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FACTORIZATIONS OF INFINITE SOLUBLE GROUPS

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Introduction. If the group $G = AB$ is the product of two of its subgroups A and B, then G is said to have a factorization with factors A and B, and G is factorized by its subgroups A and B. The main problem about factorized groups is the following question: What can be said about the structure of the factorized group $G = AB$ if the structure of its subgroups A and B is known?

The most important of the few known results of this type are the following two theorems. A theorem of Itô states that every group which is factorized by two abelian subgroups is metabelian; see [**6**] or Huppert [**5**], Satz 4.4, p. 674. A theorem of Kegel and Wielandt states that every finite group which is factorized by two nilpotent subgroups is soluble; see [**7**] and [**12**] or Huppert [**5**], Hauptsatz 4.3, p. 674. Some extensions of the second theorem to certain classes of infinite groups are given in Kegel [**8**], another improvement for finite groups was recently obtained by Pennington in [**9**].

In the following mainly soluble factorized groups $G = AB$ are considered. In order to have induction arguments available, we must assume that one of the two subgroups, B say, is polycyclic or a Černikov group. Also, in the polycyclic case some essential arguments work only if at least one of the two subgroups A and B is locally nilpotent.

If the subgroup B is polycyclic, the following theorem holds.

THEOREM A. *Let $\mathfrak{X}$ be a class of groups closed under the forming of subgroups, epimorphic images and extensions. If the soluble group $G = AB$ is factorized by an $\mathfrak{X}$-subgroup A and a polycyclic $\mathfrak{X}$-subgroup B, and if A or B is locally nilpotent, then G is an $\mathfrak{X}$-group.*

Among the possible candidates for the class $\mathfrak{X}$ in Theorem A are the class of polycyclic groups, the class of poly-π-minimax groups (for a set of primes π) and the class of groups of finite rank (in the sense of Prüfer). Here a soluble group G is called a *poly-π-minimax group* if every abelian factor F of G is an extension of a noetherian group by an artinian group and if π contains every prime p for which F has an infinite p-quotient group. A group G is said to be of *finite rank* (in the sense of Prüfer) if there exists a positive integer r such that every finitely generated subgroup of G can be generated by r (or fewer) elements.

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Another possible candidate for the class $\mathfrak{X}$ in Theorem A is the class of soluble groups with finite abelian section rank. Here a soluble group G is said to be of ***finite abelian section rank*** if every abelian factor (section) of G has finite p-rank for $p = 0$ or a prime. Recall that the p-rank (0-rank) of an abelian group X is the cardinal of a maximal independent set of elements of order a power of the prime p (of infinite order) in X.

For properties of these group classes $\mathfrak{X}$, in particular their subgroup, epimorphism and extension inheritance, the reader is referred to the books [**10**] and [**11**] of Robinson. These should also be consulted for all other notations and facts quoted.

To obtain a similar result when the subgroup B is a Černikov group, an additional requirement for the class $\mathfrak{X}$ is needed.

THEOREM B. *Let $\mathfrak{X}$ be a class of groups closed under the forming of subgroups, epimorphic images and extensions such that every soluble $\mathfrak{X}$-group has finite abelian section rank. If the soluble group $G = AB$ is factorized by an $\mathfrak{X}$-subgroup A and a Černikov $\mathfrak{X}$-subgroup B, then G is an $\mathfrak{X}$-group.*

Among the possible candidates for the class $\mathfrak{X}$ in Theorem B are the class of Černikov groups, the class of poly-π-minimax groups (for some set of primes π), the class of groups of finite rank and the class of soluble groups with finite abelian section rank.

On the other hand, the requirement that soluble $\mathfrak{X}$-groups have finite abelian section rank, excludes some interesting group classes. Such a class of groups which is closed under the forming of subgroups, epimorphic images and extensions, but which is too large to be taken for $\mathfrak{X}$ in Theorem B, is the class of groups of finite torsionfree rank, which is defined as follows; see Zassenhaus [13], exercise 21, p. 241. A group G is said to be of *finite torsionfree rank* if it contains finitely many subgroups G_i such that $1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$, where G_i is normal in G_{i+1}, and each factor G_{i+1}/G_i is a torsion group or infinite cyclic. The number of infinite cyclic factors in such a series is an invariant of G and will be called the *Hirsch number*, $h(G)$, of G.

For the class of groups with finite torsionfree rank, which contains all other special group classes mentioned so far, there is the following result.

THEOREM C. *If the soluble group $G = AB$ is factorized by a subgroup A with finite torsionfree rank and a subgroup B which is polycyclic or a Černikov group, and if in the first case A or B is locally nilpotent, then G has finite torsionfree rank with Hirsch number*

$$h(G) = h(A) + h(B) - h(A \cap B).$$

This theorem gives the number of infinite cyclic factors in certain normal series of G. In some cases a similar precise statement can be made about the quasicyclic subgroups. If X is any group, the maximal torsion normal subgroup of X is denoted by $\mathfrak{T}X$. If X is a Černikov group, denote by $\mathfrak{D}X$ its radicable radical, i.e. the subgroup of X which is generated by all quasicyclic subgroups of X. It is well-known that $\mathfrak{D}X$ is abelian and the direct product of finitely many quasicyclic groups of type p^{∞} for finitely many primes p; see for instance Robinson [10], p. 67.

THEOREM D. *If the soluble group $G = AB$ is factorized by a subgroup A with finite abelian section rank such that $\mathfrak{T}A$ is a Černikov group and a Černikov subgroup B, then G has finite abelian section rank, $\mathfrak{T}G$ is a Černikov group and*

$$\mathfrak{D}(\mathfrak{T}G) = (\mathfrak{D}(\mathfrak{T}A))(\mathfrak{D}B).$$

It would be interesting to know whether Theorem D also holds if the subgroup B is polycyclic. Note that the statement in Theorem D that G has finite abelian section rank follows already from Theorem B. However, it cannot directly be deduced from this theorem that $\mathfrak{T}G$ is a Černikov group, since the class of soluble groups G with finite abelian section rank such that $\mathfrak{T}G$ is a Černikov group is not closed under epimorphisms.

Another class of groups $\mathfrak{X}$, for which results similar to Theorems A and B hold, though it is not closed under the forming of extensions, is the class of groups which are an extension of an artinian group by a noetherian group; see Theorem 5.3.

Theorem D is essentially a corollary of Theorem B. The proofs of Theorems A, B and C are by induction on the length of the derived series of the soluble group $G = AB$. If the subgroup B is polycyclic, induction is then made on the Hirsch number and finally on the order of the finite maximal torsion normal subgroup of B; if the subgroup B is a Černikov group, induction is made on the sum of the primary ranks of B and then on the index of the radicable radical $\mathfrak{D}B$ of B. In both cases we ultimately arrive at a group factorized by two isomorphic subgroups which are factors of A and B. Since in the polycyclic case the nilpotency requirement is only needed in this situation, it follows easily from the proof that this condition could be omitted from Theorems A and C if it can be shown that every soluble group which is factorized by two isomorphic polycyclic subgroups is polycyclic. Some of the arguments do not depend on the solubility of the group G and could be used to obtain criteria for a hyperabelian group factorized by two $\mathfrak{X}$-subgroups one of which is polycyclic or a Černikov group, to be a hyper-$\mathfrak{X}$-group.

This paper is a continuation of [3] and uses and generalizes some of the results obtained there.

Notations

$X \subseteq Y =$ X is a subgroup of Y
$X \subset Y =$ X is a proper subgroup of Y
$\langle \cdots \rangle =$ subgroup generated by $\cdots$
$AB =$ set of all elements ab where $a \in A$ and $b \in B$
$X_G =$ largest normal subgroup of G which is contained in X
$\mathfrak{c}_G X =$ centralizer of X in G
$\mathfrak{n}_G X =$ normalizer of X in G
$\mathfrak{T} G =$ maximal torsion normal subgroup of G
$\mathfrak{D} G =$ semi-radicable radical of G
$G^{(0)} = G$, $G^{(i+1)} =$ commutator subgroup of $G^{(i)}$
$\pi =$ set of primes
factor = epimorphic image of a subgroup
chief factor = minimal normal subgroup of an epimorphic image
noetherian (artinian) group = group with maximum (minimum) condition for subgroups
polycyclic group = noetherian and soluble group
Černikov group = artinian group which contains an abelian (normal) subgroup of finite index
hyper-$\mathfrak{X}$-group = group whose nontrivial epimorphic images contain nontrivial normal subgroups with property $\mathfrak{X}$

1. **Groups with finite torsionfree rank.** A group G has *finite torsionfree rank* if there is a series of finite length n, $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$, where G_i is a normal subgroup of G_{i+1}, such that each factor G_{i+1}/G_i is infinite cyclic or a torsion group; see Zassenhaus [13], exercise 21, p. 241. The number of infinite cyclic factors in such a series is an invariant of G and will be called the *Hirsch number*, $h(G)$, of G. The Hirsch number has the following property: If the normal subgroup N of the group G has finite Hirsch number h and if G/N has finite Hirsch number h^*, then G has finite Hirsch number $h + h^*$.

If the group $G = AB$ with finite torsionfree rank is factorized by two subgroups A and B, we say that the *rank formula* holds if $h(G) = h(A) + h(B) - h(A \cap B)$. The following lemmas are concerned with the validity of the rank formula in a factorized group $G = AB$.

Problems about factorized groups $G = AB$ usually become trivial if at least one of the two subgroups A and B is a subnormal in G.

LEMMA 1.1. *If the group $G = AB$ is factorized by two subgroups A and B with finite torsionfree rank, one of which is subnormal in G, then G has finite torsionfree rank and the rank formula holds for $G = AB$.*

PROOF. Assume that the lemma is false, and let $G = AB$ be a counterexample where the length of a subnormal series of the subnormal subgroup A of G is minimal. If A is normal in G, then $G/A = AB/A \simeq B/(A \cap B)$. Since A and B have finite torsionfree rank, G has finite torsionfree rank. Furthermore

$$\begin{aligned} h(G) &= h(A) + h(G/A) = h(A) + h(B/(A \cap B)) \\ &= h(A) + h(B) - h(A \cap B). \end{aligned}$$

If A is not normal in G, A is normal in a subnormal subgroup S of G where the length of a subnormal series of S in $G = SB$ is less than that of A in G. We have

$$\begin{aligned} (B \cap S)/(A \cap B) &= (B \cap S)/(A \cap B \cap S) \simeq (B \cap S)A/A \\ &= (S \cap AB)/A) = S/A. \end{aligned}$$

By induction it follows that G has finite torsionfree rank and

$$\begin{aligned} h(G) &= h(S) + h(B) - h(S \cap B) \\ &= h(A) + h(S/A) + h(B) - (h(A \cap B) + h((B \cap S)/(A \cap B)) \\ &= h(A) + h(B) - h(A \cap B). \end{aligned}$$

This proves the lemma.

REMARK 1.2. Let the class of groups $\mathfrak{X}$ be closed under the forming of subgroups, epimorphic images and extensions. If the group $G = AB$ is factorized by two $\mathfrak{X}$-subgroups A and B, one of which is subnormal in G, then a similar argument shows that G is an $\mathfrak{X}$-group.

If N is a normal subgroup of the factorized group $G = AB$, then by [3], Theorem 1.7, the subgroup $X = X(N) = AN \cap BN$ is the smallest factorized subgroup of G containing N and $X = (A \cap BN)(B \cap AN)$ and $A \cap B \subseteq X$.

LEMMA 1.3. *If N is a normal subgroup of the group $G = AB$ with finite torsionfree rank which is factorized by two subgroups A and B*

and if the rank formula holds for the factorized subgroup $X(N) = AN \cap BN = (A \cap BN)(B \cap AN)$ *and for the quotient group* $G/N = (AN/N)(BN/N)$, *then the rank formula holds for* $G = AB$.

PROOF. Note first that for the subgroup $X = X(N)$ we have

$$(A \cap BN)/(A \cap N) \simeq (A \cap BN)N/N = (AN \cap BN)/N = X/N.$$

Hence $h((A \cap BN)/(A \cap N)) = h(X/N)$. Similarly $h((B \cap AN)/(B \cap N)) = h(X/N)$. This is used in the following series of equalities

$$\begin{aligned} h(G) &= h(G/N) + h(N) \\ &= h(G/N) + h(X) - h(X/N) \\ &= h(AN/N) + h(BN/N) - h(AN \cap BN/N) + h(X) - h(X/N) \\ &= h(A/(A \cap N)) + h(B/(B \cap N)) - h(X/N) + h(X) - h(X/N) \\ &= h(A) - h(A \cap N) + h(B) - h(B \cap N) - h(X/N) - h(X/N) \\ &\quad + h(A \cap BN) + h(B \cap AN) - h(A \cap B) \\ &= h(A) + h(B) - h(A \cap B) + h(A \cap BN) - h(A \cap N) - h(X/N) \\ &\quad + h(B \cap AN) - h(B \cap N) - h(X/N) \\ &= h(A) + h(B) - h(A \cap B) + h(X/N) - h(X/N) + h(X/N) \\ &\quad - h(X/N) \\ &= h(A) + h(B) - h(A \cap B). \end{aligned}$$

This proves the lemma.

LEMMA 1.4. *If* N *is a normal subgroup of the factorized group* $G = AB$ *with finite torsionfree rank which is contained in* A, *and if the rank formula holds for the factorized group* $G/N = (A/N)(BN/N)$, *then the rank formula holds for* $G = AB$.

PROOF. We have

$$\begin{aligned} A \cap BN/N = N(A \cap B)/N &\simeq (A \cap B)/(A \cap B \cap N) \\ &= (A \cap B)/(N \cap B), \end{aligned}$$

so that $h(A \cap B) = h(A \cap BN/N) + h(B \cap N)$. This can be used to conclude

$$\begin{aligned} h(G) &= h(N) + h(G/N) \\ &= h(N) + h(A/N) + h(BN/N) - h(A \cap BN/N) \end{aligned}$$

$$= h(A) + h(B/(B \cap N)) - h(A \cap B) + h(B \cap N)$$

$$= h(A) + h(B) - h(A \cap B).$$

The lemma is proved.

Let the soluble group $G = AB$ be factorized by two subgroups A and B with finite torsionfree rank. Does then G have finite torsionfree rank and does the rank formula hold for $G = AB$? The following theorem answers a special case of this problem and contains Theorem C as a corollary.

THEOREM 1.5. *If the soluble group $G = AB$ is factorized by a subgroup A with finite torsionfree rank and a subgroup B which is locally-normal-by-finite or polycyclic and if in the second case A or B is locally nilpotent, then G has finite torsionfree rank and the rank formula holds for $G = AB$.*

PROOF. Assume that the theorem is false. Among the counterexamples $G = AB$ with $h(A) + h(B)$ minimal consider one with minimal derived length. Let $K = G^{(k)} \neq 1$ be the last nontrivial term of the derived series of G. Then by induction G/K has finite torsionfree rank and the rank formula holds for $G/K = (AK/K)(BK/K)$. Without loss of generality it may be assumed that the maximal torsion normal subgroup of G is trivial. This implies that the abelian normal subgroup K of G must be torsionfree. By [3], Theorem 1.7 (b)

$$X = X(K) = AK \cap BK = KA^* = KB^* = A^*B^*$$

where $A^* = A \cap BK$ and $B^* = B \cap AK$. By Lemma 1.3 the theorem does not hold for the factorized group $X = A^*B^*$.

Since K is torsionfree, $A^* \cap K$ is a torsionfree normal subgroup of $X = KA^*$, and $B^* \cap K$ is a torsionfree normal subgroup of $X = KB^*$. Assume that $A^* \cap K = 1$ and $B^* \cap K = 1$. Then

$$A^* \simeq A^*K/K = X/K = B^*K/K \simeq B^*.$$

Hence $X = A^*B^*$ is factorized by two isomorphic subgroups A^* and B^* which are locally-normal-by-finite or polycyclic and nilpotent. In the first case X is locally finite by [3], Theorem 5.4. In the second case X is polycyclic by [3], Theorem 4.3, and by [3], Theorem 4.4 (a) the Fitting subgroup $F = \mathfrak{F}X$ of X is factorized. Since in this case F is nilpotent, the rank formula holds for $F = (A^* \cap F)(B^* \cap F)$ by Lemma 1.1. By induction the rank formula holds for the factorized group $X/F = (A^*F/F)(B^*F/F)$. It follows from Lemma 1.3 that the rank formula holds for the factorized group $X = A^*B^*$. This contradiction shows that $A^* \cap K \neq 1$ or $B^* \cap K \neq 1$.

Let $N = A^* \cap K \neq 1$, say. Thus N is a nontrivial torsionfree normal subgroup of $X = A^*B^*$ which is contained in A^*. Therefore $h(A^*/N) < h(A^*) \leqq h(A)$. It follows that $X/N = A^*B^*/N = (A^*/N)(B^*N/N)$ has finite torsionfree rank and the rank formula holds. Since N and X/N have finite torsionfree rank, X has finite torsionfree rank. By Lemma 1.4 the rank formula holds for $X = A^*B^*$. This contradiction proves the theorem.

Remark. The proof of the above theorem shows that every soluble group $G = AB$ factorized by two subgroups A and B with finite torsionfree rank has finite torsionfree rank and the rank formula holds for $G = AB$ if this is the case for every soluble group factorized by two isomorphic subgroups with finite torsionfree rank.

It can be expected that these problems are easier to handle for groups factorized by two abelian subgroups. The following theorem shows that it suffices to show that every group $G = AB \neq 1$ which is factorized by two abelian subgroups A and B with finite torsionfree rank has a nontrivial normal subgroup contained in A or B.

Theorem 1.6. *If the group $G = AB$ is factorized by two abelian subgroups A and B with finite torsionfree rank, then G has finite torsionfree rank and the rank formula holds for $G = AB$, if the following condition is satisfied:*
(+) *If $G/N = (AN/N)(BN/N) \neq 1$ is an epimorphic image of G, then there exists a normal subgroup $M/N \neq 1$ of G/N which is contained in AN/N or BN/N.*

Proof. Assume that the theorem is false, and let $G = AB$ be a counterexample where $h(A) + h(B)$ is minimal. Without loss of generality $A \cap B = 1$, and it may also be assumed that the maximal torsion normal subgroup of G is trivial. By (+) there exists a normal subgroup $M \neq 1$ of $G \neq 1$ which is contained in A, say. Then M is a torsionfree abelian group with finite rank and the rank formula holds trivially for $M = (A \cap M)(B \cap M)$. By induction $G/M = (A/M)(BM/M)$ is no counterexample. Hence G/M has finite torsionfree rank and the rank formula holds for G/M. It follows that G has finite torsionfree rank and the rank formula holds for $G = AB$ by Lemma 1.4. This contradiction proves the theorem.

2. **Proof of Theorem A.** Assume that Theorem A is false. Among the counterexamples with minimal derived length consider one $G = AB$ where the Hirsch number and then also the order of the finite maximal torsion normal subgroup of the polycyclic subgroup B is

minimal. Let $K = G^{(k)} \neq 1$ be the last nontrivial term of the derived series of G. By [3], Theorem 1.7 (b)

$$X = X(K) = AK \cap BK = KA^* = KB^* = A^*B^*$$

where $A^* = A \cap BK$ and $B^* = B \cap AK$. By induction the factorized group $G/K = (AK/K)(BK/K)$ is an $\mathfrak{X}$-group. Since $\mathfrak{X}$ is inherited by extensions, K is not an $\mathfrak{X}$-group. Since K is contained in X, X is not an $\mathfrak{X}$-group.

Let $A^*{}_X$ be the largest normal subgroup of X which is contained in A^*. Without loss of generality it may be assumed that $A^*{}_X = 1$. Since $A^* \cap K$ is a normal subgroup of $X = KA^*$, we have in particular that $A^* \cap K = 1$. If also $B^* \cap K = 1$, then

$$\begin{aligned} A^* &= A^*/(A^* \cap K) \simeq A^*K/K = X/K \\ &= B^*K/K \simeq B^*/(B^* \cap K) = B^*. \end{aligned}$$

Thus $X = A^*B^*$ is factorized by two isomorphic polycyclic nilpotent $\mathfrak{X}$-subgroups A^* and B^*. Since $X = KA^* = KB^* = A^*B^*$ has a factorization with three nilpotent subgroups A^*, B^* and K and since X is finitely generated and soluble, it follows from [3], Corollary 4.2 that X is nilpotent. Hence A^* and B^* are subnormal in X. As subgroups of A and B, A^* and B^* are $\mathfrak{X}$-groups. Now it is easy to see that X is an $\mathfrak{X}$-group; see Remark 1.2. This contradiction shows that $B^* \cap K \neq 1$. In particular $N = B^*{}_X \neq 1$.

Since B is an $\mathfrak{X}$-group, also N is an $\mathfrak{X}$-group. The group $X/N = (A^*N/N)(B^*/N)$ is factorized by an $\mathfrak{X}$-subgroup $A^*N/N \simeq A^*/(A^* \cap N)$ and a polycyclic $\mathfrak{X}$-subgroup B^*/N. Since either the Hirsch number or else the order of the finite maximal torsion normal subgroup of B^*/N is less than that of B^* and B, X/N is an $\mathfrak{X}$-group. Since N and X/N are $\mathfrak{X}$-groups, X is an $\mathfrak{X}$-group. This contradiction proves Theorem A.

3. **Proof of Theorem B.** The following lemma is needed in the proof of Theorem B.

LEMMA 3.1. *Let $\mathfrak{X}$ be a class of groups closed with respect to the forming of subgroups, epimorphic images and extensions. If the soluble Černikov group $G = AB$ is factorized by two $\mathfrak{X}$-subgroups A and B, then G is an $\mathfrak{X}$-group.*

PROOF. By [3], Lemma 5.6, the semi-radicable radical $\mathfrak{D}G$ of G is abelian and $\mathfrak{D}G = (\mathfrak{D}A)(\mathfrak{D}B)$. As subgroups of A and B, the groups $\mathfrak{D}A$ and $\mathfrak{D}B$ are $\mathfrak{X}$-groups. By the requirements for the class $\mathfrak{X}$ it follows easily that the abelian group $\mathfrak{D}G = (\mathfrak{D}A)(\mathfrak{D}B)$ is an $\mathfrak{X}$-group. Since G is a Černikov group, the quotient group $G/\mathfrak{D}G$ is finite.

Clearly $G/\mathfrak{D}G = (A\mathfrak{D}G/\mathfrak{D}G)(B\mathfrak{D}G/\mathfrak{D}G)$ is factorized by two $\mathfrak{X}$-subgroups $A\mathfrak{D}G/\mathfrak{D}G \simeq A/(A \cap \mathfrak{D}G)$ and $B\mathfrak{D}G/\mathfrak{D}G \simeq B/(B \cap \mathfrak{D}G)$. Since the class $\mathfrak{X}$ is closed under the forming of subgroups, epimorphic images and extensions, it is not hard to show that every finite soluble group factorized by two $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group. Hence $G/\mathfrak{D}G$ is an $\mathfrak{X}$-group. Since $\mathfrak{D}G$ and $G/\mathfrak{D}G$ are $\mathfrak{X}$-groups, G is an $\mathfrak{X}$-group. This proves the lemma.

Assume that Theorem B is false. Among the counterexamples with minimal derived length consider one $G = AB$ where the sum of the primary ranks and then also the index of the semi-radicable radical $\mathfrak{D}B$ of the Černikov group B are minimal. Since A is an $\mathfrak{X}$-group, A has finite abelian section rank and in particular finite torsionfree rank. Hence it may also be assumed that the Hirsch number of A is minimal.

Let $K = G^{(k)} \neq 1$ be the last nontrivial term of the derived series of G. By induction G/K is an $\mathfrak{X}$-group. By [3], Theorem 1.7

$$X = X(K) = AK \cap BK = KA^* = KB^* = A^*B^*$$

where $A^* = A \cap BK$ and $B^* = B \cap AK$. Since G/K is an $\mathfrak{X}$-group and the class $\mathfrak{X}$ is extension inherited, K is not an $\mathfrak{X}$-group. Hence $X = A^*B^*$ is also a counterexample. In particular X is not an $\mathfrak{X}$-group.

Let A^*_X be the largest normal subgroup of X which is contained in A^*. Without loss of generality it may be assumed that $A^*_X = 1$, because A^*_X is an $\mathfrak{X}$-group. Since $A^* \cap K$ is a normal subgroup of $X = A^*K$, we have in particular $A^* \cap K = 1$. If also $B^* \cap K = 1$, then

$$\begin{aligned} A^* &= A^*/(A^* \cap K) \simeq A^*K/K = X/K \\ &= B^*K/K \simeq B^*/(B^* \cap K) = B^*. \end{aligned}$$

Thus $X = A^*B^*$ is factorized by two isomorphic Černikov subgroups A^* and B^*. Since X is soluble, it follows from [3], Theorem 5.5 that X is a Černikov group. Since A^* and B^* are $\mathfrak{X}$-groups, X is an $\mathfrak{X}$-group by Lemma 3.1. This contradiction shows that $B^* \cap K \neq 1$. In particular $L = B^*_X \neq 1$.

If L is infinite, $X/L = (A^*L/L)(B^*/L)$ is an $\mathfrak{X}$-group, because in this case the sum of the primary ranks of B^*/L is less than the sum of the primary ranks of B^* and hence B. As a factor of B, the group L is an $\mathfrak{X}$-group. Since L and X/L are $\mathfrak{X}$-groups, X is an $\mathfrak{X}$-group. This contradiction shows that L is a nontrivial finite normal $\mathfrak{X}$-subgroup of X.

By the definition of $L = B^*_X$ there exists no nontrivial normal $\mathfrak{X}$-subgroup of $X/L = (A^*L/L)(B^*/L)$ which is contained in B^*/L. Furthermore

$$X/L = (A^*L/L)(KL/L) = (B^*/L)(KL/L) = (A^*L/L)(B^*/L).$$

The normal subgroup $(B^*/L) \cap (KL/L)$ of $X/L = (B^*/L)(KL/L)$ must be trivial, since it is contained in B^*/L. If also $(A^*L/L) \cap (KL/L) = 1$, then

$$\begin{aligned}(A^*L/L) &\simeq (A^*L/L)(KL/L)/(KL/L)\\ &= (B^*/L)(KL/L)/(KL/L)\\ &\simeq (B^*/L).\end{aligned}$$

Hence X/L is factorized by two isomorphic Černikov $\mathfrak{X}$-subgroups A^*L/L and B^*/L. Since X/L is soluble, X/L is a Černikov group by [3], Theorem 5.5. By Lemma 3.1 the group X/L is an $\mathfrak{X}$-group. Since L and X/L are $\mathfrak{X}$-groups, X is an $\mathfrak{X}$-group. This contradiction proves the existence of a normal subgroup N of X such that $L \subset N \subseteq A^*L$.

It follows from Dedekind's modular law that $N = N \cap A^*L = L(A^* \cap N)$. If $A^* \cap N = 1$, then $N = L$, a contradiction. Hence $A^* \cap N \neq 1$. Since $A^*{}_X = 1$, the subgroup $A^* \cap N$ of A^* is not normal in X. Since $N/L \simeq (A^* \cap N)/(A^* \cap N \cap L)$, N is an $\mathfrak{X}$-group. If N contains elements of infinite order, so does N/L. Hence the Hirsch number of $A^*N/N = A^*LN/N = A^*L/N$ is less than that of A^*L/L and A^*. It follows that $X/N = (A^*N/N)(B^*N/N)$ is no counterexample and hence an $\mathfrak{X}$-group. Since N and X/N are $\mathfrak{X}$-groups, X is an $\mathfrak{X}$-group. This contradiction shows that N must be a torsion group. N/L is a nontrivial soluble torsion $\mathfrak{X}$-group with finite abelian section rank. Then the last nontrivial term of the derived series of N/L is an abelian torsion $\mathfrak{X}$-group with finite section rank. Hence without loss of generality it may be assumed that N/L is abelian. If $(N/L)_p$ is a nontrivial primary component of N/L, then it is artinian and hence contains a nontrivial finite characteristic subgroup M/L of $(N/L)_p$ and N/L. Then M is a finite normal $\mathfrak{X}$-subgroup of X such that $L \subset M \subseteq N \subseteq A^*L$. Without loss of generality it may be assumed that $N = M$. Hence N is a nontrivial finite normal $\mathfrak{X}$-subgroup of X.

Since N is finite, the group of automorphisms induced by X in N is likewise finite. Hence $X/\mathfrak{c}_X N$ is finite. Since $A^* \cap N \subseteq N$, it follows that $\mathfrak{c}_X N \subseteq \mathfrak{n}_X(A^* \cap N)$, so that $\mathfrak{n}_X(A^* \cap N)$ is a subgroup of finite index in X. By Dedekind's modular law $\mathfrak{n}_X(A^* \cap N) \cap A^*B^* = A^*(\mathfrak{n}_X(A^* \cap N) \cap B^*)$. Thus $\mathfrak{n}_X(A^* \cap N) = A^*Y$ where $Y = \mathfrak{n}_X(A^* \cap N) \cap B^*$ is a subgroup of B^*. In particular the group A^*Y is factorized by an $\mathfrak{X}$-subgroup A^* with finite abelian section rank and a Černikov $\mathfrak{X}$-subgroup Y. Since $A^*{}_X = 1$, the subgroup $A^* \cap N \neq 1$ of A^* is not normal in X. Hence $\mathfrak{n}_X(A^* \cap N)$ is a proper subgroup of X. It follows that the index $|X : \mathfrak{n}_N(A^* \cap N)| = |A^*B^* : A^*Y|$ is finite and at least two. Then also $|A^*B^* : \mathfrak{n}_X(A^* \cap N)| = |\mathfrak{n}_X(A^* \cap N)B^* : \mathfrak{n}_X(A^* \cap N)| = |B^* : (B^* \cap \mathfrak{n}_X(A^* \cap N)| = |B^* : Y|$ is finite and at least two. Thus the index of the semi-radicable radical of the Černikov subgroup Y is less than the index of the semi-radicable radical of the Černikov group B^*. It follows that A^*Y is no counterexample and hence an $\mathfrak{X}$-group. If $(A^*Y)_X$ is the largest normal subgroup of X which is contained in A^*Y, then $(A^*Y)_X$ is likewise an $\mathfrak{X}$-group. By the theorem of Poincaré $X/(A^*Y)_X$ is finite. Hence X is an extension of the $\mathfrak{X}$-group $(A^*Y)_X$ by the finite soluble group $X/(A^*Y)_X$. Since $X/(A^*Y)_X$ is factorized by two $\mathfrak{X}$-subgroups, it is an $\mathfrak{X}$-group by Lemma 3.1. Since $(A^*Y)_X$ and $X/(A^*Y)_X$ are $\mathfrak{X}$-groups, X is an $\mathfrak{X}$-group. This contradiction proves Theorem B.

4. **Groups with finite rank, Proof of Theorem D.** Statements about factorized groups with finite rank depend very much on the kind of rank in question. Consider for instance the general rank in the sense of Mal'cev, which is defined as follows; see for instance Robinson [10]. A group G has *finite general rank* $g = g(G)$ if g is the least positive integer such that every finitely generated subgroup of G is contained in a subgroup with g generators. Clearly, every group with finite rank r in the sense of Prüfer has also finite general rank $g \leqq r$, but the converse is false. However, the two notations coincide for abelian groups.

The following lemma is almost trivial and holds even if the group G is only generated by the subgroups A and B.

LEMMA 4.1. *If the group $G = AB$ is factorized by two subgroups A and B with finite general ranks $g(A)$ and $g(B)$, then G has finite general rank $g(G) \leqq g(A) + g(B)$.*

PROOF. Let $g_1, \cdots, g_n$ be a finite subset of G. Then $g_i = a_ib_i$ where $a_i \in A$, $b_i \in B$. Then $S = \langle g_1, \cdots, g_n \rangle \subseteq \langle a_1, \cdots, a_n, b_1, \cdots, b_n \rangle = \langle A^*, B^* \rangle$ where $A^* = \langle a_1, \cdots, a_n \rangle$ and $B^* = \langle b_1, \cdots, b_n \rangle$. Since A^* can be generated by $g(A)$ elements and B^* can be generated by $g(B)$ elements, S can be generated by $g(A) + g(B)$ elements (or less). Hence $g(G) \leqq g(A) + g(B)$.

The group theoretical property of having finite rank in the sense of Prüfer is inherited by subgroups, epimorphic images and extensions; see for instance Robinson [10], Lemma 1.44, p. 34. Furthermore, it is not hard to see that the property of being a soluble group with finite abelian section rank is also inherited by subgroups, epimorphic images and extensions; see Robinson [11], section 9.3. Hence, the following theorem is a special case of Theorems A and B.

THEOREM 4.2. *If the soluble group $G = AB$ is factorized by a subgroup A with finite rank [finite abelian section rank] and a polycyclic or Černikov subgroup B, and if in the first case A or B is locally nilpotent, then G has finite rank [finite abelian section rank].*

An abelian group A has finite rank in the sense of Fuchs if and only if it has finite torsionfree rank (0-rank) and its torsion subgroup is artinian. This is the case if and only if

$$r_d(A) = r_0(A) + \sum_p r_p(A)$$

is finite, where $r_0(A)$ is the torsionfree rank of A and $r_p(A)$ is the p-rank of A for the prime p. Clearly, every abelian group with finite rank in the sense of Fuchs has also finite rank in the sense of Prüfer. But the direct product of cyclic groups, one of each prime order, has finite rank in the sense of Prüfer, but infinite rank in the sense of Fuchs.

Unfortunately the property of having finite rank in the sense of Fuchs is not inherited by epimorphic images as the additive group of rational numbers shows. It is therefore not surprising that an abelian group factorized by two subgroups with finite rank in the sense of Fuchs may have infinite rank in the sense of Fuchs, as the following example shows.

EXAMPLE 4.3. The additive group Q of rational numbers is a torsionfree group of rank 1. The direct square $H = Q \times Q = AB$ is an abelian group which is factorized by two subgroups A and B isomorphic to Q. Let D be the diagonal subgroup of H consisting of all (y, y) where $y \in Q$. Then D is isomorphic to Q. The set of all elements (x, x) where $x \in Z$, the additive group of integers, is a subgroup N of D. N is an infinite cyclic group and D/N is a direct product of quasicyclic groups of type p^∞, one for each prime p. In particular D/N has infinite rank in the sense of Fuchs. The group $G = H/N = (AN/N)(BN/N)$ is factorized by two subgroups $AN/N \simeq A/(A \cap N)$ and $BN/N \simeq B/(B \cap N) = B$, each isomorphic to Q. Thus G is factorized by two torsionfree subgroups of rank 1, but G has infinite rank in the sense of Fuchs, since it contains D/N as a subgroup.

The group G is a soluble group with finite abelian section rank and the maximal torsion normal subgroup of G is a Černikov group if and only if G possesses an abelian series of finite length whose factors have finite rank in the sense of Fuchs; see for instance [11], p. 137. These groups are dealt with in Theorem D.

Proof of Theorem D. Let $G = AB$ be a soluble group factorized by a subgroup A with finite abelian section rank such that the maximal torsion normal subgroup $S = \mathfrak{T}A$ of A is a Černikov group and a Černikov subgroup B. It follows from Theorem 4.2 that G has finite abelian section rank. By Robinson [11], Theorem 9.31, p. 129, the semi-radicable radical $\mathfrak{D}G$ of G is radicable and nilpotent. The subgroup $V = \langle \mathfrak{D}S, \mathfrak{D}B \rangle$ of G which is generated by the semi-radicable radicals of S and B is contained in $\mathfrak{D}G$, and so is the subgroup $N = V^G$ generated by all the subgroups V^g which are conjugate to V in G. Since N is contained in $\mathfrak{D}G$, N is nilpotent. Hence N is abelian, since N is generated by quasicyclic subgroups; see for instance Robinson [11], Theorem 9.23, p. 125. As a subgroup of G, N has finite section rank. Since $\mathfrak{D}S$ and $\mathfrak{D}B$ are artinian, there are only finitely many nontrivial primary components in V and hence in N. Since N is a torsion group with finite section rank, this implies that N is artinian. The group $G^* = G/N = (AN/N)(BN/N)$ is factorized by its subgroups $A^* = AN/N \simeq A/(A \cap N)$ and $B^* = BN/N \simeq B/(B \cap N)$. Clearly A^* has finite abelian section rank and since $A \cap N$ is a torsion group, the maximal torsion normal subgroup of A^* is a Černikov group. Since every quasicyclic subgroup of the Černikov group B is contained in $B \cap N$, B^* is finite. Hence, by the theorem of Poincaré, there exists a normal subgroup M of G^* contained in A^* with finite G^*/M. As a subgroup of A^*, M has finite abelian section rank and $\mathfrak{T}M$ is a Černikov group. It follows that G is an extension of the radicable artinian abelian normal subgroup N by the group G/N which contains a subgroup M of finite index; M has finite abelian section rank and $\mathfrak{T}M$ is a Černikov group. In particular G has finite abelian section rank and $T = \mathfrak{T}G$ is a Černikov group.

By [3], Theorem 1.7 (b)

$$X = X(T) = AT \cap BT = T(A \cap BT) = T(B \cap AT)$$

$$= (A \cap BT)(B \cap AT).$$

Since T and $B \cap AT$ are torsion groups, X is a torsion group. Hence X is factorized by two Černikov subgroups $A \cap BT$ and $B \cap AT$. By [3], Theorem 5.5 X is a Černikov group. Since X is factorized, $X = (A \cap X)(B \cap X)$; see [3], Lemma 1.1. Thus by [3], Lemma 5.6, $\mathfrak{D}X = \mathfrak{D}(A \cap X)\mathfrak{D}(B \cap X)$. Every quasicyclic subgroup of G is contained in $\mathfrak{T}(\mathfrak{D}G) \subseteq T \subseteq X$. Hence it is easy to see that $\mathfrak{D}(\mathfrak{T}G) = \mathfrak{D}(\mathfrak{T}A)(\mathfrak{D}B)$. This proves Theorem D.

5. **Polyminimax groups.** An abelian group A is a *minimax group* if it is an extension of a noetherian group by an artinian group. The set of all primes p such that A has an infinite p-quotient group is called the *spectrum* of A. If π is a set of primes, a minimax group A is a π-minimax group if its spectrum belongs to π. If π is the empty set of primes, a group is a π-minimax group if and only if it is a noetherian abelian group. If π is the set of all primes, a group is a torsion π-minimax group if and only if it is an artinian abelian group.

A soluble group is a *poly-π-minimax group* (*polyminimax group*) if its abelian factors are π-minimax groups (minimax groups).

For properties of these group classes see Robinson [11], section 10.3.

The following theorem is a special case of Theorems A and B.

THEOREM 5.1. *If the soluble group $G = AB$ is factorized by a poly-π-minimax subgroup A and a polycyclic or Černikov subgroup B with spectrum in π, and if in the first case A or B is locally nilpotent, then G is a poly-π-minimax group.*

A group is *min-by-max* (*max-by-min*) if it is an extension of an artinian (noetherian) group by a noetherian (artinian) group. It is not hard to see that every abelian min-by-max group is the direct product of an artinian and a noetherian group. Hence every abelian min-by-max group is also max-by-min, i.e. a minimax group. The converse is false, as the additive group of rational numbers whose denominators are powers of a fixed prime p shows.

There exist groups $G = AB$ which are factorized by an abelian min-by-max (normal) subgroup A and an artinian abelian subgroup B, which are not max-by-min; see Baer [4], Bemerkung 4.7, p. 30. However, the following shows that such a group is always min-by-max.

PROPOSITION 5.2. *A soluble group $G = AB$ with finite abelian section rank which is factorized by two min-by-max subgroups A and B, one of which is locally nilpotent, is an extension of a radicable artinian abelian characteristic subgroup by a polycyclic group.*

PROOF. Since G is a soluble group with finite abelian section rank, the semi-radicable radical $\mathfrak{D}G$ of G is radicable and nilpotent; see Robinson [11], Theorem 9.31, p. 129. The subgroup $V = \langle \mathfrak{D}A, \mathfrak{D}B \rangle$ of G which is generated by the semi-radicable radicals $\mathfrak{D}A$ of A and $\mathfrak{D}B$ of B, is contained in $\mathfrak{D}G$. The subgroup $N = V^G$ which is generated by all the subgroups V^g which are conjugate to V in G is also contained in $\mathfrak{D}G$ and therefore nilpotent. Since N is generated by quasicyclic subgroups, N is abelian; see for instance Robinson [11], Theorem 9.23, p. 125. Since $\mathfrak{D}A$ and $\mathfrak{D}B$ are artinian, there are only finitely many nontrivial primary components in V and N. Since the abelian group N is generated by elements of finite order, N is a torsion group. As an abelian torsion group with finite section rank and only finitely many nontrivial primary components N is artinian. The soluble group $G/N = (AN/N)(BN/N)$ which is factorized by two polycyclic subgroups $AN/N \simeq A/(A \cap N)$ and $BN/N \simeq B/(B \cap N)$ one of which is locally nilpotent, is polycyclic by Theorem A. Now it is easy to see that $N = \mathfrak{D}G$. Hence G is an extension of the artinian radicable abelian characteristic subgroup $\mathfrak{D}G$ by the polycyclic group $G/\mathfrak{D}G$.

As an immediate consequence of Proposition 5.2 there is the following result.

THEOREM 5.3. *If the soluble group $G = AB$ is factorized by a min-by-max subgroup A and a polycyclic or Černikov subgroup B and if A or B is locally nilpotent, then G is min-by-max.*

Some special cases of this theorem have been obtained in [1] and [2].

If the group $G = AB$ is factorized by two noetherian abelian subgroups A and B, then G is metabelian by Itô's theorem and polycyclic by Theorem A. By Theorem C the Hirsch number of G is $h(G) = h(A) + h(B) - h(A \cap B) = h(G^{(1)}) + h(G/G^{(1)})$. The following example shows that $h(G) = h(G^{(1)})$ can happen.

EXAMPLE. Consider the group $G = \langle a, b \rangle$ with defining relations

$$(ab)^2 = (ab^{-1})^2 = 1.$$

G has a factorization as a product of two infinite cyclic subgroups: $G = AB$ where $A = \langle a \rangle$ and $B = \langle b \rangle$. Neither A nor B is normal in G. The center of G is trivial, so that also $A \cap B = 1$. We have $A = \mathfrak{c}A$ and $B = \mathfrak{c}B$. The commutator subgroup of G is $G^{(1)} = \langle a^2b^2, a^4 \rangle$. The Fitting subgroup of G is $F = \langle a^2, b^2 \rangle$; F is abelian, so that $F = \mathfrak{c}F$. $G/F = \langle Fa, Fb \rangle$ is a four group, $F/G^{(1)} = \langle G^{(1)}a^2 \rangle$ is cyclic of order 2. Thus $G/G^{(1)}$ is an abelian group of order 8. The Hirsch number of G is $h(G) = h(F) = h(G^{(1)}) = 2$. F and $G^{(1)}$ are torsionfree groups of rank 2. Note that the elements of the coset Fab of F in G have order 2, while all the other elements of G have infinite order. F is a factorized subgroup in the sense of [3], while $G^{(1)}$ is not.

References

1. B. Amberg and W. R. Scott, *Products of abelian subgroups,* Proc. Amer. Math. Soc. **26** (1970), 541-547.

2. B. Amberg, *Abelian factorizations of infinite groups,* Math. Z. **123** (1971), 201-214.

3. ——, *Artinian and noetherian factorized groups,* (to appear in Rend. Sem. Mat. Univ. Padova).

5. B. Huppert, *Endliche Gruppen I,* Springer-Verlag, Berlin (1967).

6. N. Itô, *Uber das Produkt von zwei abelschen Gruppen,* Math. Z. **62** (1955), 400-401.

7. O. H. Kegel, *Produkte nilpotenter Gruppen,* Arch. Math. **12** (1961), 90-93.

8. ——, *On the solubility of some factorized linear groups,* Illinois J. of Math. 9 (1965), 535-547.

9. E. Pennington, *On products of finite nilpotent groups,* Math. Z. **134** (1973), 81-83.

10. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups, Part* 1, Springer-Verlag, Berlin (1972).

11. ——, *Finiteness conditions and generalized soluble groups, Part 2,* Springer-Verlag, Berlin (1972).

12. H. Wielandt, *Uber Produkte von nilpotenten Gruppen,* Illinois J. of Math. 2 (1958), 611-618.

13. H. Zassenhaus, *The theory of groups,* second edition, Chelsea (1958).

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