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FACTORIAL MODULES

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Introduction. In [7] the notion of unique factorization domains (UFD's) was generalized to torsion free modules over integral domains, called factorial modules, and some basic theorems for factorial modules were proved.

The purpose of this article is to extend the investigation of factorial modules to wider classes of modules in order to search for more resemblance between the theory of factorial modules and the theory of UFD's.

§ 1 initiates the study by giving formal definitions of factorial modules and related terminologies. In § 2 we prove Theorem 2.1, which is the most fundamental theorem of factorial modules; it states five conditions each of which is equivalent to that a module is factorial over a UFD. In § 3 we introduce the notion of prime submodules which is similar to prime ideals of rings. It is proved that a module M over a UFD R, which is not a field, is factorial if and only if M possesses non-zero prime submodules each of which contains an element of the form $p\eta$ for certain irreducible elements $\eta \in M$ and $p \in R$ respectively. An overring of a UFD R is called a factorial extension if it is a factorial R-module. In § 4 we study a necessary and sufficient condition in order that an overring of a UFD be a factorial extension. The result yields a series of corollaries such as that the completion of a regular local ring R is a factorial extension of R. § 5 consists of various transition theorems which deal with the inheritance problem of the factorial property between, respectively: (i) a module M and modules M_S of fractions of M and (ii) a topological module and its completion. Thus, an analogue of Nagata's theorem and that of Mori's theorem for UFD's are considered for modules. Finally § 6, in which results of previous sections are fully applied, is devoted to the study of the factorial property of an $R[x_i \mid i \in I]$-module $M[x_i \mid i \in I]$ and an $R[[x_i \mid i \in I]]$-module $M[[x_i \mid i \in I]]$ for any I.

In this paper we confine ourselves exclusively to commutative unitary rings and torsion free unitary modules; all subrings are assumed to possess the identity element of the containing ring.

1. **Definitions.** Let M be a non-zero module over an integral domain R, and U the group of units of R. Let m and m' be two elements of M. We say that m *divides* m' *in* M and write $m \mid m'$ if there exists an ele-

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ment $r \in R$ such that $m' = rm$. If $m \mid m'$, then m is called a *factor* or *divisor of* m' *in* M. Similarly, an element $d \in R$ is said to *divide* m *in* M, written $d \mid m$, if there exists an element $m_0 \in M$ such that $m = dm_0$. If $d \mid m$, then d is called a *divisor of* m *in* R. Note that including the division $a \mid a'$ of elements a and a' in R, we are here dealing with three types of divisions. If $m \mid m'$ and $m' \mid m$, then we shall say that these elements are *associates in* M and write $m \sim m'$. Clearly, $m \sim m'$ if and only if $m = um'$ and $m' = vm$ for some u and v in U. If $m \mid m'$ and m is not an associate of m', then we say that m is a *proper factor of* m' *in* M.

Definition 1.1. A non-zero element $m \in M$ is said to be *irreducible in* M if m has no proper factor in M.

It is clear that $m \neq 0$ is irreducible if and only if $m = am'$ for $a \in R$ and $m' \in M$ implies that $a \in U$.

Definition 1.2. A non-zero element $m \in M$ is said to be *primitive* in M if, whenever $m \mid am'$ for $0 \neq a \in R$ and $m' \in M$, then $m \mid m'$.

Every primitive element m is irreducible in M because $m = am'$ implies that $m \sim m'$, i.e., a is a unit. An element r of R, regarded as a module over itself, is primitive if and only if r is a unit, since $r \mid r \cdot 1$.

A submodule N of M is called a *pure submodule* if $rM \cap N = rN$ for every $r \in R$.

Proposition 1.1. *Let* m *be a non-zero element of an* R*-module* M. *Then the following statements are equivalent:*

(1) m is primitive;
(2) the cyclic submodule Rm is pure;
(3) if $x \in M$, then either $Rx \cap Rm = (0)$ or $Rx \subseteq Rm$.

Corollary. *Two primitive elements* m *and* m' *of an* R*-module* M *are non-associates if and only if* $Rm \cap Rm' = (0)$.

Definition 1.3. An element $p \in R$ is said to be *prime to* an R-module M if

(i) p is irreducible in R, and
(ii) $p \mid am$ for $a \in R$ and $m \in M$ implies that $p \mid a$ in R or $p \mid m$ in M.

Definition 1.4. A (torsion free) module M over an integral domain R is called a *unique factorization module* (UFM) or a *factorial module* if the following two conditions are satisfied:

[UF1] Every non-zero element $x \in M$ has an irreducible factorization—that is, $x = a_1a_2 \cdots a_nm$, where $a_1, a_2, \cdots, a_n$ are irreducible in R and m is irreducible in M.

[UF2] If $x = a_1a_2 \cdots a_nm = b_1b_2 \cdots b_km'$ are two irreducible factorizations of x, then $n = k$, $m \sim m'$ in M, and we can rearrange the order of the b_i's so that $a_i \sim b_i$ in R for every $i \in \{1, 2, \cdots, n\}$.

As pointed out in [7, p. 37], Property 2.2, if an R-module M is a UFM, then R is necessarily a UFD. Therefore when looking for factorability of an R-module we may assume from the start that the ring R is a UFD. We remark that the following condition [UF1′] implies [UF1] if R is a UFD:

> [UF1′] M satisfies the ascending chain condition (a.c.c.) for cyclic submodules.

Definition 1.5. Let $a \in R$ and $m \in M$.

(1) An element $d \in R$ is a *greatest common divisor* (g.c.d.) of a and m if

(i) $d \mid a$ in R and $d \mid m$ in M, and

(ii) any element $c \in R$, such that $c \mid a$ in R and $c \mid m$ in M, is a divisor of d.

We denote it by (a, m) or g.c.d. $\{a, m\}$.

(2) An element $m^* \in M$ is a *least common multiple* (l.c.m.) of a and m if

(i) $a \mid m^*$ and $m \mid m^*$ in M respectively, and

(ii) any element $w \in M$ such that $a \mid w$ and $m \mid w$ in M has m^* as a factor.

We denote it by $[a, m]$ or l.c.m. $\{a, m\}$.

Proposition 1.2. *Let M be a module over an integral domain R. Let m and m^* be elements of M, and $a \in R$. Then*

(1) $m^* \sim$ l.c.m. $\{a, m\}$ if and only if $aM \cap Rm = Rm^*$.

(2) Let p be an irreducible element of R such that an l.c.m. $\{p, m\}$ exists in M. If $p \nmid m$, then $pM \cap Rm = Rpm$.

Proof. (1) is trivial. (2) Let $m^* \sim$ l.c.m. $\{p, m\}$, then $0 \neq m^* = am = pm_0$ and $pm = bm^*$ for some non-zero elements $a, b \in R$ and $m_0 \in M$. We have that $p = ab$ and so $m = bm_0$. Since $p \nmid m$ and $b \mid m$, $p \nmid b$. Hence $p \sim a$ and b is a unit, i.e., $m^* \sim pm$ so that $pM \cap Rm = Rpm$ by (1).

PROPOSITION 1.3. *Let M be a module over a* G.C.D.*-domain R such that a* g.c.d. $\{a, m\} = (a, m)$ *exists for every $a \in R$ and $m \in M$. Then, for any $b \in R$,*

(1) $((a, b), m) \sim (a, (b, m))$,

(2) $(ba, bm) \sim (b(a, m))$,

(3) *if $a \mid bm$ and $(a, m) = 1$, then $a \mid b$.*

2. **Factorial Modules.** In the present section we investigate the following fundamental characterization of a factorial module and its application to various examples.

THEOREM 2.1. *Let M be a module over a* UFD *R which satisfies* [UF1]. *Then the following statements are equivalent:*

(1) *M is factorial over R;*

(2) *Every irreducible element of M is primitive;*

(3) *For any pair of elements $a \in R$ and $m \in M$, a* g.c.d. $\{a, m\}$ *exists in R;*

(4) *For any pair of elements $a \in R$ and $m \in M$, an* l.c.m. $\{a, m\}$ *exists in M, i.e., the submodule $aM \cap Rm$ is cyclic;*

(5) *Every irreducible element p of R is prime to M;*

(6) (i) *If a and b are elements of R such that $aM \subseteq bM$, then $b \mid a$, and*

(ii) *for every pair $a, b \in R$, there exists an element $c \in R$ such that $aM \cap bM = cM$*([7, p. 41, Theorem 4.3]).

REMARK. If M is a UFM, any element c satisfying (i) and (ii) is necessarily an l.c.m. $\{a, b\}$ in R.

PROOF. (a) It is trivially true that (1)$\Longleftrightarrow$(2).

(b) (2)$\Longrightarrow$(3): If $m = 0$, then g.c.d. $\{a, m\} \sim a$ for every $a \in R$. If $m = bm_0 \neq 0$, where $b \in R$ and m_0 is an irreducible element of M, then we claim that g.c.d. $\{a, m\} \sim$ g.c.d. $\{a, b\} = d \in R$. Clearly d is a common divisor of a and m. Assume that d' is another common divisor of a and m, and put $m = bm_0 = d'm'$ for some $m' \in M$. Then $d' \mid b$ so that $d' \mid d$, since m_0 is primitive and $d =$ g.c.d. $\{a, b\}$. Consequently $d \sim$ g.c.d. $\{a, m\}$; thus (3) holds.

(c) (3)$\Longrightarrow$(4): If $a = 0$, then l.c.m. $\{a, m\} \sim 0 \in M$ for every m $\in M$. For any pair $0 \neq a \in R$ and $m \in M$, let $d \sim$ g.c.d. $\{a, m\}$. Then $a = da'$ and $m = dm'$ for some $a' \in R$ and $m' \in M$ such that g.c.d. $\{a', m'\} \sim 1$ by Proposition 1.3, (2). Now it can be verified that $m^* = a'm$ is an l.c.m. $\{a, m\}$ with the aid of Proposition 1.3, (3).

(d) (4)$\Longrightarrow$(5): Let p be an irreducible element of R such that $p \mid am$ for $a \in R$ and $m \in M$. If $p \nmid m$, then $am \in pM \cap Rm = Rpm$ by Proposition 1.2, (2). Therefore $p \mid a$, which means that (5) is true.

(e) (5)$\Longrightarrow$(6): (i) Let a and b be two elements of R such that aM

$\subseteq bM$. If $b = 0$, then $a = 0$ so that $b \mid a$. Assume that $b \neq 0$, then $b \mid am_0$ for any irreducible element $m_0 \in M$, whence $p \mid am_0$ for each prime factor p of b. By (5), $p \mid a$ for each prime factor p of b; hence $b \mid a$. (ii) If $a = 0$ or $b = 0$, then clearly $c = 0$. Assume that $a \neq 0$ and $b \neq 0$, and put $c \sim$ l.c.m. $\{a, b\}$ and $d \sim$ g.c.d. $\{a, b\}$. Then $aM \cap bM \supseteq cM$ and $c = a'b = ab'$, where $a' = a/d$ and $b' = b/d$. If w is any non-zero element of M such that $w = am = bm' \in aM \cap bM$, then $a'm = b'm'$. Since $a' \mid b'm'$ and $(a', b') \sim 1$, $a' \mid m'$ by the same argument as in the proof of part (i). Consequently, $c = a'b$ divides $w = bm'$. Thus $aM \cap bM \subseteq cM$, and (ii) is true.

(d) (6) $\Longrightarrow$ (2) $\Longrightarrow$ (1): If m is an irreducible element of M such that $am' = bm$ for some $a, b \in R$ and $m' \in M$; then $am' = bm \in aM \cap bM = cM$ for some $c \in R$ by (6), (ii), and in view of (6), (i), $b \mid c$ and $a \mid c$. Since m is irreducible, $b \sim c$ whence $a \mid b$ and $m \mid m'$. That is, m is primitive. Thus (2) is true and (2) $\Longrightarrow$ (1) by part (a) above. This completes the proof of Theorem 2.1.

Our next results give some basic information about factorial modules. Most of them were already discussed and proved in [7]; however, some proofs are rather unnecessarily lengthy. Here, we intend to give simpler proofs by applying Theorem 2.1. We assume that every module in the results is a non-zero module.

RESULT 2.1. *Every cyclic module Rm over a* UFD *R is a* UFM *in which every primitive element is an associate of m.*

RESULT 2.2. *Every vector space is a (trivial)* UFM *in which every non-zero vector is primitive.*

RESULT 2.3. *Let K be the field of quotients of a* UFD *R, and M an R-submodule of K. Then M is factorial if and only if it is cyclic. Hence an ideal of R is a* UFM *over R if and only if it is a principal ideal.*

PROOF. For any pair of non-zero elements $x = a/b$ and $y = c/d$ of M, we have $0 \neq bcx = ady \in Rx \cap Ry$. In view of the Corollary to Proposition 1.1, M has at most one cyclic submodule generated by a primitive element. Now, Result 2.3 is easy to see.

RESULT 2.4. *Every pure submodule N of a factorial module M over a* UFD *R is also a factorial R-module, in which every irreducible element remains irreducible in M.*

PROOF. Clearly N satisfies [UF1$'$]. For any $a \in R$ and $m \in M$, $aN \cap Rm = (aM \cap N) \cap Rm = (aM \cap Rm) \cap N = Rx \cap N = Rx$ for some $x \in M$ by (4) of Theorem 2.1 and the purity of N. Hence N is factorial. The second statement is obvious.

RESULT 2.5. *Let $\{M_i \mid i \in I\}$ be a set of modules over a* UFD *R. Then the following statements are equivalent:*
(1) $\prod_{i\in I} M_i$ *is factorial over R;*
(2) $\bigoplus_{i\in I} M_i$ *is factorial over R;*
(3) *each M_i is factorial over R.*

PROOF. Result 2.4 implies that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume (3) and put $\prod_{i\in I} M_i = M$. If $m = (m_i)_{i\in I} \in M$, where each $m_i = a_i m_i'$ for some $a_i \in R$ and an irreducible element $m_i' \in M_i$; then m is irreducible if and only if the set of elements $\{a_i \mid i \in I\}$ has no g.c.d. in R. Hence we can see that M satisfies [UF1]. Let p be any irreducible element of R such that $p \mid am$ in M for some $a \in R$ and $m = (m_i)_{i\in I} \in M$, then $p \mid am_i$ in M_i for every i. If $p \nmid a$ in R, then $p \mid m_i$ in M_i for every i by (5) of Theorem 2.1, as each M_i is factorial. Consequently $p \mid m$ and, therefore, $\prod_{i\in I} M_i = M$ is factorial due to (5) of Theorem 2.1 again. Thus (3) $\Rightarrow$ (1).

COROLLARY. (1) *Every free module over a* UFD *is factorial.* (2) *Every projective module over a* UFD *is factorial.* (3) *If R is a* UFD, *then the R-modules $R[x_i \mid i \in I]$ and $R[[x_i \mid i \in I]]$ are factorial for any index set I.*

RESULT 2.6. *Let M be a module over a unique factorization Bézout domain R, in particular, a principal ideal domain. Then* (1) *M is factorial if and only if it satisfies* [UF1], *and* (2) *if M is a* UFM, *then it is a faithfully flat R-module* (cf. [7, p. 45, Property 5.2]).

PROOF. (1) The necessity is trivial. To prove the sufficiency, let p be an irreducible element of R such that $p \mid am$ for $a \in R$ and $m \in M$. Suppose that $p \nmid a$, then there exist some s and t in R such that $(p, a) = 1 = ps + at$. Now we can see that $p \mid m$ since $m = psm + atm$; therefore, M is factorial by (5) of Theorem 2.1. (2) follows from [1, p. 44, Proposition 1, (d)].

3. **Prime submodules of factorial modules.** In this section we consider two families of submodules of a factorial module which play roles similar to those principal prime ideals do in a UFD.

DEFINITION 3.1. A proper submodule N of a module M which may not be torsion free over a ring R is called a *prime submodule* if $x \in M$, $a \in R$, and $ax \in N$ implies that either $x \in N$ or $a \in N : M$.

Evidently, every prime ideal P of a ring R is a prime submodule of the R-module R with $P : R = P$. It is also clear that every torsion free module contains the prime submodule (0).

In the following Result 3.1–Result 3.3, the modules M need to be torsion free.

RESULT 3.1. *If N is a prime submodule of an R-module M, then $N : M$ is a prime ideal of R.*

RESULT 3.2. *Every maximal submodule is prime.*

RESULT 3.3. *A proper submodule N of a module M is pure if and only if it is a prime submodule with $N : M = (0)$.*

RESULT 3.4. *If M is a free R-module, then PM is a prime submodule of M with $PM : M = P$ for each prime ideal P of R.*

PROPOSITION 3.1. *Let M be a module over an integral domain R and $m \in M$ such that $Rm \neq M$. Then m is primitive if and only if Rm is a prime submodule with $Rm : M = (0)$.*

This is a combined result of Proposition 1.1 and Result 3.3.

PROPOSITION 3.2. *Let M be a module over an integral domain R such that $pM \neq M$ for every non-unit element $p \in R$. Then the following two statements are equivalent:*

(1) *p is prime to M;*
(2) *pM is a prime submodule of M with $pM : M = (p)$.*

PROOF. It is straightforward to show that (1) implies (2). To prove the converse, firstly we notice that p is irreducible. For otherwise, we are led to the contradiction $pM = M$. Now the rest of the proof is easy.

THEOREM 3.1. *Let M be a factorial module over a* UFD *R. Then*

(1) *Rm is a prime submodule with $Rm : M = (0)$ for every irreducible element m of M such that $Rm \neq M$, and*

(2) *pM is a prime submodule with $pM : M = (p)$ for every irreducible element p of R.*

COROLLARY. *Let M be a cyclic module Rx over a* UFD *R and $m \in M$ such that $m \not\sim x$. Then,*

Rm is a prime submodule

$\Longleftrightarrow$ *$m \sim px$ for some irreducible element p of R,*

$\Longleftrightarrow$ *$Rm = pM$ for some irreducible element p of R.*

Proof. Put $m = ax$, where $a \in R$, then $Rm : Rx = (a)$. If Rm is prime, then (a) is a prime ideal by Result 3.1. Hence, $m \sim px$ for an irreducible element $p \in R$. Now, the proof of the corollary can be completed easily by applying Theorem 3.1, (2).

Definition 3.2. In a torsion free module, a non-zero prime submodule is said to be *minimal* if it contains properly no prime submodule other than (0).

Theorem 3.2. *Let N be a non-zero submodule of a factorial R-module M. Then* (1) *N is a minimal prime submodule with $N : M = (0)$ if and only if $N = R\eta \subsetneqq M$ for a primitive element η of M, and* (2) *N is a minimal prime submodule with $N : M \neq (0)$ if and only if $N = pM$ for an irreducible element $p \in R$.*

Proof. (1) If N is a minimal prime submodule with $N : M = (0)$, then N is pure and contains a primitive element $\eta \in M$. Since N is minimal and $R\eta$ is prime by Theorem 3.1, $N = R\eta$. The converse is easy to see because any non-zero prime submodule of M contained in $R\eta$, where η is primitive, is pure and therefore contains η. (2) The necessity follows from the fact that the non-zero prime ideal $N : M$ must contain an irreducible element p of R so that $pM \subseteq N$, where pM is a prime submodule by Theorem 3.1. To prove the sufficiency, let N' be a prime submodule of M contained in $N = pM$ for some irreducible element p of R. In Theorem 3.1 we have seen that N is a prime submodule with $N : M = (p)$, hence N contains no primitive element of M. We assert that $N' : M = (p)$; for otherwise, $N' : M = (0)$ and $N' \subseteq N$ contains a primitive element. Now, it is clear that $pM = N = N'$. Thus $N = pM$ is a minimal prime submodule with $N : M = (p) \neq (0)$.

It is known that an integral domain is a UFD if and only if every non-zero prime ideal contains a principal prime ([5, p. 4, Theorem 5]). We shall prove that a similar statement holds for modules in the next Theorem 3.3, where we exclude a trivial case of factorial modules — vector spaces.

Theorem 3.3. *Let R be a* UFD *which is not a field. An R-module M is a* UFM $\Longleftrightarrow$ (i) *M contains non-zero prime submodules, and* (ii) *each non-zero prime submodule possesses an element of the form $p\eta$ for a primitive element $\eta \in M$ and a prime element $p \in R$.*

Proof. The necessity follows from Theorem 3.1 and the fact that every non-zero prime submodule of a factorial module contains a minimal prime submodule. Note that either type of minimal prime submodule (cf. Theorem 3.2) contains an element of the form $p\eta$ as described in the theorem. To prove the sufficiency let S be the set of all primitive elements in M and put $S^* = \{a\eta \mid \eta \in S \text{ and } a \in R - \{0\}\}$; then $S^* \neq \emptyset$ since $S \neq \emptyset$ by the hypothesis. We remark that M is factorial if and only if $S^* = M - \{0\}$. Suppose that $S^* \subsetneq M - \{0\}$ and let e be a non-zero element of $M - S^*$, then $Re \subseteq M - S^*$. For otherwise, $e \in S^* \cap (M - S^*) = \emptyset$, because some primitive element divides e in M if $Re \cap S^* \neq \emptyset$. Next, we extend Re to a submodule N of M maximal with respect to exclusion of S^*; such N must exist by Zorn's Lemma. We assert that N is pure. If it is not so, then there exist some $0 \neq a \in R$ and $m \in M - N$ such that $am \in aM \cap N$. Since $(N + Rm) \cap S^* \neq \emptyset$, there exists an element $w \in S^*$ such that $w = x + rm$ for some $x \in N$ and $r \in R$, which implies the contradiction $aw = ax + ram \in N \cap S^* = \emptyset$. Thus N is pure, so that it is prime. Due to the hypothesis, we have that $N \cap S^* \neq \emptyset$, a contradiction. Therefore $S^* = M - \{0\}$, hence, M is factorial. This completes the proof.

Remark. In the special case $M = R$, Theorem 3.3 is identical with the above mentioned Theorem 5 of [5, p. 4] for UFD. Recall that η is primitive in the R-module $M = R$ if and only if η is a unit.

4. **Factorial Extensions.** Let B be a unitary overring of a UFD A. We call B a factorial extension of A if B is factorial as an A-module (cf. [7, p. 48, §7]).

Let K be the field of quotients of A. We can see easily that $B \cap K = A$ if and only if $a'B \subseteq aB$ implies that $a \mid a'$ in A for any a and $a' \in A$. Thus in view of Theorem 2.1, (6), we have

Theorem 4.1. *Let B be a unitary overring of a* UFD *A which is a torsion free A-module. Then B is a factorial extension of A if and only if* (1) *the A-module B satisfies* [UF1], (2) $B \cap K = A$, *and* (3) *for any pair of elements a_1, $a_2 \in A$, there exists an element $a_3 \in A$ such that $a_1B \cap a_2B = a_3B$.*

Suppose that B is an integral domain containing a UFD A and that B satisfies the a.c.c. for principal ideals. If $B \cap K = A$, then B also satisfies [UF1′], the a.c.c. for cyclic A-submodules. This gives rise to the following series of corollaries to Theorem 4.1:

Corollary 1. *Let B be an integral domain containing a* UFD *A. If* (1) *B satisfies the* a.c.c. *for principal ideals and* (2) *B is a faithfully flat A-module, then B is a factorial extension of A.*

Corollary 2. *Let R be a* UFD *which is a Zariski ring with respect to an ideal m such that the m-adic completion $\hat{R}$ of R is an integral domain. Then $\hat{R}$ is a factorial extension of R* ([2, p. 72, Proposition 9]).

Corollary 3. *If R is a regular local ring, then the completion $\hat{R}$ of R for the natural topology is a factorial extension of R.*

Corollary 4. *Let A and B be two* UFD's *such that $A \subseteq B$. Then B is a factorial extension of A if and only if* (1) $B \cap K = A$ *and* (2) *for any pair a_1, $a_2 \in A$,* l.c.m. $\{a_1, a_2\} \in A$ *and* l.c.m. $\{a_1, a_2\} \in B$ *are associates in B* ([7, p. 49, Theorem 7.5]).

Theorem 4.2. *Let A and B be two* UFD's *such that $A \subseteq B$. Suppose that a B-module M is a* UFM, *then M is factorial over A if and only if B is a factorial extension of A.*

Proof. (Necessity): If $a'/a = b \in B \cap K$, then $a'M = abM \subseteq aM$ which implies that $a \mid a'$ in A as M is factorial over A. Therefore, $B \cap K = A$. Next, for any pair a_1, $a_2 \in A$, l.c.m. $\{a_1, a_2\} = a_3 \in A$ and l.c.m. $\{a_1, a_2\} = b \in B$ are associates in B since $a_1M \cap a_2M = a_3M = bM$ from the hypothesis that M is factorial over both A and B. Hence B is a factorial extension of A by Corollary 4 to Theorem 4.1.

(Sufficiency): We assume that M is factorial over B and B is factorial over A. Let $0 \neq m \in M$, then $m = b\eta$, where $b \in B$ and $\eta \in M$ is irreducible over B. Let $b = ab'$, where $a \in A$ and $b' \in B$ is irreducible over A, then $m = a(b'\eta)$, and it can be verified that the element $b'\eta$ is a primitive element of the A-module M. This proves that M is factorial over A.

Corollary. *Let A, B, C be rings such that $A \subseteq B \subseteq C$ and that C is a factorial extension of B. Then C is a factorial extension of A if and only if B is that of A.*

5. **Factorization in modules of fractions and topological modules.** It is well known that if R is a UFD, then the ring R_S of fractions is also a UFD for any multiplicatively closed set S of R such that $0 \notin S$. For modules, similarly, we have

Theorem 5.1. *Let S be a multiplicatively closed set of a* UFD R *such that $0 \notin S$. If M is a factorial R-module, then the module of fractions M_S is a factorial R_S-module.*

Proof. M_S satisfied [UF1] since every irreducible element of M remains irreducible in M_S. Let P be a system of representatives of irreducible elements of R_S; we may choose each representative from R so that $P \subseteq R$. Since M is factorial over R, each $p \in P$ is prime to M by Theorem 2.1, (5); moreover $p \nmid s$ for each $s \in S$ (cf. [9, p. 29, Theorem 4]). Applying these properties of elements of P, we can prove that every $p \in P$ is prime to the R_S-module M_S. Hence, M_S is factorial over R_S by Theorem 2.1, (5) again.

We remark that the ring of fractions R_S of a UFD R is not a factorial extension of R in general. For example, no element of the field of quotients of R is irreducible over R.

Next, we consider a modification to UFM's of Nagata's theorem for UFD's ([9, p. 31, Theorem 5]).

THEOREM 5.2. *Let M be a module over a* UFD *R satisfying* [UF1]. *Let S be the multiplicatively closed set of R generated by any family P' of elements which are prime to M. If the R_S-module M_S is factorial, so is the R-module M.*

PROOF. Let P and P'' be systems of representatives of irreducible elements of R and R_S respectively, then $P = P' \cup P''$. In view of Theorem 2.1, (5), it is sufficient to prove that every irreducible element of R is prime to M. This is trivially true for each $p \in P'$. Suppose that $p \in P''$ and $p \mid am$ in M for some $a \in R$ and $m \in M$. Since M_S is a UFM, $p \mid a$ in R_S or $p \mid m$ in M_S. Applying the facts that $(s, p) = 1$ in R for every $s \in S$ and that every irreducible factor of $s \in S$ is prime to M, we can prove that $p \mid a$ in R or $p \mid m$ in M. Thus every p in P'', hence in P, is prime to the R-module M.

THEOREM 5.3. *Let R be a noetherian* UFD *and m an ideal of R such that the completion $\hat{R}$ for the m-adic topology is a* UFD. *Let M be a finitely generated R-module and $\hat{M}$ the completion of M for the m-adic topology. If M is a factorial module over R, then $\hat{M}$ is a factorial module over $\hat{R}$.*

PROOF. Since the noetherian UFD $\hat{R}$ is a flat R-module and M is torsion free over R, the $\hat{R}$-module $\hat{M} = \hat{R} \otimes_R M$ is torsion free over $\hat{R}$ by [1, p. 65, Ex. 20]. Clearly, $\hat{R}$ is noetherian as R is a Hausdorff space. According to [7, p. 43, Theorem 4.6], it suffices to show that M is a reflexive module over R. This however, is easy to see from [3, p. 53, Proposition 8].

A ring with a linear topology is called a *Gelfand ring* if its radical is open. For example, all Zariski rings are Gelfand rings. A generalization of Mori's theorem for Zariski rings to Gelfand rings was studied in [6]. Applying its results, here we investigate a similar problem for modules over Gelfand rings.

THEOREM 5.4. *Let $(R, (q_n)_{n=0}^{\infty})$ be a filtered Gelfand domain such that the completion $\hat{R}$ of R is an integral domain which is a faithfully flat R-module. Let $(M, (q_n M)_{n=0}^{\infty})$ be a finitely generated filtered R-module, which is a Hausdorff space. If the completion $\hat{M}$ of M is factorial over $\hat{R}$, then M is factorial over R.*

PROOF. Firstly, we note that $\hat{M} = \hat{R}M$. Since $\hat{M}$ is a UFM over $\hat{R}$, $\hat{R}$ is a UFD. According to [6, p. 374, Corollary 1 to Theorem 1], R is a UFD. Moreover $\hat{R}$ is a factorial extension of R by Corollary 1 to Theorem 4.1. Consequently, $\hat{M}$ is factorial over R due to Theorem 4.2. Next, we assert that M is a pure R-submodule of $\hat{M}$. For, if $r \in R$, then $r\hat{M} = r(\hat{R}M) = \hat{R}(rM)$ so that $\hat{R}(rM) \cap M = r\hat{M} \cap M = rM$ by [1, p. 52, Proposition 10, (ii)]. Now it is clear that M is factorial over R from Result 2.4.

COROLLARY. *Let M be a finitely generated module over a regular local ring R and $\hat{R}$ the completion of R for the natural topology. Then M is factorial over R if and only if the completion $\hat{M}$ of M is factorial over $\hat{R}$.*

6. *$M[x_i \mid i \in I]$ and $M[[x_i \mid i \in I]]$.* Let I be any nonempty set. Following the definition and notation in [2, p. 112 Ex. 22], the ring $R[[x_i \mid i \in I]]$ of formal power series in indeterminates $\{x_i\}_{i \in I}$ over a ring R can be identified with $R^{N^{(I)}}$, which we simply write R_I. If, in particular, $I = \{1, 2, \cdots, n\}$, then $R^{N^{(I)}} = R^{N^I} = R[[x_1, x_2, \cdots, x_n]]$ and the ring $R[x_1, x_2, \cdots, x_n]$ of polynomials is its subring. We extend these notions to $M_I = M^{N^{(I)}}$ for any R-module M and call each element of M_I a power series in the indeterminates $\{x_i\}_{i \in I}$ with coefficients in M; we also denote M_I by $M[[x_i \mid i \in I]]$, and put $N^{(I)} = I^*$. Clearly M_I is an abelian group under termwise addition and has a structure of R_I-module if for each $f = (a_i)_{i \in I^*} \in R_I$ and $F = (m_i)_{i \in I^*} \in M_I$ we define $fF = (w_i)_{i \in I^*}$ such that $w_i = \sum_{s+t=i} a_s m_t$ for every i. Accordingly, $M^{N^I} = M[[x_1, x_2, \cdots, x_n]]$ for $I = \{1, 2, \cdots, n\}$ is a module over $R[[x_1, x_2, \cdots, x_n]]$, and the subset $M[x_1, x_2, \cdots, x_n]$ of polynomials in $x_1, x_2, \cdots, x_n$ with coefficients in M forms a module over $R\,[x_1, x_2, \cdots, x_n]$ (cf. [8, p. 29]). We remark that $M[x_1, x_2, \cdots, x_n] \cong R[x_1, x_2, \cdots, x_n] \otimes_R M$ for each $n \geqq 1$. If M is a torsion free R-module, then it is easy to see that R_I-modules M_I and $R[x_1, x_2, \cdots, x_n]$-modules $M[x_1, x_2, \cdots, x_n]$ are also torsion free for any I and $n \geqq I$. They are known to be noetherian modules for any finite set I and $n \geqq 1$ when M is noetherian (cf. [8, p. 30, Theorem 10 and p. 68, Ex. 10]).

We now begin an investigation of the factorial property of $M[x]$ and $M[[x]]$. If R is an integral domain satisfying the a.c.c. for principal ideals, then both $R[x]$ and $R[[x]]$ satisfy the a.c.c. for principal ideals. Similar statements hold for $M[x]$ and $M[[x]]$ respectively. More generally, we have

PROPOSITION 6.1. *Let R be an integral domain satisfying the* a.c.c. *for principal ideals and M an R-module. If M satisfies the* a.c.c. *for* *cyclic submodules, so do both the* $R[x_i \mid i \in I]$*-module* $M[x_i \mid i \in I]$ *and the* $R[[x_i \mid i \in I]]$*-module* $M[[x_i \mid i \in I]]$ *for any* I.

If M is a UFM over a UFD R, then every irreducible element p of R is prime to M. We remark that p remains prime to the $R[x]$-module $M[x]$. Therefore $S = R - \{0\}$ is a multiplicatively closed set in $R[x]$ generated by the family $P = \{p \mid p \in R$, and p is irreducible in $R[x]\}$ of elements which are prime to $M[x]$.

THEOREM 6.1. *If* M *is a factorial module over a* UFD R, *then the* $R[x]$*-module* $M[x]$ *is also factorial.*

PROOF. Let $S = R - \{0\}$ and K the field of quotients of R. Applying Theorem 5.1 and Proposition 6.1 we can verify that the module $(M[x])_S = M_S[x]$ over the principal ideal domain $(R[x])_S = K[x]$ satisfies [UF1′]; hence it is factorial by Result 2.6. It follows immediately that $M[x]$ is factorial over $R[x]$ from Theorem 5.2.

COROLLARY. *If* M *is factorial over a* UFD R, *then* $M[x_i \mid i \in I]$ *is factorial over* $R[x_i \mid i \in I]$ *for any set* I.

THEOREM 6.2. *Let* R *be a noetherian* UFD *such that* $R[[x]]$ is a UFD. *If a finitely generated* R*-module* M *is factorial, so is the* $R[[x]]$*-module* $M[[x]]$.

PROOF. We note that $M[[x]]$ is the completion of the finitely generated $R[x]$-module $M[x]$ for the (x)-adic topology. Since $M[x]$ is factorial over $R[x]$ by Theorem 6.1, $M[[x]]$ is factorial over $R[[x]]$ due to Theorem 5.3.

COROLLARY 1. *Let* R *be a noetherian* UFD *such that* $R[[x_1, x_2, \cdots, x_n]]$ *is a* UFD *for an* $n \geqq 1$ *and* M *a finitely generated* R*-module. If* M *is a* UFM, *so is the* $R[[x_1, x_2, \cdots, x_n]]$*-module* $M[[x_1, x_2, \cdots, x_n]]$.

COROLLARY 2. *Let* M *be a finitely generated module over a regular* UFD R. *If* M *is a* UFM, *so is the* $R[[x_1, x_2, \cdots, x_n]]$*-module* $M[[x_1, x_2, \cdots, x_n]]$ *for every* $n \geqq 1$ (cf. [5, p. 137, Theorem 188]).

Let M be a free module of rank I over an integral domain R. Let $A = R[[x_1, x_2, \cdots, x_n]]$ and $E = M[[x_1, x_2, \cdots, x_n]]$ for any $n \geqq 1$, where A is equipped with the $(x_1, x_2, \cdots, x_n)$-adic topology. If I is finite, then clearly $E \cong A^I$ as an A-module; but if I is infinite, then $E \ncong A^I$. In fact, E is isomorphic to the submodule $A_{(I)}$ of the A-module A^I which consists of families $(f_i)_{i \in I}$ of elements $f_i \in A$ such that $\lim f_i = 0$ with respect to the filter of complements of finite parts of I([2, p. 121, Ex. 16, Ex. 17]).

PROPOSITION 6.2. *$A_{(I)} \cong M[[x_1, x_2, \cdots, x_n]]$ is a pure submodule of the free A-module A^I.*

PROOF. Let $0 \neq g \in A$ and $(f_i)_{i \in I} \in A^I$. Then $g(f_i)_{i \in I} = (gf_i)_{i \in I} \in gA^I \cap A_{(I)} \Longleftrightarrow \lim gf_i = g \lim f_i = 0 \Longleftrightarrow \lim f_i = 0 \Longleftrightarrow (f_i)_{i \in I} \in A_{(I)}$. This proves that $gA^I \cap A_{(I)} = gA_{(I)}$ for every $g \in A$.

Proposition 6.2 implies that $A_{(I)}$ is a factorial A-module if A is a UFD, by Result 2.4 and Result 2.5. Thus we have

THEOREM 6.3. *Let R be a* UFD *such that $R[[x_1, x_2, \cdots, x_n]]$ is a* UFD *for an $n \geqq 1$. If M is a free (respectively, projective) R-module, then the $R[[x_1, x_2, \cdots, x_n]]$-module $M[[x_1, x_2, \cdots, x_n]]$ is factorial.*

THEOREM 6.4. *Let I be any infinite index set. Let M be a factorial module over a* UFD *R such that $M[[x_i \mid i \in J]]$ is a factorial module over $R[[x_i \mid i \in J]]$ for every finite subset J of I. Then $M[[x_i \mid i \in I]]$ is factorial over $R[[x_i \mid i \in I]]$.*

PROOF. $M[[x_i \mid i \in I]] = M_I$ satisfies [UF1′] by Proposition 6.1. Let Λ be the set of all finite subsets of I. Since $M[[x_i \mid i \in J]] = M_J$ is a UFM over $R[[x_i \mid i \in J]] = R_J$ for every $J \in \Lambda$, R_J is a UFD for every J by [7, p. 37, Property 2.2]. Therefore, $R[[x_i \mid i \in I]] = R_I$ is a UFD by [4]. Let P be an irreducible element of R_I such that $P \mid fG$ in M_I for $f \in R_I$ and $G \in M_I$. Let $(P)_J$ be the F-projection of P in R_J (cf. [4, p. 48]); that is, $(P)_J$ is the element of R_J obtained from P by substituting 0 for all x_i such that $i \notin J$. We can define the F-projection $(G)_J$ of G in M_J in a similar way and see easily that $(fG)_J = (f)_J(G)_J$ for every $J \in \Lambda$. According to [4, p. 56, L. 9], there exists a $J' \in \Lambda$ such that $(f)_{J'}$ is irreducible in $R_{J'}$. If $P \nmid f$ in R_I and $P \nmid G$ in M_I, then there exists a $J \in \Lambda$ so large that $J' \subset J$, $(P)_J$ is irreducible in R_J, and $(P)_J \nmid (f)_J$ in R_J and $(P)_J \nmid (G)_J$ in M_J. Since M_J is a UFM over R_J, this is a contradiction in view of Theorem 2.1, (5). Hence $P \mid f$ or $P \mid G$, so that M_I is factorial over R_I.

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