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SOME SINGULAR PERTURBATION PROBLEMS FOR WAVE MOTIONS IN SIMPLE MODEL ATMOSPHERES

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Wave propagation problems in simple atmospheric models frequently lead to boundary value problems of the following type:

$$[\epsilon L_2 + e^{-z}L_1]\ Y(z) = 0,\ 0 \leqq z < \infty,$$

with certain boundary and normalization or radiation conditions as $z \to \infty$. Here L_1 and L_2 are differential operators with constant coefficients. The function e^{-z} multiplying the second term represents the dimensionless fluid density, which is exponential in simple models such as the isothermal atmosphere. The dimensionless parameter ϵ is proportional to the viscosity or the thermal conductivity of the fluid.

A problem of this type is singular since the solution $Y(z, \epsilon)$ in general cannot be approximated uniformly on $[0, \infty)$ by any solution of the reduced equation $L_1 Y = 0$. There is, however, a solution of $L_1 Y = 0$ which approximates $Y(z, \epsilon)$ in the region where e^{-z}/ϵ is large (except possibly in a boundary layer), and a solution of $L_2 Y = 0$ which approximates $Y(z, \epsilon)$ in the region where e^{-z}/ϵ is small. However, there is no simple way to connect these solutions since none of the terms in general can be neglected in the region where $e^{-z}/\epsilon \sim 1$. Under appropriate conditions waves generated below will be reflected downward in such a region.

Several problems of the above type are listed below. Here $D = d/dz$, and k, r, γ, and σ are constants. Boundary conditions at $z = 0$ and a normalization condition of the form $\int_0^\infty ||Y'(z)|^2\, dz < \infty$ are imposed.

1. Two dimensional waves in a viscous incompressible fluid with exponential density (see [1]).

$$[4\epsilon (D^2 - k^2)^2 + ie^{-z}(D^2 - D + r)]\ Y = 0.$$

2. Vertical oscillations in a viscous isothermal atmosphere ([2]).

$$\epsilon D^2 Y + 4ie^{-z}(D^2 - D + \sigma^2/4)\ Y = 0.$$

3. Vertical oscillations in a thermally conducting isothermal atmosphere ([3]).

$$\epsilon D^2(D^2 + D + \gamma\sigma^2/4)Y + i\sigma\gamma\, e^{-z}(D^2 - D + \sigma^2/4)Y = 0.$$

A radiation condition is required in addition to the other conditions.

To solve these problems it is convenient to introduce a new independent variable, $\xi = e^{-z/\epsilon}$. The resulting equations have regular singularities at $\xi = 0$, and solutions can be obtained in the form of series. The asymptotic behavior of these series can be deduced by converting them to integrals, or directly from some theorems of Ford [4].

Problems 2 and 3 do not yield the same limiting result even in the region where $e^{-z/\epsilon}$ is large. To resolve the difficulty a somewhat more complicated problem in which the fluid is both viscous and thermally conducting has to be examined [3].

References

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