ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 4, Fall 1976

A GENERALIZATION OF THE REDUCED EQUATION IN SINGULARLY PERTURBED SYSTEMS

M. B. BALACHANDRA

In this presentation, based on the results of [1], it is shown that under suitable conditions, the solutions of singularly perturbed systems may approach asymptotic limits other than the solutions of the classical "reduced" systems.

Considering the real, singularly perturbed system

$$\frac{dx}{d\tau} = f(\tau, x, y, \epsilon), \epsilon \frac{dy}{d\tau} = g(\tau, x, y, \epsilon), \tag{1}$$

where x and y are vectors of arbitrary dimensionality and $\epsilon > 0$ is a small parameter, the classical results for the initial value problem show that the solutions of (1) approach, as ϵ tends to zero, the solutions of the "reduced" system:

$$\frac{dx}{d\tau} = f(\tau, x, y, 0) \qquad 0 = g(\tau, x, y, 0). \tag{2}$$

The convergence here is uniform over a given finite τ-interval for x and uniform, except near the initial time, $\tau = 0$, for y. The results (e.g., [2] through [6]) involve consideration of the "boundary layer" equation

$$\frac{dz}{dt} = g(\tau, x, z, 0) \tag{3}$$

and assumptions of a stability nature on the constant solution $z = \phi(\tau, x)$ of (3). (Note that $\phi(\tau, x)$ is a solution of the algebraic equation $g = 0$ in (2)).

The present work shows that if (3) possess a nonconstant solution $z = \phi(t, \tau, x)$ that is bounded in a certain sense, then in the limit as $\epsilon \to 0$, $x \to \zeta(\tau)$ and $y \to \phi(\tau/\epsilon, \tau, \zeta(\tau))$, where $\zeta(\tau)$ is the solution of the "Averaged" Equation,

$$\frac{d\zeta}{d\tau} = f_0(\tau, \zeta) \equiv \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} f(\tau, x, \phi(s, \tau, x), 0)\, ds \tag{4}$$

having the same initial value as x in (1). The results depend on assumptions about the limit in (4) and on the solution ϕ of (3).

It may be seen that if ϕ is constant, (4) reduces to (2). In this sense, the present result is a generalization of the classical result of Tikhonov [4].

It is also shown that a similar generalization of the concept of the reduced equation is obtained for the existence of almost periodic and bounded solutions of (1) when the functions on the right side have the appropriate properties. In this case also, the limiting system is analogous to (4) and is different from (2) whenever ϕ is not constant.

References

1. M. Balachandra, *Some new Basic Results for Singularly Perturbed Ordinary Differential Equations,* J. Math. Anal. Appl. 49 (1975), 302–316.

2. W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations,* Chapter X, Wiley Interscience, New York, 1965.

3. R. E. O'Malley, *Topics in singular perturbations,* in "Lectures in Ordinary Differential Equations" R. W. McKelvey, ed., pp. 150–260, Academic Press, New York, 1970.

4. A. N. Tikhonov, *On a system of differential equations containing parameters,* Mat. Sb. **27** (1950), 147–156.

5. A. B. Vasil'eva, *Asympotic behavior of solutions to certain problems involving nonlinear differential equations containing a small parameter multiplying the highest derivatives,* Russian Math. Surveys **18** (1963), 13–84.

6. F. C. Hoppensteadt, *Singular perturbations on the infinite interval,* Trans. Amer. Math. Soc. 123 (1966), 521–535.

7. L. Flatto and N. Levinson, *Periodic solutions of singularly perturbed systems,* J. Rational Mech. Anal. 4 (1955), 943–950.

8. J. K. Hale and G. Seifert, *Bounded and almost periodic solutions of singularly perturbed equations,* J. Math. Anal. Appl. 3 (1961), 18–24.

9. L. S. Pontryagin, *System of ordinary differential equations with a small parameter attached to the highest derivatives,* Trudy III Vsesoyuznogo Matematicheskogo S, zeda **2** (1956), 93; 3 (1958), 570.

10. M. Balachandra and P. R. Sethna, *A generalization of the method of averaging for systems with two time scales,* Arch. Rational Mech. Anal. **58** (1975), 261–283.

11. J. K. Hale, *Ordinary Differential Equations,* Wiley Interscience, New York, 1969.

12. W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations,* D. C. Heath and Co., Boston, 1965, p. 37.

Agbabian Associates, 250 North Nash Street, El Segundo, California 90245