$ be as above. Then ψ is open but not* 1-1 *and its restriction to $a_2(G)$ is* 1-1 *but not open.*

PROOF. That ψ is open has been established; if it were 1-1 it would be a homeomorphism, which would contradict the existence of characters continuous with respect to τ_2 and not continuous with respect to τ_1. The restriction of ψ to $a_2(G)$ is 1-1, since the characters of any Hausdorff locally compact abelian topological group separate the points of the group, and cannot be open; for, this would imply that $G_a{}^2$ was homeomorphic to $G_a{}^1$, which would follow from the uniqueness of the completion of a Hausdorff totally bounded topological group (Theorem 5).

3. **The Bohr topology.** Various authors have given various characterizations of the Bohr topology $\mathcal{B}$ on a topological group G [2, 12]. One due to Alfsen and Holm [2] asserts that $\mathcal{B}$ is the topology of the finest uniform structure $\mathcal{U}$ on G whose topology makes G a totally bounded topological group and is coarser than the intiial topology of G. (The conditions for $\mathcal{U}$ just stated are a little less stringent than

those used by Alfsen and Holm, but the next sentence makes it clear that they are equivalent to Alfsen and Holm's conditions.) If we let a be the canonical continuous homomorphism from G into G_a, the almost periodic compactification of G, which we regard as the spectrum of the almost periodic functions $\mathrm{AP}(G)$, then it follows from the universal mapping property of G_a [**6**, Theorem 6.2] and Theorems 4 and 5 that Alfsen and Holm's conditions for $\mathcal{B}$ are equivalent to either of the following (obviously equivalent) conditions, the first of which we take to be the *definition of* $\mathcal{B}$.

(i) $\mathcal{B} = \{a^{-1}(U) \mid U \subset G_a \text{ is open}\}$.

(ii) $\mathcal{B}$ is the weak topology induced on G by $\mathrm{AP}(G)$; i.e., a subbase for $\mathcal{B}$-neighbourhoods of $s \in G$ consists of sets of the form $\{t \in G \mid |f(s) - f(t)| < \epsilon\}$, where $f \in \mathrm{AP}(G)$ and $\epsilon > 0$.

Definition. A function f defined on a topological group G is called *Bohr continuous* if it is continuous for the Bohr topology $\mathcal{B}$.

Arguments as in Loomis [**13**, p. 112] show that f is Bohr continuous if and only if it is of the form a^*g where g is a function continuous on $a(g)$, the image of G in G_a.

Now suppose G is a semitopological group. We wish to assert that *the definitions and characterizations of Bohr topology and Bohr continuous function given above for a topological group go over without change for* G. (To see this it helps to recall that G_a is always a *topological* group [**18**].)

We state the following two theorems. The proof of one part of the first one is trivial, the other part follows by contradiction. We omit the details.

Theorem 8. *Let G be a topological group. Then G is totally bounded if and only if every continuous complex-valued function on G is Bohr continuous.*

Theorem 9. *The weak topology induced on G by the Bohr continuous functions is the same as the Bohr topology.*

4. **Reduction to metrizable groups.** In this section we prove a theorem (whose full generality we do not need) that generalizes a number of theorems in the literature. The first (chronologically) theorem of this type was proved by Kakutani and Kodaira [**11**]. Other theorems of this type were proved by Ross and Stromberg [**20**] and Comfort and Ross [**5**]. We present a complete proof here for several reasons, among others because our proof is quite short and self-contained (as compared with that of Comfort and Ross whose theorem is closest to ours). Also, in neither of [**9**, **17**] is this theorem (or related theorems) proved in even the generality in which Kakutani and Kodaira first proved it.

THEOREM 10. *Suppose $\bar{G}$ is a Hausdorff topological group that has totally bounded neighbourhood V' of the identity e, suppose $\exists$ a sequence $\{r_m\}_1^\infty \subset G \ni G = \bigcup_1^\infty r_m V'$, and let $\{f_n\}_1^\infty$ be a sequence of continuous functions on G. Then $\exists$ a closed normal subgroup N of G $\ni$ for each n, f_n is constant on the cosets of N and G/N is metrizable.*

PROOF. The hypotheses are designed to ensure that G can be homeomorphically and isomorphically embedded as a dense subgroup of a locally compact, σ-compact, topological group $\bar{G}$ [26], $\bar{G} = \bigcup_1^\infty V_k$ where each V_k is compact and may be taken to be equal to $\bigcup_1^k r_m \overline{V'}$ (closure in $\bar{G}$). For the moment, we deal with one function f_1.

Let C_0 be a countable dense subset of the complex numbers. Given $x \in C_0$ and an integer $n > 0$, $\exists$ an open $U(x, n) \subset \bar{G} \ni U(x, n) \cap G = \{s \in G \mid |f(s) - x| < 1/n\}$. Since $\bar{G}$ is σ-compact, $\exists$ an increasing sequence of compact sets $\{K_{jxn}\}_{j=1}^\infty \ni K_{jxn} \subset U(x, n)$ $\forall j$ and $\bigcup_1^\infty K_{jxn}$ is dense in $U(x, n)$. It follows that, for each j, $\exists$ a neighbourhood $U_j(x, n)$ of $e \ni U_j(x, n)K_{jxn} \subset U(x, n)$ [10, p. 70]. Thus we get a countable family of neighbourhoods of e, $\{U_j(x, n) \mid x \in C_0, j, n = 1, 2, 3, \cdots\}$, which we relabel as a single sequence $\{W_m\}_1^\infty$. A variant of a standard argument [9, p. 71; 17, p. 58] shows that $\bigcap_1^\infty W_m$ contains a compact normal G_δ subgroup of $\bar{G}$. We let W_1' be a relatively compact symmetric neighbourhood of $e \in \bar{G} \ni xW_1'x^{-1} \subset W_1 \cap W_1^{-1}$ $\forall x \in V_1$. (Such a neighbourhood exists [17, p. 55]. Recall $\bar{G} = \bigcup_1^\infty V_k$ and each V_k is compact.) If W_1', W_2', $\cdots$, W_{m-1}' have been chosen, choose W_m', a symmetric neighbourhood of $e \in \bar{G}$, $\ni (W_m')^2 \subset W_{m-1} \cap W_{m-1}'$ and $xW_m'x^{-1} \subset W_{m-1}$ $\forall x \in V_m$. It is easy to check that $\bigcap_1^\infty W_m' = N_1$ is a compact normal G_δ subgroup of $\bar{G}$, from which it follows that $\bar{G}/N_1$ is first countable, i.e., the topology of $\bar{G}/N_1$ has a countable basis for open sets at each point, and hence $\bar{G}/N_1$ is metrizable [17, p. 34]. Also, $N_1 K_{jxn} \subset U(x, n)$ $\forall j, n = 1, 2, 3, \cdots$ and $\forall x \in C_0$; and $N_1(\bigcup_{j=1}^\infty K_{jxn}) = N_1 U(x, n) = \overline{U(x, n)}$ $\forall n = 1, 2, 3, \cdots$ and $\forall x \in C_0$, since N_1 is compact and $\bigcup_{j=1}^\infty K_{jxn}$ is dense in $U(x, n)$.

We now show that, if $s, t \in G$ and $st^{-1} \in N_1$, then $f(s) = f(t)$. Since f is continuous, it is sufficient to show that $s \in \overline{U(x, n)}$ whenever $t \in U(x, n)$, and this follows immediately; for, if $t \in U(x, n)$, $s \in N_1 t \subset N_1 U(x, n) \subset \overline{U(x, n)}$.

For f_1 we have the compact G_δ subgroup N_1. We can get such a subgroup N_n for each f_n. The desired subgroup then is $N = \bigcap_1^\infty N_n$, which is a G_δ, so G/N is metrizable.

The following corollary is obviously proved by taking one more intersection of a countable number of G_δ subgroups.

COROLLARY 11. *The statement of Theorem* 10 *is still true if the functions* $\{f_n\}_1^\infty$ *are merely required to be pointwise limits of sequences of continuous functions.*

We state another corollary.

COROLLARY 12. *Let G be a group as in the theorem and let X be a norm-separable subset of $C(G)$. Then $\exists$ a closed normal subgroup N of $G \ni$ each $f \in X$ is constant on the cosets of N and G/N is metrizable.*

Part II.

1. **Equivalent conditions.** We recall that a subset E of a group G is called *left relatively dense* if $\exists$ a finite subset $\{s_m\}_{m=1}^M \subset G \ni \bigcup_1^M s_m E = G$.

THEOREM 13. *Suppose G is a semitopological group and $f \in C(G)$. Then each of the following conditions on f is equivalent to any of the others.*

1 (2). *f is left (right) almost automorphic.*

3. *f is continuous in the Bohr topology.*

4. *Given $\epsilon > 0$ and finite $N \subset G$, $\exists$ a left relatively dense subset $E \subset G \ni E^{-1}E \subset \{s \in G \mid \sup_{r,t \in N} |f(rst) - f(rt)| < \epsilon\}$.*

5 (6). *Given $\epsilon > 0$ and $t \in G$, $\exists$ a left relatively dense subset $E \subset G \ni E^{-1}E \subset \{s \in G \mid |f(ts) - f(t)| < \epsilon\}$ $(E^{-1}E \subset \{s \in G \mid \ |f(st) - f(t)| < \epsilon\})$.*

7. *Every net $\alpha' \subset G$ has a subnet $\alpha \ni$ the joint limit* $\lim_{i,j} f(s\alpha_i\alpha_j^{-1}t) = f(st)$ $\forall s, t \in G$.

8 (9). *Every net $\alpha' \subset G$ has a subnet $\alpha \ni$ the joint limit* $\lim_{i,j} f(\alpha_i\alpha_j^{-1}t) = f(t)$ $(\lim_{i,j} f(t\alpha_j^{-1}\alpha_i) = f(t))$ $\forall t \in G$.

10. *Every sequence $n' \subset G$ has a subsequence $n \ni$ the joint limit*

11 (12). *Every sequence n' has a subsequence $n \ni$ the joint limit* $\lim_{i,j} f(n_i n_j^{-1}t) = f(t)$ $(\lim_{i,j} f(tn_j^{-1}n_i) = f(t))$ $\forall t \in G$.

PROOF. It is obvious that 4 implies 5 and 6, 7 implies 8 and 9 and 10 implies 11 and 12. The proofs that 3 implies 4–12 inclusive (and 1 and 2 as well) are easy exercises once we know we can do the work on a metrizable totally bounded topological group (Theorems 5 and 10) which has a Weil compactification (Theorem 4).

Bearing in mind the symmetry of 3, we can consider ourselves done when we show each of 5, 6, 8 and 11 implies 1 and 1 implies 3.

8 *implies* 1. If f does not satisfy 1, $\exists$ a net $\alpha \subset G$ and $t \in G \ni$

$$\text{(A)} \qquad T_\alpha f = g,\ T_{\alpha^{-1}} g = h \text{ and } f(t) \neq h(t).$$

If α' is a subnet of α, the equations (A) still hold with α replaced by α' and it follows from the double limit theorem of general topology that the double limit $\lim_{i,j} f(\alpha_i'(\alpha_j')^{-1}t)$ is equal to $h(t)$ and hence not to $f(t)$, if it exists at all. Thus, f does not satisfy 8.

11 *implies* 1. Continuing in the same setting, we choose a sequence inductively using the fact that $\lim_j \lim_i f(\alpha_i\alpha_j^{-1}t) = \lim_j g(\alpha_j^{-1}t) = h(t)$ with $|h(t) - f(t)| = \epsilon > 0$, say. Choose $i_1 \ni i \geqq i_1$ implies $|g(\alpha_i^{-1}t) - h(t)| < \epsilon/4$. Having chosen $i_1 < i_2 < i_3 < \cdots < i_k$, choose $i_{k+1} > i_k \ni i \geqq i_{k+1}$ implies $|f(\alpha_i\alpha_{i_k}^{-1}t) - g(\alpha_{i_k}^{-1}t)| < \epsilon/4$. Then $\forall k \geqq m$, $|f(t) - f(\alpha_{i_k}\alpha_{i_m}^{-1}t)| \geqq |f(t) - h(t)| - |h(t) - g(\alpha_{i_m}^{-1}t)| - |g(\alpha_{i_m}^{-1}t) - f(\alpha_{i_k}\alpha_{i_m}^{-1}t)| \geqq \epsilon/2$. Thus f does not satisfy 11.

5(6) *implies* 1. Suppose f satisfies 5 (6), α' is a net in $G \ni T_{\alpha'}f = g$, $T_{(\alpha')^{-1}}g = h$ and $t \in G$. We must show $f(t) = h(t)$. Suppose $\epsilon > 0$ is given. By assumption, $\exists$ a left relatively dense subset $E \subset G \ni E^{-1}E \subset \{s \in G \mid |f(ts) - f(s)| < \epsilon\}$. Since $G = \bigcup_1^M s_mE$ we can write $(\alpha_i')^{-1} = s_{m(i)}\tau_i'$ $(\alpha_i' = s_{n(i)}\nu_i')$ where $\tau_i' \in E$ $(\nu_i' \in E)$ for each i, and conclude that there is a subnet α of $\alpha' \ni \alpha_i^{-1} = s_{m_0}\tau_i$ $(\alpha_i = s_{n_0}\nu_i)$ for each i, and $m_0(n_0)$ is independent of i. Of course $T_\alpha f = g$ and $T_{\alpha^{-1}}g = h$ still, and it follows that

$$\begin{aligned} |h(t) - f(t)| &= |\lim_j \lim_i f(\alpha_i\alpha_j^{-1}t) - f(t)| \\ &= |\lim_j \lim_i f(\tau_i^{-1}s_{m_0}^{-1}s_{m_0}\tau_jt) - f(t)| \leqq \epsilon \\ (|h(t) - f(t)| &= |\lim_j \lim_i f(t\alpha_j^{-1}\alpha_i) - f(t)| \\ &= |\lim_j \lim_i f(t\nu_j^{-1}\nu_i) - f(t)| \leqq \epsilon), \end{aligned}$$

since $\tau_i^{-1}\tau_j \in E^{-1}E$ $(\nu_j^{-1}\nu_i \in E^{-1}E)$ and hence $|f(\tau_i^{-1}\tau_jt) - f(t)| < \epsilon$ $(|f(t\nu_j^{-1}\nu_i) - f(t)| < \epsilon)$, $\forall i, j$.

1 *implies* 3. Let $\mathcal{T}$ denote the topology induced on G by the left almost automorphic functions. Thus sets of the form $\{t \in G \mid |f(t) - f(s)| < \epsilon\}$ where $s \in G$, $\epsilon > 0$ and f is left almost automorphic form a subbase for $\mathcal{T}$. Of course, the left almost automorphic functions are $\mathcal{T}$-continuous. Once we prove that $(G, \mathcal{T})$ is a totally bounded topological group, we will be done, by Theorem 8.

It follows immediately from the translation invariance of the left almost automorphic functions that $(G, \mathcal{T})$ is a semitopological group and we complete the proof that $(G, \mathcal{T})$ is a topological group by contradiction; we assume α and β are nets in G $\mathcal{T}$-converging to $e \in G$ and suppose $\exists \epsilon > 0$ and left almost automorphic f such that the following statement holds: $\forall i_0$, j_0 $\exists i \geqq i_0$, $j \geqq j_0 \ni |f(\alpha_i\beta_j^{-1}) - f(e)| \geqq \epsilon$, i.e., the product net $\{\alpha_i\beta_j^{-1}\}$ does not $\mathcal{T}$-converge jointly to e.

We define two sequences $\{s_n\}$ and $\{t_n\}$ by induction. Choose i_1,

$j_1 \ni |f(\alpha_{i_1}\beta_{j_1}^{-1}) - f(e)| \geqq \epsilon$ and put $s_1 = \alpha_{i_1}$, $t_1 = \beta_{j_1}^{-1}$. Having chosen $s_1, s_2, \cdots, s_n$ and $t_1, t_2, \cdots, t_n$, choose $i_0, j_0 \ni \forall i \geqq i_0$, $j \geqq j_0$ $|f(\alpha_i(\prod_m^n s_k)t_m) - f((\prod_m^n s_k)t_m)| \leqq 1/2^{n+1}$ and $|f(\beta_j(\prod_m^n s_k)t_m) - f((\prod_m^n s_k)t_m)| \leqq 1/2^{n+1}$, $m = 1, 2, \cdots, n$. (Here $\prod_m^n s_k = s_n s_{n-1} s_{n-2} \cdots s_m$ for $m \leqq n$.) Then pick $i_{n+1} \geqq i_0$ and $j_{n+1} \geqq j_0 \ni |f(\alpha_{i_{n+1}}\beta_{j_{n+1}}^{-1}) - f(e)| \geqq \epsilon$ and put $s_{n+1} = \alpha_{i_{n+1}}$, $t_{n+1} = \beta_{j_{n+1}}$. Now define a new sequence $\{r_n\}$, $r_n = t_{n+1}^{-1} (\prod_1^n s_k)$. Note that, if $n > m$, $r_n r_m^{-1} = t_{n+1}^{-1} (\prod_{m+1}^n s_k) t_{m+1}$. Hence, if $n > m$,

$$\begin{aligned} &|f(r_n r_m^{-1}) - f(s_{m+1}t_{m+1})| \\ &\leqq \left| f\left(t_{n+1}^{-1} \left(\prod_{m+1}^{n} s_k\right) t_{m+1}\right) - f\left(\left(\prod_{m+1}^{n} s_k\right) t_{m+1}\right)\right| \\ &+ \sum_{k=m+1}^{n-1} \left| f\left(\left(\prod_{m+1}^{k+1} s_p\right) t_{m+1}\right) - f\left(\left(\prod_{m+1}^{k} s_p\right) t_{m+1}\right)\right| \\ &\leqq \sum_{m+2}^{n+1} 1/2^k < 1/2^{m+1}. \end{aligned}$$

For m so large that $1/2^{m+1} < \epsilon/2$, we have $\forall n > m$ $|f(e) - f(r_n r_m^{-1})| \geqq |f(e) - f(s_{m+1}t_{m+1})| - |f(s_{m+1}t_{m+1}) - f(r_n r_m^{-1})| \geqq \epsilon - \epsilon/2 = \epsilon/2$. Hence for no subnet γ of the sequence $\{r_n\}$ can $(T_{\gamma^{-1}} T_\gamma f)(e) = f(e)$, which implies f is not left almost automorphic, the desired contradiction.

If $(G, \mathcal{T})$ is not totally bounded, then $\exists$ a basic $\mathcal{T}$-neighbourhood $U = \{t \in G \mid |f_k(ts) - f_k(s)| < \epsilon, \; k = 1, 2, \cdots, p\} = \{t' \in G \mid |f_k(t') - f_k(s)| < \epsilon, k = 1, 2, \cdots, p\}s^{-1}$ and a sequence $n = \{n_j\}_{j=1}^\infty \ni$

$$\text{(A)} \qquad \forall j, n_{j+1} \notin \bigcup_{m=1}^{j} U n_m.$$

Let γ be a subnet of $n \ni T_\gamma f_k = g_k$, say, and $T_{\gamma^{-1}} g_k = f_k$, $k = 1, 2, \cdots, p$. Hence, $\exists i_0 \ni$

$$\text{(B)} \qquad |g_k(\gamma_{i_0}^{-1} s) - f_k(s)| < \epsilon/2, k = 1, 2, \cdots, p.$$

Since γ is a subnet of n, $\gamma_{i_0} = n_{j_0}$ for some j_0 and $\exists i_1 \ni$

$$\text{(C)} \qquad \gamma_{i_1} = n_{j_1} \text{ with } j_1 > j_0$$

and $|f_k(\gamma_{i_1}\gamma_{i_0}^{-1}s) - g_k(\gamma_{i_0}^{-1}s)| = |f_k(n_{j_1}n_{j_0}^{-1}s) - g_k(\gamma_{i_0}^{-1}s)| < \epsilon/2$. This last assertion, along with (B), implies $n_{j_1}n_{j_0}^{-1} \in U$. (C) then contradicts (A).

The diagonalization process can be used, along with conditions 10–12 of Theorem 13, to prove the following corollaries.

COROLLARY 14. *Let* $A \subset C(G)$ *be a countable family of functions each member of which satisfies one (hence all) of the conditions of Theorem* 13. *Then*

10′. *Every sequence* $n' \subset G$ *has a subsequence* n *such that the joint limit* $\lim_{i,j} f(sn_i n_j^{-1}t) = f(st)$ $\forall s, t \in G$ *and* $\forall f \in A$.

11′ (12′). *Every sequence* $n' \subset G$ *has a subsequence* n *such that the joint limit* $\lim_{i,j} f(n_i n_j^{-1}t) = f(t)$ $(\lim_{i,j} f(tn_j^{-1}n_i) = f(t))$, $\forall t \in G$ *and* $\forall f \in A$.

COROLLARY 15. *The assertions of Corollary* 14 *remain true if* A *is a norm-separable subset of* $C(G)$, *instead of just a sequence.*

DEFINITION. We call a function in $C(G)$ satisfying one, hence all, of the conditions of Theorem 13 *almost automorphic* and denote the class of such functions by AA(G). (Thus AA(G) = LAA(G); we drop this latter notation.)

REMARKS. (a) We believe that, even for topological groups, the equivalence of conditions 1 and 10–12 of Theorem 13 is new (Reich proved the equivalence of 1 and 10 for a special case in [19]). The conditions 5 and 6 are simpler (and weaker) forms of conditions used by Reich [19], who calls functions Levitan almost periodic that satisfy condition 4, which is formally a little weaker than the conditions used by Veech [23, 24] to define Bohr almost automorphy.

(b) In |19], Reich proved the equivalence of conditions 1–4, 7 in the setting of topological groups. The proofs given above are different from Reich's, except for the proof that 5(6) implies 1, the germ of which appears in Reich's Hilfssatz 1 [19]. The key idea in the only really tricky part of the proof, namely the proof that 1 implies 3, is due to Veech [23]; our proof that 1 implies 3 is an adaptation of proofs by Terras in [21, 22]. The method of our proof that 11 implies 1 is also taken from [22].

(c) We find it interesting that the "left" conditions, 1, 5, 8, and 11 are equivalent to their companion "right" conditions, 2, 6, 9, and 12, respectively. AP(G) and the weakly almost periodic subspace of $C(G)$ WAP(G) each have equivalent "left" and "right" conditions that can be used to define them. For example, if βG is the spectrum of $C(G)$, which is just the Stone-Čech compactification of G if G is completely regular, then WAP(S) = $\{f \in C(G) \mid \{f_s| \ s \in G\}$ is relatively $\sigma(C(G), \beta G)$-compact$\}$ = $\{f \in C(G) \mid \{f^s \mid s \in G\}$ is relatively $\sigma(C(G), \beta G)$-compact$\}$. However, $\{f \in C(G) \mid \{f_s \mid s \in G\}$ is relatively compact for the topology of pointwise convergence on $G\}$ and $\{f \in C(G) \mid \{f^s \mid s \in G\}$ is relatively compact for the topology of pointwise convergence on $G\}$ are equal to the right and left uniformly continuous subspaces RUC(G) and LUC(G) of $C(G)$, respectively, for many groups G, including locally compact groups [**16**], and hence do not always define the same subspace of $C(G)$.

(d) Veech [**24**, Theorem 6.1] knew that an almost automorphic function could be represented on a totally bounded metrizable group.

(e) One might wonder whether the difficult part of the proof that 1 implies 3, namely the proof that $(G, \mathcal{T})$ is a *topological group*, could be proved in a different way, by proving that every totally bounded Hausdorff semitopological group is a topological group. This statement is true if "totally bounded" is replaced by "compact" (and is a special case of a theorem of Ellis [**6**, appendix, **10**]), but is false in general: the real numbers (modulo 1) $[0, 1)$ with the topology for which sets of the form $[a, b)$, $0 \leqq a \leqq b \leqq 1$, form a base is a totally bounded Hausdorff semitopological group with discontinuous inversion and jointly continuous multiplication (which is addition here). However, inversion in $(G, \mathcal{T})$ is readily seen to be continuous [**22**, Lemma 2.1] and one might hope to show that every totally bounded Hausdorff semitopological group with continuous inversion is a topological group. This hope is also ill-founded as the following example shows. The real numbers (modulo 1) $[0, 1)$ with the topology for which sets of the form $V = U \backslash U_1$, where U is an ordinary open subset of $[0, 1)$ and U_1 is countable, form a base is a Hausdorff semitopological group with inversion continuous but multiplication only separately continuous. That the topology is totally bounded can be shown by reducing to the case where the open set U in question is dense in $[0, 1)$ and observing that a linear independence argument then shows that $[0, 1)$ can be covered by two translates of U.

In connection with Remark (e) and the examples given there, it is interesting that a case more special than the one mentioned above of a theorem of Ellis can now be quickly proved.

THEOREM 16. *A compact Hausdorff semitopological group G with continuous inversion is a topological group.*

PROOF. If $f \in C(G)$ and α' is a net in G, then $\exists$ a subnet α of α' converging to $s \in G$. Then α^{-1} converges to s^{-1} and it follows immediately that $T_\alpha f = f_s$ and $T_{\alpha^{-1}}(f_s) = (f_s)_{s^{-1}} = f$. Thus $C(G) = \mathrm{AA}(G)$ and the original topology of G is equal to $\mathcal{T}$. But $(G, \mathcal{T})$ is a topological group.

It has been observed that an almost automorphic function must be uniformly bounded. However, if a function f defined on G is merely continuous and not necessarily bounded, and satisfies one of the conditions 3–12 of Theorem 13, then its truncation, f_k, (k a positive integer), defined by

$$f_k(s) = \begin{cases} f(s) \text{ if } |f(s)| \leqq k \\ \\ kf(s)/|f(s)| \text{ if } |f(s)| > k, \end{cases}$$

also satisfies that condition and is a member of $C(G)$. Hence each f_k satisfies all the conditions of Theorem 13; and we state

THEOREM 17. *If f is a continuous function on G, then each of the conditions* 3–12 *of Theorem* 13 *on f is equivalent to any other of these conditions.*

If a continuous, complex-valued function f on G is called *sequentially almost automorphic* provided it satisfies the condition that every sequence $n' \subset G$ has a subsequence $n \ni T_n f = g$ and $T_{n^{-1}} g = f$, one might expect (this author did) that one would have here one more condition to add to the equivalent conditions of Theorem 13. This possibility seems all the more attractive when one bears in mind that some of the conditions of Theorem 13 *are* phrased in terms of sequences. However, the following example of Terras [22, p. 771] (who does prove that every sequentially almost automorphic function is a member of AA(G)), which was communicated to him as an example of a function that is almost automorphic and not almost periodic, shows that not every $f \in$ AA(G) is sequentially almost automorphic. We present this example here because we believe Terras' formulae in [22] are wrong, although his verbal description makes his intent clear, and also we want to apply the sequential condition 11 of Theorem 13 to show the function constructed is almost automorphic. (We note that it is also easy to show that this function is Bohr continuous.)

EXAMPLE. Write the additive reals R as the disjoint union $\bigcup_{k=1}^{\infty} V_k$, where $V_k = \bigcup_{m=-\infty}^{\infty} ([0, 1) + s_k + 2^k m)$, $s_k = ((-2)^{k-1} - 1)/3$, and define $f \in C(R)$ by $f(t) = \sin 2^k \pi t$ on V_k, $k = 1, 2, \cdots$. Suppose $n' = \{n_i'\}$ is a sequence in R. For each i, write $n_i' = n_i' \pmod 1 + b_i'' = n_i'' + b_i''$, say, where b_i'' is an integer. Without loss, we may assume that $n'' = \{n_i''\}$ converges. Also, since every sequence of integers b'' has, for a given positive integer k, a subsequence $b' \ni 2^k$ divides $b_i' - b_j'$ $\forall i, j$, we can use the diagonal process to get a subsequence b of b'' such that, for *every* positive integer k, 2^k divides $b_i - b_j$ $\forall$ large enough i, j. Let $n = \{n_i\}$ be the corresponding subsequence of n'. It follows that the joint limit $\lim_{i,j} f(n_i - n_j + t) = f(t)$ $\forall t \in R$. Hence f is almost automorphic. However, if one considers the sequence of translates $\{f_{s_k}\}$ of f restricted to the interval $[0, 1]$, one has the sequence of functions $\{t \to \sin 2^k \pi t\}$, no subsequence of which is going to converge pointwise on $[0, 1]$. Terras [22] has given a proof of this as a special case of the theorem one naturally arrives at if one attempts to write out the details of the proof.

The local compactness of R is essential for this example; for, the restriction of f to the rationals is sequentially almost automorphic on the rationals. This is because every $f \in \mathrm{AA}(G)$ is sequentially almost automorphic if G is countable [22].

2. **The approximation theorems.** In this section we quickly derive Veech's theorem [24, Theorem 7.1] concerning the approximation of almost automorphic functions by almost periodic functions and a related theorem.

Suppose $f \in \mathrm{AA}(G)$. Then, by Theorems 10 and 13 and § 3 of I, f can be represented as a continuous function on a totally bounded metrizable group. To be specific, $\exists$ a metrizable quotient group K of $a(G)$ and a function $g \in C(K) \ni f = a_0{}^*g$, where a_0 is the canonical continuous homomorphism of G onto K. Suppose f, hence g, is real valued as well and that $\{U_k\}_{k=1}^{\infty}$ is a basis for neighbourhoods of the identity $e \in \bar{K}$, the Weil compactification of K. We modify an idea of Veech [24, p. 128] and observe that g can be extended to $\bar{K}$ (we are regarding K as a subset of $\bar{K}$) in such a way that the extension g^* (g_*) is upper (lower) semicontinuous and hence measurable. By definition, $g^*(s) = \lim_k \sup_{t \in sU_k \cap K} g(t)$ and $g_*(s) = \lim_k \inf_{t \in sU_k \cap K} g(t)$. Clearly, $g_*(s) = g^*(s) = g(s)$ $\forall s \in K$, since $g \in C(K)$. $\forall k$, let $h_k = \chi_{U_k}/\mu(U_k)$, where μ is Haar measure on $\bar{K}$. Then, for example, $\{h_k * g^*\} \subset C(\bar{K})$ and $(h_k * g^*)(s) \to g(s)$ $\forall s \in K$. Since $a_0{}^*(C(\bar{K})) \supseteq \mathrm{AP}(G)$, and $a_0{}^*(h_k * g^*) \to a_0{}^*g = f$ pointwise on G, we are able to state the following theorems. Theorem 18 was first proved by Veech [24, p. 133]; Theorem 19 is a variant.

THEOREM 18. *Let G be a semitopological group. A function $f \in C(G)$ is almost automorphic if and only if $\exists$ a uniformly bounded sequence $\{f_k\} \subset \mathrm{AP}(G)$ such that the following statement holds: given $\epsilon > 0$ and $s \in G$, $\exists$ a Bohr neighbourhood V of s and k_0 such that $k \geqq k_0$ implies $|f_k(t) - f(t)| < \epsilon$, $\forall t \in V$.*

THEOREM 19. *Let G be a semitopological group. A function $f \in C(G)$ is almost automorphic if and only if for each positive integer m $\exists$ an increasing sequence $\{A_{m_k}\}_{k=1}^{\infty}$ of subsets of G and $\exists$ functions $\{f_{m_k}\}_{k=1}^{\infty} \subset \mathrm{AP}(G) \ni$*

(i) *the functions $\{f_{m_k} \mid m, \ k = 1, 2, 3, \cdots\}$ are uniformly bounded.*

(ii) *for each m and each $s \in G$, $\exists k_0 = k_0(m, s)$ such that $k \geqq k_0$ implies that A_{m_k} is a Bohr neighbourhood of s. (This implies that $\bigcup_{k=1}^{\infty} A_{m_k} = G$ for each m.)*

(iii) $|f_{m_k}(t) - f(t)| < 1/m$, $\forall t \in A_{m_k}$.

REMARKS.

(a) For both theorems the backward implication is the obvious one.

(b) Veech [23] calls the form of convergence in Theorem 18 *jointly almost automorphic convergence.*

(c) Both theorems can be rephrased to yield approximation theorems for Bohr continuous (not necessarily bounded) functions. Specifically, the theorems remain true if f is required to be merely Bohr continuous (rather than almost automorphic) and the approximating almost periodic functions are not required to be uniformly bounded.

In order to show that the functions g^* and g_* used in this section can be quite different, for a given g, we present an example for which we thank Dr. David Borwein. We have a dense subgroup K of a compact group $\bar{K}$ and, given $\epsilon > 0$, we find a function $g \in C(K) \ni \|g\| = 1$ and $\mu(g^* - g_*) > 2 - \epsilon$. ($\mu$ is normalized Haar measure on $\bar{K}$.)

EXAMPLE. Let K be the rationals modulo 1; then $\bar{K}$ is the reals modulo 1. Let $\{s_k\}_{k=1}^{\infty}$ be an enumeration of the members of K and put $V = \bigcup_1^{\infty} V_k$ where $V_k \subset \bar{K}$, $V_k = (s_k - \epsilon/2^{k+3}, \ s_k + \epsilon/2^{k+3})$ (mod 1). Then V is open in $\bar{K}$ and is a union of a countable number of pairwise disjoint open intervals, $V = \bigcup_1^{\infty} (r_k, t_k)$. Also $\mu(V) \leqq \epsilon/8$ and $K \subset V$. Define a function g' on V by $g' = \sum_1^{\infty} g_k'$ where $g_k'(s) = \sin((s - r_k)^{-1}(s - t_k)^{-1}$ if $s \in (r_k, t_k)$, $g_k'(s) = 0$ if $s \notin (r_k, t_k)$. Then the restriction g of g' to K is the desired function. Clearly $\|g\| = 1$ and the set $F = \{r_k, t_k \mid k = 1, 2, 3, \cdots\}$ is dense in the complement in $\bar{K}$ of V. This complement has measure not less than $1 - \epsilon/8$ and on it $g^* = 1$, $g_* = -1$ since every open set containing a member of $\bar{K}\backslash V$ contains a neighborhood of a member of F. Hence $\mu(g^* - g_*) \geqq 2(1 - \epsilon/8) - 2(\epsilon/8) > 2 - \epsilon$.

Part III.

1. The existence of almost automorphic functions that are not almost periodic. For this section a definition and a theorem which is due to Comfort and Ross [5] are of prime importance.

DEFINITION. A topological group G is called *pseudocompact* if every continuous complex-valued function on G is bounded.

THEOREM 20. *Let G be a totally bounded Hausdorff topological group and let $\bar{G}$ be its Weil compactification. Then the following assertions are equivalent*:

(a) *G is pseudocompact.*

(b) *each nonempty G_δ subset of $\bar{G}$ meets G.*

(c) *each continuous real-valued function on G admits a continuous extension to $\bar{G}$.*

(d) *every continuous real-valued function on G is uniformly continuous.*

It is not hard to prove that pseudocompact groups are totally bounded; that compact groups are pseudocompact is trivial. The examples of pseudocompact non-compact groups that have been given are certain subgroups of uncountable products of compact metric groups (see [5] for a discussion and further references). In particular, if $G_0 = \times_{\alpha \in I} G_\alpha$, then $H = \{x = \{x_\alpha\} \in G_0 \mid x_\alpha$ is the identity of G_α for all but countably many $\alpha\}$ is pseudocompact and not compact. This is not hard to prove using Theorem 10 and the equivalence of (a) and (c) of Theorem 20, once it has been observed that every metrizable quotient group of H must be compact. A particular case of a result of Hewitt [8; p. 67] now follows without difficulty.

THEOREM 21. *A Hausdorff topological group G is pseudocompact if and only if every real-valued $f \in C(G)$ has closed range.*

Using the fact that a compact group can be written as a subgroup of a product of compact metric groups (see [17, § 2.7], for example), one can prove the following theorem which shows that all pseudocompact groups are similar in form to the example above.

THEOREM 22. *Every pseudocompact group G can be written as a subgroup of a product of compact metric goups $\times_{\alpha \in I} G_\alpha$ in such a way that the canonical projection of G onto G_α is onto for each α and the canonical projection of G onto $\times_{\alpha \in B} G_\alpha$ for any countable subset B of I has closed range. Another way of saying this last assertion is to say that every metrizable quotient group of G is compact.*

The connection between the existence of functions that are almost automorphic but not almost periodic and pseudocompact groups is obvious. The second assertion of the following theorem follows from Theorem 20.

THEOREM 23. *Let G be a semitopological group. Then* AP(G) = AA(G) *if and only if $a(G)$, the canonical image of G in its almost periodic compactification, is pseudocompact. In this case every Bohr continuous function on G is also almost periodic.*

It seems likely that AP(G) $\neq$ AA(G) for most familiar non-compact groups. However, non-compact groups for which equality holds are not hard to find. For example, AP(G) = AA(G) if G is pseudocompact. An example of quite a different kind (Veech [25]) is the group H of finite permutations of the natural numbers. In this case H_a has only two members. Hence, if G is the product of H and a pseudocompact infinite group, then G is an example of a non-discrete non-pseudocompact topological group for which AP(G) = AA(G).

At this point, one can use the equivalence of (a) and (b) of Theorem 20 and Theorem 23 to prove the following variant of a theorem of Terras [22, Theorem 2.11].

THEOREM 24. *Let G be a σ-compact, completely regular semitopological group. Then G is compact if and only if $C(G)$ =* AP(G).

Now, if G is a semitopological group and $a(G)$ is not pseudocompact, then $\exists$ functions in AA(G)$\backslash$AP(G). According to Theorem 20, they correspond to the functions in $C(a(G))$ that are not uniformly continuous, i.e., $f \in$ AA(G)$\backslash$AP(G) if and only if f is bounded and continuous, but not uniformly continuous, with respect to the Bohr topology. A common way to construct functions in AA(G)$\backslash$AP(G) is to find an almost periodic function on G with non-closed range and then take a bounded continuous, but not uniformly continuous, function of that function (see [23, 24, 25], for example). On the integers, the function $n \to 2 + e^{in} + e^{in/\sqrt{2}}$ is almost periodic and never assumes the value 0, but has 0 in the closure of its range; hence $n \to (2 + e^{in} + e^{in/\sqrt{2}})/|2 + e^{in} + e^{in/\sqrt{2}}|$ is almost automorphic and not almost periodic. Hewitt's theorem (Theorem 21) is explicit about why this type of construction is impossible if $a(G)$ is pseudocompact.

If $f \in$ AP(G), then it is obvious that the limit $T_\alpha f$ is in AA(G) whenever it exists. On the other hand, suppose $f \in$ AA(G)$\backslash$AP(G). If we consider f as a function on $a(G) \subset G_a$, then $\exists x \in G_a$ and nets α, $\beta \subset a(G)$ converging to x such that $\lim f(\alpha_i) = c_1 \neq c_2 = \lim f(\beta_j)$. Without loss we may assume $T_\alpha f$ exists on $a(G)$. Then $T_\alpha f(e) = \lim f(\alpha_i) = c_1$. But then the product net $\{\alpha_i^{-1}\beta_j\}$ converges to e and we see that $\lim T_\alpha f(\alpha_i^{-1}\beta_j) = c_2$, using joint continuity of multiplication in $a(G)$. Hence $T_\alpha f \notin C(a(G))$ and $f \notin$ AA(G), and we have the following theorem of Veech [23, Theorem 3.3.1].

THEOREM 25. *Let G be a semitopological group and let $f \in \mathrm{AA}(G)$. Then $f \in \mathrm{AP}(G)$ if and only if $T_\alpha f \in \mathrm{AA}(G)$ whenever it exists.*

2. **Extension properties of almost automorphic functions.** A title indicating a negative aspect would be more suitable for this section; for, what we do here is present four examples, in increasing order (we believe) of interest and novelty, of functions almost automorphic on a subgroup that do not extend to functions almost automorphic on the containing group.

EXAMPLE 1. Let G be the additive rationals modulo 1. G is a dense subgroup of the reals modulo 1 $[0, 1)$ and, if $E = \{s \in G \mid 1/\sqrt{3} < s < 1/\sqrt{2}\}$, then $\chi_E \in \mathrm{AA}(G)$, but does not extend to a function continuous on $[0, 1)$, and, in particular, does not extend to a function in $\mathrm{AA}([0, 1))$.

EXAMPLE 2. In a similar vein, a continuous periodic function on the rationals that does not extend to a function in $C(R)$, in particular, does not extend to a function in $\mathrm{AA}(R)$, is easily given. The periodic function on the rationals of period $\sqrt{2} \ni y = x$ for $0 \leqq x < \sqrt{2}$ will do.

Example 3 is due to Terras [22] and Berg [3]. A function has been constructed on the rationals that is almost automorphic and extends to a function continuous but not almost automorphic on R. The reader is referred to the papers of Terras and Berg for details.

In the first three examples, the subgroup was dense but not closed. We now give an example of a locally compact group G with a (non-dense) closed normal abelian subgroup $H \ni f \in \mathrm{AA}(H)$ extends to a function in $\mathrm{AA}(G)$ if and only if f is constant. (In [15] we showed for this G and H that no non-trivial character on H extends to a function that is uniformly continuous on G.)

EXAMPLE 4. Let $G = \{(x, y) \mid x, y \in R, x > 0\}$ with the multiplication $(a, b)(x, y) = (ax, ay + b)$, and let $H = \{(1, y) \mid y \in R\}$. Gelfand and Naimark [7] have shown that the only finite dimensional irreducible representations of G are one dimensional and of the form $(x, y) \to x^{ix_0}$ for some $x_0 \in R$. (It is probably faster and certainly more instructive to read about these matters in the context of induced representations [14]). It follows that the Bohr topology of G does not separate points that have the first coordinate and hence that the only functions in $\mathrm{AA}(H)$ that extend to functions in $\mathrm{AA}(G)$ are the constant functions.

3. **Invariant means.** The main purpose of this last section is to ask the

QUESTION. Does AA(G) always admit an invariant mean? (Of course, it does if G is amenable.)

One might think that an invariant mean on AA(G) could be constructed using techniques like those used in II, § 2, i.e., that a map like $f \to \mu(g^* + g_*)/2$ would be an invariant mean. (Notation here is as in II, § 2. μ is normalized Haar measure on $\bar{K}$.) However, this construction fails because the map $f \to (g^* + g_*)/2$ can fail to be additive. For example, this failure occurs if G is the rationals (mod 1).

Terras [22] has shown that one can construct an almost automorphic function f on the integers for which the limit $\lim_{N\to\infty} 1/(2N + 1) \sum_{i=+N}^{N} f(i)$ does not exist. This shows that invariant means on almost automorphic functions need not be unique. This is in contrast with the weakly almost periodic subspace WAP(G), for which an invariant mean always exists and is unique.

In [1] Alexandroff studied invariant "measures" (which are only required to be finitely additive) on totally bounded topological groups and found a criterion that singled out a particular invariant measure for each such group. There are two problems with this measure:

(i) It can differ from Haar measure if the group is compact.

(ii) There can exist continuous bounded functions that are not measurable.

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