ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 2, Spring 1976

CATEGORIES OF ACTIONS AND MORITA EQUIVALENCE

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0. **Introduction.** Let A be a small category. By a (right) A-action is meant a (covariant) functor $A \to \text{Ens}$, where Ens is the category of sets, and by a morphism of A-actions a morphism of the corresponding functors. The category of A-actions will be denoted by Ens^A, and A will be called the operator category. An abstract characterization of categories of actions is given in [3]. Two (small) categories A and B are called Morita-equivalent ($A \simeq_M B$) if their categories of actions are equivalent in the usual sense ($\text{Ens}^A \simeq \text{Ens}^B$); i.e., if there exist functors F and G between Ens^A and Ens^B such that FG and GF are isomorphic to the corresponding identity functors. The following questions arise:

I. How can we characterize Morita equivalence intrinsically?

II. How can we construct all categories Morita-equivalent to a given one?

Answers are given to I and II in [5], when the operator categories are (finite) groups (in which case Morita equivalence implies isomorphism), and in [1] and [7], when they are arbitrary monoids. The situation is analogous to the Morita theory for modules over a ring, described, e.g., in [2], [4].

In the present paper we provide answers to I and II for the case of arbitrary operator categories. One form of the answer can be stated in terms of the notion of *weak equivalence* $\simeq_w$ between categories. A weak functor $f: A \to B$ is like an ordinary functor, but without the requirement that the image of an identity morphism be an identity morphism (it is then perforce an idempotent). There is a corresponding notion of weak functor morphism, and the result can be stated (Theorem 4.4):

> *Two (small) categories are Morita-equivalent if and only if they are weakly equivalent.*

An alternative formulation (Theorem 3.6′) is in terms of the *idempotent completion* $\bar{A}$ constructed in §3 for any category A:

> *Two (small) categories are Morita-equivalent if and only if their idempotent completions are equivalent (in the usual sense).*

Received by the editors on September 3, 1974.

The idempotent completion $\bar{A}$ also provides a means of answering question II (Theorem 3.9).

Our initial approach to the problem made use of an Eilenberg-Gabriel-Watts theorem for actions, describing an arbitrary cocontinuous functor from Ens^A to Ens^B in terms of a tensor product construction. The final version circumvents this idea, but an account of the construction is included, both because of its relevance and because of its independent interest. After this paper had been written, we discovered in [8] that P. Freyd had treated question I with Ens replaced by *Ab*, the category of abelian groups, using the ideas of idempotent completion and amenable category. This does not seem to diminish the desirability of the present exposition.

1. **Preliminaries.** Functional operation will generally be written on the right; in particular, composition of functions (and morphisms) is from left to right. An action $F : A \to \mathrm{Ens}$ is equivalent, in the category Ens^A, to one in which the sets aF, $a \in |A|$, are disjoint. Thus, an action amounts to a graded set $X = \{X_a\}$, $a \in |A|$, with operations $X_\alpha : x \to x\alpha$ for every morphism $\alpha : a \to a'$ in A, taking $x \in X_a$ into $x\alpha \in X_{a'}$ and satisfying the axioms

$$\text{(i) } x1_a = x \quad (x \in X_a) \quad \text{and}$$

$$\text{(ii) } (x\alpha)\alpha' = x(\alpha\alpha').$$

Viewed in this manner, the action will be called a (*right*) *A-set*. The category Ens^A can therefore be regarded as the category of right A-sets, and when it is so regarded a morphism in Ens^A will be called an A-map. An A-map $f : X \to Y$ is then a map of graded sets such that $(x\alpha)f = (xf)\alpha$. Similarly, a *left A-set* means a graded set $X = \{X^a\}$, $a \in |A|$, with operations X^α taking $x \in X^{a'}$ into $\alpha x \in X^a$ for any morphism $\alpha : a \to a'$, and satisfying the axioms

$$\text{(i) } 1_a x = x \quad (x \in X^a) \quad \text{and}$$

$$\text{(ii) } \alpha(\alpha' x) = (\alpha\alpha')x.$$

A left A-set corresponds to a left A-action — i.e., a contravariant functor from A to Ens — and amounts to a right A^o-set, where A^o is the category opposite to A. "A-set", unmodified, will mean "right A-set". Note that because of our convention concerning order of composition, the notation is counter to the common one for semisimplicial complexes; in the present language, these are left Δ-sets, where Δ is the category of finite ordinals.

If B is another small category, we mean by an A-B-biset a bigraded set $X = \{X_b{}^a\}$, $a \in |A|$, $b \in |B|$, which is simultaneously a left A-set and a right B-set, such that $\alpha(x\beta) = (\alpha x)\beta$ whenever either side of this equality is defined. This amounts to a right $(A^o \times B)$-set, or a left $(A \times B^o)$-set. It also amounts to a contravariant functor X^* from A to the category Ens^B of right B-sets, with $X^* : a \mapsto X^a$, where X^a is the B-set $X^a = \bigcup \{X_b{}^a | b \in |B|\}$; and, of course, equally well to a covariant functor X_* from B to the category Ens_A of left A-sets, with $X_* : b \mapsto X_b = \bigcup \{X_b{}^a | a \in |A|\}$. To indicate that X is an A-B-biset we shall sometimes write it as $X_B{}^A$. In particular, the usual hom functor on A, being a covariant functor from $A^o \times A$ to Ens, is equivalent to an A-A-biset $A_A{}^A$, which we will denote by A all over again. Its components $A_{a'}^a$ are the hom-sets $A(a, a')$.

Just as two-sided operation can always be regarded as one-sided, so can one-sided operation be regarded as two-sided: the operator category on the other side is the unit category, $\mathbb{1}$, consisting of one object and its identity morphism. We return for the time being to a description of the "right" theory.

Among the covariant functors $A \to \mathrm{Ens}$ are the representable ones, viz., the functors $A(a, -)$. In the above notation, these are the right A-sets A^a, which we call the *models* in the category Ens^A. The Yoneda lemma asserts that for any A-set X, the collection of A-maps $\mathrm{Ens}^A(A^a, X)$ is in 1-1 correspondence with the set X_a; namely, to each element $x \in X_a$ corresponds the unique A-map $\varphi_x : A^a \to X$ such that

$$1_a\varphi_x = x \tag{1.1}$$

If $\alpha : a \to a'$ is a morphism in A, so that α is an element of the A-set A^a, then

$$\varphi_{x\alpha} = \varphi_\alpha \cdot \varphi_x; \tag{1.2}$$

and if $f : X \to Y$ is an A-map, then

$$\varphi_{xf} = \varphi_x \cdot f. \tag{1.3}$$

The Yoneda imbedding $\mathrm{Yon}^o : A^o \to \mathrm{Ens}^A$ is the contravariant functor from A to Ens^A given by $a \mapsto A^a$, $\alpha \mapsto A^\alpha$, and constitutes an anti-isomorphism between the category A and the (full) subcategory $M^o(A)$ of models in Ens^A. The map φ_α above is the same as A^α. We shall in general make no distinction between the subcategory $M^o(A)$ and the subcategory of all isomorphic copies of members of $M^o(A)$. In the "left" theory, the Yoneda imbedding is a covariant functor $\mathrm{Yon} : A \to \mathrm{Ens}_A$, whose image $M(A)$ is isomorphic to A.

We shall often write $[x]$ for A^a if $x \in X_a$. For indexing purposes it is useful to regard $[x]$ as the set of pairs (x, α), $\alpha \in A^a$, with $(x, \alpha)\alpha' = (x, \alpha\alpha')$ and $(x, \alpha)\varphi_x = x\alpha$; and in that case we shall proliferate the notation and also write $[x, \alpha]$ for the map $\varphi_\alpha = A^\alpha : [x\alpha] \to [x]$, taking the element $(x\alpha, \alpha') \in [x\alpha]$ into the element $(x, \alpha\alpha') \in [x]$.

The category Ens^A is complete and cocomplete, with

$$(\varinjlim_i X^{(i)})_a = \varinjlim_i X_a{}^{(i)},$$
$$(\varprojlim_i X^{(i)})_a = \varprojlim_i X_a{}^{(i)}.$$

In particular, monomorphisms and epimorphisms are monomorphisms and epimorphisms for each a; which is to say, injections and surjections. An A-map $f : X \to Y$ has an epi-mono factorization

$$X \xrightarrow{p} I \xrightarrow{m} Y,$$

the image I being unique up to isomorphism.

To any A-set X is associated a canonical diagram D^X of models, indexed over the comma category (Yon^o, X), such that X is the colimit of D^X in Ens^A. An object in the index category (Yon^o, X) is an element x of X, and an arrow (= morphism) is a pair $(x, \alpha) : x\alpha \to x$. Composition is given by the formula

$$(x\alpha\alpha' \xrightarrow{(x\alpha, \alpha')} x\alpha \xrightarrow{(x, \alpha)} x) = (x\alpha\alpha' \xrightarrow{(x, \alpha\alpha')} x).$$

The diagram D^X assigns to each object x the model $[x]$, and to each arrow (x, α) the A-map $[x, \alpha] : [x\alpha] \to [x]$. One sees, using (1.2), that X is the colimit of D^X, the cone from D^X to X being given by the family of maps $\{\varphi_x\}$, $x \in X$; and, using (1.3), that any A-map $f : X \to Y$ is the morphism of colimits induced by the morphism of diagrams $D^f : D^X \to D^Y$ given by the functor $(\mathrm{Yon}^o, f) : x \mapsto xf$, $(x, \alpha) \mapsto (xf, \alpha)$ on index sets. (A morphism of diagrams $D \to D'$, where D and D' are over index categories J and J', is a functor $F : J \to J'$ together with a functor morphism from D to FD'. In the case of D^f, the functor morphism consists objectwise of isomorphisms.) The assignment $X \mapsto D^X$, $f \mapsto D^f$ constitutes a functor from Ens^A to the category of diagrams in $M^o(A)$.

The index category (Yon^o, X) for the diagram D^X will be called the *fundamental category* of the A-set X.

By the *cocompletion* of a category A is meant a category $\tilde{A} \supset A$ such that

(i) $\tilde{A}$ is cocomplete, and
(ii) any functor $\varphi : A \to C$, where C is an arbitrary cocomplete category, extends to a cocontinuous functor $\tilde{\varphi} : \tilde{A} \to C$, the extension being unique up to an isomorphism (which is the identity on A).

These properties determine the cocompletion, if it exists, up to equivalence; and, indeed, the category Ens^A is the cocompletion of A^o, the latter being imbedded in Ens^A as $M^o(A)$. (For (ii), see [9], p. 108; the extension $\tilde{\varphi}$ is dictated by the diagrams D^X.)

For any A-set X, let $\sim$ be the equivalence relation on X generated by setting $x' \sim x$ whenever $x' = x\alpha$ for some α. Each equivalence class $X^{(i)}$ is itself an A-set, with $X_a{}^{(i)} = X^{(i)} \cap X_a$, so that X has a decomposition

$$X = \coprod_i X^{(i)} \tag{1.4}$$

as a coproduct of its *orbits* (or *connected components*) $X^{(i)}$. Up to isomorphism, the representation (1.4) is the finest decomposition, in an obvious sense, of X as a coproduct in Ens^A. X is called *indecomposable* (or *connected*) if it has just one orbit; thus, X is indecomposable if every representation of X as a coproduct (of nonvoid A-sets) consists of a single component. In particular, each model A^a is indecomposable, since it has the single generator 1_a.

The following fact is obvious:

1.1 LEMMA. *Under any A-map $f : X \to Y$, the image of any orbit of X is contained in some orbit of Y.*

We shall be dealing with epi-mono factorizations of idempotents in an arbitrary category. Concerning these, we make the following easily verified observations.

1.2. LEMMA. *For any morphisms*

$$a \xrightarrow{\pi} b \xrightarrow{\mu} a, \tag{1.5}$$

the following two conditions are equivalent:
(a) *π is epic, μ monic, and $\epsilon = \pi\mu$ idempotent.*
(b) $\mu\pi = 1$.

The equivalent conditions of Lemma 1.2 are referred to by saying that the idempotent ϵ *splits* and the factorization (1.5) is called a *splitting* of the idempotent ϵ under these conditions.

1.3. LEMMA. *For any idempotent* $\epsilon : a \to a$, *denote by* $D(\epsilon)$ *the diagram consisting of the single object* a *and the morphism* ϵ. *Then* ϵ *has an epi-mono factorization* (1.5) *if and only if* b *is the colimit of the diagram* $D(\epsilon)$, *with* π *as corresponding "cone"; or, equally well, if and only if* b *is the limit of* $D(\epsilon)$, with μ *as corresponding "cone". Hence, also, the image* b *of* ϵ *is unique up to isomorphism.*

1.4. LEMMA. *The image, under any functor* $\varphi : A \to B$, *of a splitting of an idempotent* ϵ *in* A *is a splitting of an idempotent* $\epsilon\varphi$ *in* B.

Lemma 1.4 asserts that the (co)limit of a diagram $D(\epsilon)$ is absolute, i.e., is preserved under any functor.

2. **Projectives in** $\mathbf{Ens}^A$. To any A-set X is associated a canonical covering by models. We take their colimit

$$P_X = \coprod_{x \in X} [x] \xrightarrow{p_X} X, \tag{2.1}$$

where, as before, $[x]$ means the model A^a such that $x \in X_a$. The map p_X is determined by the condition that

$$i_x p_X = \varphi_x, \tag{2.2}$$

where $i_x : [x] \to P_X$ is the inclusion and φ_x is as in (1.1). The elements of P_X are pairs (x, α), with $(x, \alpha)\alpha' = (x, \alpha\alpha')$ and $(x, \alpha)p_X = x\alpha$. Obviously p_X is surjective, and therefore epic.

2.1. LEMMA. X *is projective if and only if there exists an* A-*map* $m : X \to P_X$ *such that* $mp_X = 1$.

PROOF. If X is projective, then by definition there must be such an m. Conversely, suppose $mp_X = 1$. To show X is projective, it suffices to exhibit in the diagram

$$\begin{tikzcd} P_X \arrow[r, dashed, "g"] \arrow[d, shift left, "p_X"] & Z \arrow[d, "p"] \\ X \arrow[u, shift left, "m"] \arrow[r, "f"'] & Y \end{tikzcd},$$

where p is epic, a map g such that $gp = p_X f$. Any map $g : P_X \to Z$ is uniquely determined by specifying for each index x an element of Z (of the appropriate grade — i.e., in Z_a if $x \in X_a$), this to be the value of g on $(x, 1_a)$. For each index, select the element $(x, 1_a)g$ so that $(x, 1_a)gp = xf$. But then $(x, 1_a)gp = (x, 1_a)\ p_X f$, and since this equality holds for all generators $(x, 1_a)$ of P_X, we have $gp = p_X f$ identically, as required.

In particular, every model A^a is projective, since it obviously satisfies the condition of the lemma.

2.2. Proposition. *Let $X = \coprod_i X_i$ be a representation of X as a coproduct in* $\mathrm{Ens}^{\mathbf{A}}$. *Then X is projective if and only if each X_i is projective. In particular, X is projective if and only if each of its orbits is projective.*

Proof. If each X_i is projective, then so is $X = \coprod_i X_i$, since this is the case in an arbitrary category. Conversely, suppose X is projective, and let $m : X \to P_X$ be an in Lemma 2.1. Since clearly $P_X = \coprod_i P_{X_i}$, and $p_X = \coprod_i p_{X_i}$, the map m must be of the form $\coprod_i m_i$, with $m_i p_{X_i} = 1$ for each i. By Lemma 2.1, this implies that each X_i is projective.

Thus, in order to describe all projectives in $\mathrm{Ens}^{\mathbf{A}}$, it remains to characterize the indecomposable ones.

2.3. Proposition. *An $\mathbf{A}$-set X is an indecomposable projective if and only if X is the idempotent image of a model, i.e., the image of a model under an idempotent $\mathbf{A}$-map.*

Proof. Since an anti-isomorphism preserves idempotents, this amounts to the condition that X be the image of an $\mathbf{A}$-map $A^\epsilon : A^a \to A^a$ for some idempotent $\epsilon : a \to a$ in $\mathbf{A}$. Suppose X is an indecomposable projective, and let $m : X \to P_X$ be a map such that $m \cdot p_X = 1$ (Lemma 2.1). By Lemma 1.1, there is an index x in the decomposition (2.1), and a map $j : X \to [x] = A^a$, such that $j \cdot i_x = m$, where i_x is as in (2.2). Hence $j \cdot \varphi_x = j \cdot i_x \cdot p_X = mp_X = 1$. By Lemma 1.2, on the one hand, the $\mathbf{A}$-maps j and φ_x are monic and epic, respectively; and on the other hand, the map $\varphi_x \cdot j : A^a \to A^a$ is idempotent. Since the category of models is anti-isomorphic to the category $\mathbf{A}$, there must be an idempotent $\epsilon : a \to a$ in $\mathbf{A}$ such that $\varphi_x \cdot j = A^\epsilon$. Since the factorization

$$A^a \xrightarrow{\varphi_x} X \xrightarrow{j} A^a$$

of A^ϵ is epi-mono, it follows that X is the image of A^ϵ.

Conversely, let $\epsilon : a \to a$ be an idempotent in $\mathbf{A}$, and let X be the image of A^ϵ. Then in $\mathrm{Ens}^{\mathbf{A}}$ the idempotent A^ϵ has an epi-mono factorization

$$A^a \xrightarrow{q} X \xrightarrow{j} A^a,$$

and by Lemma 1.2, $jq = 1$. Thus X is a retract of a projective, and therefore itself projective. Since it is also the image of an indecomposable, it is also indecomposable, as follows from Lemma 1.1.

For an arbitrary category A, we denote by $I^o(A)$ the (full) subcategory of Ens^A whose objects are the images of models under idempotent maps; by Proposition 2.3, this is the same as the subcategory of indecomposable projectives. Thus $I^o(A)$ contains the category of models $M^o(A)$. We now state the condition for equality.

2.4. DEFINITION. Idempotents split in the category A if each idempotent in A has a splitting in A. Such categories will be called proper categories.

2.5. PROPOSITION. *Let A be a small category. Then* $M^o(A) = I^o(A)$ *if and only if idempotents split in the category A.*

PROOF. It suffices to observe that, because of the anti-isomorphism between A and $M^o(A)$, and in view of Lemma 1.2, the image of an idempotent $A^\epsilon : A^a \to A^a$ is of the form A^b for some $b \in A$ if and only if ϵ has an epi-mono factorization

$$a \xrightarrow[\pi]{} b \xrightarrow[\mu]{} a$$

in A, and then the epi-mono factorization of A^ϵ is

$$A^a \xrightarrow[A^\mu]{} A^b \xrightarrow[A^\pi]{} A^a.$$

3. **Idempotent completion and Morita equivalence.** We begin this section by obtaining an invariant characterization of the category $I^o(A)$ for arbitrary A. This has indeed already been done from the point of view of regarding $I^o(A)$ as a subcategory of Ens^A : by Prop. 2.3, $I^o(A)$ is characterized as the subcategory of indecomposable projectives. This implies that if Ens^A and Ens^B are equivalent; so are $I^o(A)$ and $I^o(B)$. In fact, we can already state:

3.1. PROPOSITION. *Suppose the small categories A and B are proper. Then*

$$\mathrm{Ens}^A \simeq \mathrm{Ens}^B \Longleftrightarrow A \simeq B.$$

PROOF. As just observed, $\mathrm{Ens}^A \simeq \mathrm{Ens}^B \Longrightarrow I^o(A) \simeq I^o(B)$, and, by Prop 2.5, this implies $A \simeq B$. On the other hand, an equivalence between A and B implies an equivalence between the functor categories Ens^A and Ens^B.

Our object now is to characterize $I^o(A)$ from the opposite point of view; namely, as a supercategory of A.

3.2. DEFINITION. A category $\tilde{A} \supset A$ is called the *idempotent completion* of A if

(i) idempotents split in $\tilde{A}$, and

(ii) any functor $\varphi : A \to C$, where idempotents split in C, extends to a functor $\tilde{\varphi} : \tilde{A} \to C$, the extension being unique up to an isomorphism (which is the identity on A).

Obviously the idempotent completion, if it exists, is determined up to equivalence. A proper category is its own idempotent completion. The notion is of course precisely analogous to that of (full) cocompletion (§ 1): to say $\tilde{A}$ is proper is to say that it has colimits for all diagrams of the form $D(\epsilon)$ (Lemma 1.3), while the preservation of such colimits by $\tilde{\varphi}$ is automatic, since they are absolute. More generally, for any class $\mathcal{I}$ of small categories, regarded as the "permissible" index categories, one has the obvious notions of $\mathcal{I}$-cocomplete category, $\mathcal{I}$-cocontinuous functor and $\mathcal{I}$-cocompletion. The present case is, so to speak, the smallest nontrivial one: $\mathcal{I}$ consists of just one index category, which itself consists of a single object and an idempotent. In this case, moreover, cocompletion is the same as completion, since colimit and limit are the same for a diagram $D(\epsilon)$; hence our omission of the prefix "co".

3.3. Lemma. *Suppose*

(i) *A is a full subcategory of $\tilde{A}$, and*

(ii) *every object in $\tilde{A}$ is the image of an idempotent in A. Then $\tilde{A}$ satisfies condition* (ii) *of Def.* 3.2.

Proof. Select for each object $a \in \tilde{A}$ an idempotent ϵ_a in A of which it is the image, and a corresponding epi-mono factorization $\epsilon_a = \pi_a \mu_a$, where π_a and μ_a are morphisms in $\tilde{A}$. In particular, if $a \in A$, take $\epsilon_a = \pi_a = \mu_a = 1_a$. Now suppose given a functor $\varphi : A \to C$, where C is proper. Under any extension $\tilde{\varphi}$ of φ, this factorization goes into an epi-mono factorization in C, so that the object $a\tilde{\varphi}$ is determined up to equivalence as the image of the idempotent $\epsilon_a\varphi$. Pick an epi-mono factorization $\bar{\pi}_a \bar{\mu}_a$ of $\epsilon_a\varphi$, and take $a\tilde{\varphi}$ to be the corresponding image; again, if $a \in A$, take $\bar{\pi}_a = \bar{\mu}_a =$ identity morphism, so that $\tilde{\varphi}$ agrees with φ on objects. For any morphism $\alpha : a \to a'$ in $\tilde{A}$, the morphism $\rho_\alpha = \pi_a \alpha \mu_{a'}$ belongs to A, because A is a full subcategory. Let $\alpha\tilde{\varphi} = \bar{\mu}_a \cdot \rho_\alpha\varphi \cdot \bar{\pi}_{a'}$. Observe that: if $\alpha' : a' \to a''$ is a second morphism in A, then $\rho_\alpha \epsilon_{a'}\, \rho_{\alpha'} = \rho_\alpha\, \rho_{\alpha'}$; if α is an identity morphism I_a, then $\rho_\alpha = \epsilon_a$; and if α is in A, then $\rho_\alpha = \alpha$. It follows easily that $\tilde{\varphi}$ is a functor, with $\tilde{\varphi}\,|_A = \varphi$. Furthermore, if α is either π_a or μ_a, then $\rho_\alpha = \epsilon_a$; and this implies that $\pi_a\tilde{\varphi} = \bar{\pi}_a$ and $\mu_a\tilde{\varphi} = \bar{\mu}_a$. Now let $\tilde{\varphi}'$ be a second extension of φ, and put $\bar{\pi}_a' = \pi_a\tilde{\varphi}'$, $\bar{\mu}_a' = \mu_a\tilde{\varphi}'$, so that also $\bar{\pi}_a' \bar{\mu}_a'$ is an epi-mono factorization of $\epsilon_a\varphi$. Then the family of morphisms $\{\bar{\mu}_a \bar{\pi}_a' \mid a \in A\}$ constitutes a morphism from $\tilde{\varphi}$ to $\tilde{\varphi}'$, as follows from the commutativity of the squares

$$\text{(3.1)}\qquad \begin{tikzcd} {} \arrow[r, "\mu_a"'] \arrow[d, "\alpha"'] & {} \arrow[d, "\rho_\alpha"] \\ {} \arrow[r, "\mu_a{}'"] & {} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {} \arrow[r, "\pi_a"'] \arrow[d, "\rho_\alpha"'] & {} \arrow[d, "\alpha"] \\ {} \arrow[r, "\pi_a{}'"] & {} \end{tikzcd}$$

in A, and therefore of the diagram

$$\begin{tikzcd} {} \arrow[r, "\bar{\mu}_a"'] \arrow[d, "\alpha\tilde{\varphi}"'] & {} \arrow[r, "\bar{\pi}_a{}'"'] \arrow[d, "\rho_\alpha\varphi"] & {} \arrow[d, "\alpha\tilde{\varphi}'"] \\ {} \arrow[r, "\bar{\mu}_{a'}"'] & {} \arrow[r, "\bar{\pi}_{a'}{}'"'] & {} \end{tikzcd}$$

in C. Finally, this is an isomorphism of functors, since the morphism $\bar{\mu}_a\bar{\pi}_a{}'$ has as inverse the morphism $\bar{\mu}_a{}'\bar{\pi}_a$.

3.4. COROLLARY. *Suppose*
(i) *A is a full subcategory of $\tilde{A}$;*
(ii) *every object in $\tilde{A}$ is the image of an idempotent in A;*
(iii) *every idempotent in A has an image (i.e., has an epi-mono factorization) in $\tilde{A}$.*
Then $\tilde{A}$ is the idempotent completion of A.

PROOF. All that has to be shown is that idempotents split in $\tilde{A}$; i.e., that every idempotent in $\tilde{A}$ has an image. Let $\epsilon : a \to a$ be an arbitrary idempotent in $\tilde{A}$. The object a is the image of an idempotent in A, with corresponding epi-mono factorization $\pi\mu$. Then $\pi\epsilon\mu$ is also an idempotent in A, and therefore has an epi-mono factorization $\pi'\mu'$. But then ϵ has the factorization $\epsilon\mu\pi' \cdot \mu'\pi$, which is epi-mono because $\mu'\pi \cdot \epsilon\mu\pi' = \mu'\pi'\mu'\pi' = 1$.

Since the category $I^o(A)$ obviously satisfies the conditions of the corollary — except, of course, that it is $A^o \approx M^o(A)$ which is the subcategory — it follows that $I^o(A)$ is a realization of the idempotent completion of A^o. But then any other realization must satisfy the conditions of the corollary (in terms of A^o), since under an equivalence between supercategories of A^o these conditions are preserved. Thus the conditions of the corollary are necessary as well as sufficient. In the "left" theory, the corresponding category $I(A)$ is precisely the idempotent completion of A.

It is convenient to give a more explicit construction of the idempotent completion (cf. [6], p. 61). We call the new version $\bar{A}$. An object in $\bar{A}$ is an idempotent $\epsilon : a \to a$ in A. A morphism in $\bar{A}$, with domain $\epsilon : a \to a$ and codomain $\epsilon' : a' \to a'$, is a triple $(\epsilon, \alpha, \epsilon')$ such that $\epsilon\alpha = \alpha\epsilon' = \alpha$. Thus, a morphism is given by a commutative diagram

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "\epsilon"'] \arrow[dr, "\alpha"] & a' \arrow[d, "\epsilon'"] \\
a' \arrow[r, "\alpha"'] & a'
\end{tikzcd}$$

Composition is given by the rule $(\epsilon, \alpha, \epsilon')(\epsilon', \alpha', \epsilon'') = (\epsilon, \alpha\alpha', \epsilon'')$. Note that the identity morphism 1_ϵ at the object ϵ is $(\epsilon, \epsilon, \epsilon)$. Note also that for a given pair ϵ, ϵ', the morphisms α such that the above diagram is commutative are precisely those of the form $\epsilon\alpha'\epsilon'$, for an arbitrary morphism $\alpha' : a \to a'$.

The functor $\varphi_A : A \to \bar{A}$ given by $a \mapsto 1_a$, $(\alpha : a \to a') \mapsto (1_a, \alpha, 1_a)$, imbeds A as a full subcategory of $\bar{A}$.

3.5. Proposition. *$\bar{A}$ is the idempotent completion of A.*

Proof. Any idempotent in $\varphi_A(A)$ is of the form $(1_a, \epsilon, 1_a)$, where $\epsilon : a \to a$ is an idempotent in A. In $\bar{A}$ it has the factorization

$$(1_a, \epsilon, 1_a) = (1_a, \epsilon, \epsilon)(\epsilon, \epsilon, 1_a),$$

and this is epi-mono, because

$$(\epsilon, \epsilon, 1_a)(1_a, \epsilon, \epsilon) = (\epsilon, \epsilon, \epsilon) = 1_\epsilon.$$

The same factorization also shows that any object ϵ in $\bar{A}$ is the image of an idempotent in $\varphi_A(A)$. Finally, as already observed, $\varphi_A(A)$ is a full subcategory of $\bar{A}$. Hence $\bar{A}$ is the idempotent completion of A, by Cor. 3.4.

Of course, the two completions $\bar{A}$ and $I(A)$ are in general just equivalent, not isomorphic.

We have already observed that if Ens^A and Ens^B are equivalent, so are $I^o(A)$ and $I^o(B)$. We assert now that the converse is also true.

3.6. THEOREM. *Let A and B be arbitrary small categories. Then an equivalence* $\mathrm{Ens}^A \simeq \mathrm{Ens}^B$ *induces an equivalence* $I^o(A) \simeq I^o(B)$. *Conversely, an equivalence* $I^o(A) \simeq I^o(B)$ *determines an equivalence* $\mathrm{Ens}^A \simeq \mathrm{Ens}^B$, *unique up to isomorphism.*

PROOF. It remains to prove the second half. Let $\varphi : I^o(A) \to I^o(B)$ and $\psi : I^o(B) \to I^o(A)$ be an equivalence pair, and denote by φ_M, ψ_M their restrictions to $M^o(A)$ and $M^o(B)$, regarded as maps into Ens^B and Ens^A. Any equivalence pair $\Phi : \mathrm{Ens}^A \to \mathrm{Ens}^B$ and $\Psi : \mathrm{Ens}^B \to \mathrm{Ens}^A$ must be cocontinuous, so that if they are to be extensions of φ and ψ they are determined, up to isomorphism, as the cocontinuous extensions of φ_M and ψ_M (see § 1). Consider, therefore, cocontinuous extensions Φ and Ψ of φ_M and ψ_M. Let X be any object in $I^o(A)$, so that X is the image of an idempotent $A^\epsilon : A^a \to A^a$ in $M^o(A)$. Lemma 1.4 implies that the object $X\Phi$ is the image of the idempotent $A^\epsilon\Phi = A^\epsilon\varphi_M$ in Ens^B; that the object $X\varphi$ is the image of the idempotent $A^\epsilon\varphi = A^\epsilon\varphi_M$ in $I^o(B)$; and that these two images are isomorphic objects in Ens^B. Hence $X\varphi$ belongs to $I^o(B)$. Since $I^o(B)$ is a full subcategory of Ens^B, the functor Φ takes $I^o(A)$ into $I^o(B)$. Since Φ agrees on $M^o(A)$ with φ_M, and since $I^o(A)$ is the idempotent completion of $M^o(A)$, and $I^o(B)$ is proper, it follows from Def. 3.2 that $\Phi|I^o(A)$ is isomorphic within $I^o(A)$ to the functor φ. We can therefore assume that Φ coincides with φ on $I^o(A)$. Similarly, let Ψ coincide with ψ on $I^o(B)$. Then $\Phi\Psi$ is a cocontinuous extension of $(\varphi\psi)_M$. Since $\varphi\psi : I^o(A) \to I^o(A)$ is isomorphic to the identity functor on $I^o(A)$, $(\varphi\psi)_M : M^o(A) \to \mathrm{Ens}^A$ is isomorphic to the inclusion functor $M^o(A) \subset \mathrm{Ens}^A$. This implies that $\Phi\Psi$ is isomorphic to the identity functor on Ens^A. Similarly, $\Psi\Phi$ is isomorphic to the identity functor on Ens^B. This concludes the proof of the theorem.

The anti-isomorphism between A and A^o extends to an anti-isomorphism between $\bar{A}$ and $\bar{A}^o$. It follows that an equivalence between $I^o(A)$ and $I^o(B)$ amounts to an equivalence between $\bar{A}$ and $\bar{B}$. Hence:

3.6′. THEOREM. *The statement of Theorem* 3.6 *holds with* $I^o(A)$ *and* $I^o(B)$ *replaced by* $\bar{A}$ *and* $\bar{B}$. *Thus, denoting by* $\simeq_M$ *Morita equivalence,*

$$A \simeq_M B \Longleftrightarrow \bar{A} \simeq \bar{B}.$$

The fact that $\bar{A}^o \approx (\bar{A})^o$ also implies:

3.7. COROLLARY. $A \simeq_M B \Longleftrightarrow A^o \simeq_M B^o$.

The following lemma will be used in § 6.

3.8. Lemma. *Two categories A and B have equivalent idempotent completions if and only if there exists a category C such that*

(i) *A and B are full subcategories of C, and*

(ii) *every object in C is the image of an idempotent in A, and also the image of an idempotent in B.*

Proof. Let C be such a category , and $\tilde{C}$ its idempotent completion. Then $\tilde{C}$ is also the idempotent completion of both A and B, as follows from the characterization in Cor. 3.4 of the idempotent completion. Conversely, suppose A and B have equivalent idempotent completions $\tilde{A} \supset A$ and $\tilde{B} \supset B$. There exists a category C containing both $\tilde{A}$ and $\tilde{B}$, such that the inclusions $\tilde{A} \subset C$ and $\tilde{B} \subset C$ are equivalences. (For example, if the functor $f : \tilde{A} \to \tilde{B}$ is one of an equivalence pair, let C be the "mapping cylinder" C_f obtained by identifying a morphism $(\alpha, 1)$ in the base $\tilde{A} \times 1$ of the cylinder $\tilde{A} \times I$, where I is the category consisting of two objects 0 and 1 and one isomorphism between them, with the morphism $f(\alpha)$ in $\tilde{B}$.) Then C is an idempotent completion of both A and B, and therefore satisfies conditions (i) and (ii).

Finally, we can answer question II of the Introduction in the following way. Given a category A, consider the explicit idempotent completion $\bar{A}$. If B is any category such that $\bar{B}$ is equivalent to $\bar{A}$, then the equivalence takes B into a full subcategory B' of $\bar{A}$ such that $\bar{A}$ is the idempotent completion of B'. This means that every object in $\bar{A}$ is the image of an idempotent in B'. In particular, every object in $\varphi_A(A)$ is the image of an idempotent in B'. Conversely, if a full subcategory B' of $\bar{A}$ has this latter property, then every object in $\bar{A}$ is the image of an idempotent in B', and it follows by Cor. 3.4 that $\bar{A}$ is the idempotent completion of B'. Thus:

3.9. Theorem. *For a given category, A, a category B is Morita-equivalent to A if and only if B is equivalent to a full subcategory B' of $\bar{A}$ such that every object in $\varphi_A(A)$ is the image of an idempotent in B'.*

4. **Weak equivalence.** We describe now a simple reformulation of the condition $\bar{A} \simeq \bar{B}$. Essentially, it consists in regarding a functor from $\bar{A}$ to $\bar{B}$ as a *weak functor* from A to B.

4.1. Definition. For any categories A and B, a *weak functor* $f : A \to B$ is a function that takes objects to objects, morphisms to morphisms, and preserves composition. For two weak functors $f, g : A \to B$, a *weak functor morphism* $t : f \to g$ is a weak functor from $\mathbf{2} \times A$ to B (where $\mathbf{2}$ means the category given by the ordered set $\langle 0, 1 \rangle$) that agrees with f and g on $0 \times A$ and $1 \times A$.

A weak functor differs from an ordinary one, which for emphasis we shall also call a strong functor, only in that identity morphisms need not be preserved; but since an identity morphism satisfies the equality $1_a 1_a = 1_a$, under a weak functor it goes into an idempotent. As for a morphism $t : f \to g$ between weak functors, it is readily seen to amount to a family $\{t_a \mid a \in A\}$ of morphisms in B such that

(i) for any morphism $\alpha : a \to a'$ in A, $\alpha f \cdot t_{a'} = t_a \cdot \alpha g$, and

(ii) for any object a in A, $1_a f \cdot t_a = t_a \cdot 1_a g = t_a$.

The first equality in (ii) is already contained in (i). If f and g are strong functors, the notion of weak functor morphism reduces to the ordinary one.

Composition of weak functor morphisms is as in the ordinary case: compose objectwise. This yields a category $\mathrm{Funct}_w(A, B)$ of weak functors from A to B. Observe that for a weak functor $f : A \to B$, the identity morphism $1_f : f \to f$ is given by $(1_f)_a = 1_a f$; the assignment $a \mapsto 1_{af}$ is in general not a morphism from f to f. The multiplication

$$\mathrm{Funct}_w(A, B) \times \mathrm{Funct}_w(B, C) \to \mathrm{Funct}_w(A, C),$$

which on functors is composition, is defined in the same way as in the ordinary case. One has the obvious extension of the notion of equivalence $\simeq$ between categories to that of weak equivalence $\simeq_w$.

For any category A, let $\tilde{A}$ be a category satisfying the hypothesis of Lemma 3.3, and denote by $i_A : A \subset \tilde{A}$ the inclusion functor. As in the proof of Lemma 3.3, select for each object $a \in \tilde{A}$ an idempotent ϵ_a of which it is the image, and a corresponding epi-mono factorization $\epsilon_a = \pi_a \mu_a$, with $\epsilon_a = \pi_a = \mu_a = 1_a$ if $a \in A$. Define $r_A : \tilde{A} \to A$ as follows: for any object $a \in \tilde{A}$, ar_A = domain (= codomain) of ϵ_a; and for any morphism $\alpha : a \to a'$, $\alpha r_A = \pi_a \alpha \mu_{a'}$ (called ρ_α in the proof of Lemma 3.3). Then r_A is a weak functor; it is not in general a (strong) functor, since it takes an identity morphism 1_a into the morphism ϵ_a.

4.2. Lemma. *Let $\tilde{A} \supset A$ be any category satisfying the hypothesis of Lemma 3.3; in particular, $\tilde{A}$ can be the idempotent completion of A. Then the inclusion $i_A : A \subset \tilde{A}$ determines a weak equivalence $\tilde{A} \simeq_w A$.*

Proof. The weak functor in the other direction is r_A. Since $i_A \cdot r_A = 1_A$, it remains only to show that $r_A \cdot i_A$ is isomorphic to the identity functor $1_{\tilde{A}}$. We assert that the isomorphism is given by $\mu : 1_{\tilde{A}} \to r_A \cdot i_A$ and $\pi : r_A \cdot i_A \to 1_{\tilde{A}}$, where $\mu = \{\mu_a\}, \pi = \{\pi_a\}, a \in \tilde{A}$. That μ and π are weak functor morphisms follows from the commutativity of the diagrams (3.1). (The fact that one of the functors involved — namely, $1_{\tilde{A}}$ — is a strong functor automatically implies the extra equality required for the case $\alpha = 1_a$.) For the composite $\mu\pi : 1_{\tilde{A}} \to 1_{\tilde{A}}$, $(\mu\pi)_a = \mu_a\pi_a = 1_a$. For the composite $\pi\mu$, $(\pi\mu)_a = \pi_a\mu_a = \epsilon_a = 1_a(r_A \cdot i_A)$, and as already noted, this means that $\pi\mu$ is the identity morphism at $r_A \cdot i_A$.

4.3. Lemma. *Let A be any category, B a proper category, and f a weak functor from A to B. Then f is isomorphic to a (strong) functor.*

Proof. Select for each object $a \in A$ an epi-mono factorization $\pi_a\mu_a$ of the idempotent 1_af, and denote the corresponding image by ag:

$$af \underset{\pi_a}{\to} ag \underset{\mu_a}{\to} af.$$

For any morphism $\alpha : a \to a'$ in A, let $\alpha g = \mu_a \cdot \alpha f \cdot \pi_{a'}$. Then $g : A \to B$ is a strong functor; for, firstly, if $\alpha' : a' \to a''$, then

$$\begin{aligned}(\alpha g)(\alpha' g) &= \mu_a \cdot \alpha f \cdot \pi_{a'} \cdot \mu_{a'} \cdot \alpha' f \cdot \pi_{a''}\\ &= \mu_a \cdot \alpha f \cdot 1_{a'}f \cdot \alpha' f \cdot \pi_{a''}\\ &= \mu_a \cdot (\alpha \cdot 1_{a'} \cdot \alpha')f \cdot \pi_{a''}\\ &= \mu_a \cdot (\alpha\alpha')f \cdot \pi_{a''} = (\alpha\alpha')g;\end{aligned}$$

and secondly,

$$1_ag = \mu_a \cdot 1_af \cdot \pi_a = \mu_a \cdot \pi_a\mu_a \cdot \pi_a = 1_{ag}.$$

We assert that the maps π_a, $a \in A$, determine an isomorphism $\pi : f \to g$, and the maps μ_a, $a \in A$, an isomorphism $\mu : g \to f$. This follows from the commutativity of the diagram

$$\begin{tikzcd} af \arrow[r, "\pi_a"'] \arrow[d, "\alpha f"'] & ag \arrow[r, "\mu_a"'] \arrow[d, "\alpha g"] & af \arrow[d, "\alpha f"] \\ a'f \arrow[r, "\pi_{a'}"] & a'g \arrow[r, "\mu_{a'}"] & a'f \end{tikzcd}$$

for any $\alpha : a \to a'$; again, the extra equality needed for the case $\alpha = 1_a$ is automatically satisfied, because g is a strong functor. Finally, $(\mu\pi)_a = \mu_a\pi_a = 1_{ag} = 1_ag$, while $(\pi\mu)_a = \pi_a\mu_a = 1_af$; so that both $\mu\pi$ and $\pi\mu$ are identity morphisms.

It follows from Lemma 4.3 that if B is proper, the category $\mathrm{Funct}_w(A, B)$ is equivalent to its subcategory $\mathrm{Funct}(A, B)$. This is because the latter is a full subcategory.

Now consider any categories A and B together with their idempotent completions $\tilde{A} \supset A$ and $\tilde{B} \supset B$. On the one hand, the equivalences $A \simeq_w \tilde{A}$, $B \simeq_w \tilde{B}$ imply $\mathrm{Funct}_w(A, B) \simeq \mathrm{Funct}_w(\tilde{A}, \tilde{B})$. On the other hand, by Lemma 4.3, $\mathrm{Funct}_w(\tilde{A}, \tilde{B}) \simeq \mathrm{Funct}(\tilde{A}, \tilde{B})$. There is thus an equivalence $\mathrm{Funct}_w(A, B) \simeq \mathrm{Funct}(\tilde{A}, \tilde{B})$. Furthermore, if C is a third category, with idempotent completion $\tilde{C} \supset C$, then clearly the revelant equivalences induce an equivalence between the multiplications

$$\mathrm{Funct}_w(A, B) \times \mathrm{Funct}_w(B, C) \to \mathrm{Funct}_w(A, C)$$

and

$$\mathrm{Funct}(\tilde{A}, \tilde{B}) \times \mathrm{Funct}(\tilde{B}, \tilde{C}) \to \mathrm{Funct}(\tilde{A}, \tilde{C});$$

in particular, the monoidal categories $\mathrm{Funct}_w(A, A)$ and $\mathrm{Funct}(\tilde{A}, \tilde{A})$ are equivalent. As a consequence, we have:

4.4. THEOREM. $A \simeq_M B \Longleftrightarrow A \simeq_w B$.

For the explicit completions $\bar{A}$ and $\bar{B}$ the equivalence pair of functors $w : \mathrm{Funct}(\bar{A}, \bar{B}) \to \mathrm{Funct}_w(A, B)$ and $s : \mathrm{Funct}_w(A, B) \to \mathrm{Funct}(\bar{A}, \bar{B})$ are found to be as follows. If $F : \bar{A} \to \bar{B}$, its image F^w under w is determined by the equalities $(1_a, \alpha, 1_{a'})F = (1_a F, \alpha F^w, 1_{a'} F)$ for every morphism $\alpha : a \to a'$ in A. (The object $1_a F$ in $\bar{A}$ is then the same as the morphism $1_a F^w$ in A.) If $T : F \to G$ is a morphism in $\mathrm{Funct}(\bar{A}, \bar{B})$ (so that T is a family $\{T_\epsilon\}$, where ϵ runs through the idempotents in A), its image $T^w : F^w \to G^w$ is determined by the equalities $T_{1a} = (1_a F,\ T_a{}^w,\ 1_a G)$. If $f : A \to B$, its image f^s under s is given by $(\epsilon,\ \alpha,\ \epsilon')f^s = (\epsilon f,\ \alpha f,\ \epsilon' f)$; and if $t : f \to g$ is a morphism in $\mathrm{Funct}_w(A, B)$, its image t^s is given by $t_\epsilon{}^s = (\epsilon f,\ t_a \cdot \epsilon g = \epsilon f \cdot t_a,\ \epsilon g)$, where ϵ runs through idempotents in A.

5. **Tensor products and the Eilenberg-Gabriel-Watts theorem.** For any A-B-biset $X = X_B{}^A$ and B-C-biset $Y = Y_C{}^B$, we define an A-C-biset $X \otimes Y = X_B{}^A \otimes Y_C{}^B$ as follows. The set $(X \otimes Y)_c^a$ consists of all pairs (x, y) such that $x \in X_b{}^a$ and $y \in Y_c{}^b$ for some b, subject to the equivalence relation $\sim$ generated by the requirement $(x\beta, y) \sim (x, \beta y)$, where β is a morphism in B. We shall make no notational distinction between a pair (x, y) and the corresponding equivalence class. Left and right operations by A and C are defined unambiguously by the formulas $\alpha(x, y) = (\alpha x, y)$, $(x, y)\gamma = (x, y\gamma)$. If Z is a C-D-biset, then clearly $(X \otimes Y) \otimes Z \approx X \otimes (Y \otimes Z)$.

The A-C-biset $X \otimes Y$ can be regarded as a colimit in the category $\mathrm{Ens}_A{}^C$ of A-C-bisets in the following way. Consider the fundamental category (§ 1) of the object X in the category $\mathrm{Ens}_A{}^B$. In left-right language, it can be described as follows: an object is an element $x \in X$; an arrow is a triple $(\alpha, x, \beta) : \alpha x\beta \to x$; and the composite of

$$\alpha'\alpha x\beta\beta' \xrightarrow{(\alpha', \alpha x\beta, \beta')} \alpha x\beta \xrightarrow{(\alpha, x, \beta)} x$$

is

$$\alpha'\alpha x\beta\beta' \xrightarrow{(\alpha'\alpha, x, \beta\beta')} x.$$

The diagram D^X assigns to the object x the A-B-biset $A_a \times B^b$ (where $x \in X_b{}^a$), which we can regard as the set $[x]$ of triples (α, x, β), $\alpha \in A_a$, $\beta \in B^b$, with $\alpha'(\alpha, x, \beta)\beta' = (\alpha'\alpha, x, \beta\beta')$; and to the arrow (α, x, β) the A-B-map $A_\alpha \times B^\beta$:

$$(\alpha', \alpha x\beta, \beta') \mapsto (\alpha'\alpha, x, \beta\beta').$$

Take now the same index category, but assign to the object $x(\in X_b{}^a)$ the A-C-biset $A_a \times Y^b$, and to the arrow (α, x, β) the A-C-map $A_\alpha \times Y^\beta$. Thus to $x \in X$ is assigned an A-C-biset whose elements can be written as (α, x, y), $\alpha \in A_a$, $y \in Y^b$; and to the arrow (α, x, β) is assigned the A-C-map

$$(\alpha', \alpha x\beta, y) \mapsto (\alpha'\alpha, x, \beta y).$$

One readily sees that $X \otimes Y$ is the colimit of this diagram. It is equally well the colimit of another diagram based on the fundamental category of the object Y in the category $\mathrm{Ens}_B{}^C$.

In particular, if X is an A-B-biset and Y is the B-B-biset B, then the description of $X \otimes Y$ as a colimit over the diagram D^X coincides with the description of X as a colimit over the same diagram. Thus, $X \otimes B \approx X$. Similarly, for any B-C-biset Y, $B \otimes Y \approx Y$.

An A-B-map $f : X \to X'$ and a B-C-map $g : Y \to Y'$ determine the A-C-map $f \otimes g : X \otimes Y \to X' \otimes Y'$ given by $(x, y)(f \otimes g) = (xf, yg)$. This defines a bifunctor $- \otimes -$, covariant in both variables, from $\mathrm{Ens}_A{}^B \times \mathrm{Ens}_B{}^C$ to $\mathrm{Ens}_A{}^C$.

Next, we define for an A-B-biset $X = X_B{}^A$ and a C-B-biset $Y = Y_B{}^C$ a C-A-biset $(X, Y) = (X_B{}^A, Y_B{}^C)$. The set $(X, Y)_a^c$ consists of all (right) B-maps $\varphi : X^a \to Y^c$. Left and right operations by C and A are defined by the formulas $\gamma\varphi = \varphi \cdot Y^\gamma$ and $\varphi\alpha = X^\alpha \cdot \varphi$. In particular, for the case that X is the B-B-biset B, it is easily seen that $(B, Y) \approx Y$. Notice, also, that if X and Y are both A-B-bisets, the A-A-biset (X, Y) is not in general the same thing as the hom-set $\mathrm{Ens}_A{}^B(X, Y)$; but it is, if A is the unit category $\mathbb{1}$.

An A-B-map $f : X' \to X$ and a C-B-map $g : Y \to Y'$ determine the C-A-map $(f, g) : (X, Y) \to (X', Y')$ given by

$$\varphi \mapsto f^a \cdot \varphi \cdot g^c, \quad \varphi \in (X, Y)_a^c,$$

where f^a and g^c mean the restrictions $f|X'^a$ and $g|Y^c$. This defines a bifunctor $(-\,,-)$, contravariant in the first variable and covariant in the second, from $\mathrm{Ens}_A{}^B \times \mathrm{Ens}_C{}^B$ to $\mathrm{Ens}_C{}^A$.

The following relation between the bifunctors $- \otimes -$ and $(-\,,-)$ is easily verified by means of the preceding description of the tensor product as a colimit.

5.1. Lemma. *Let $X_A{}^D$, $Y_B{}^C$, $W_B{}^A$ be bisets with operator categories as indicated by the notation. Then*

$$(X_A{}^D \otimes W_B{}^A, Y_B{}^C) \approx (X_A{}^D, (W_B{}^A, Y_B{}^C)),$$

and this isomorphism (of C-D-bisets) is natural in X, Y and W.

In addition, there exists the canonical evaluation map

$$e : (X_B{}^A, Y_B{}^C) \otimes X_B{}^A \to Y_B{}^C$$

(of C-B-bisets) given by $(\varphi, x) \mapsto \varphi(x)$, and the canonical section map

$$s : X_B{}^A \to (Y_C{}^B, X_B{}^A \otimes Y_C{}^B)$$

(of A-B-bisets) given by $x \mapsto s_x$, where $s_x : y \mapsto (x, y)$.

All of the preceding construction can be altered — by replacing each operator category by its opposite and shifting the operation to the other side — to obtain a version which is formally different but in substance identical. In this version one has the $(C$-$A)$-biset $X_A{}^B \otimes Y_B{}^C$, where $(X \otimes Y)_a^c$ consists of all pairs (x, y) such that $x \in X_a{}^b$ and $y \in Y_b{}^c$ for some b, with the equivalence relation $(\beta x, y) \sim (x, y\beta)$ and the operations $\gamma(x, y) = (x, \gamma y)$, $(x, y)\alpha = (x\alpha, y)$; and the A-C-biset $(X_A{}^B, Y_C{}^B)$, where $(X, Y)_c{}^a$ consists of all B-maps $\varphi : X_a \to Y_c$, and the operations are given by $\alpha\varphi = X_\alpha \cdot \varphi$, $\varphi\gamma = \varphi \cdot Y_\gamma$. The isomorphism of Lemma 5.1, for example, then reads:

$$(X_D{}^A \otimes W_A{}^B, Y_C{}^B) \approx (X_D{}^A, (W_A{}^B, Y_C{}^B)).$$

To use both versions side by side, however, would entail a notational ambiguity: $X_B{}^A \otimes Y_A{}^B$ would mean two different things, and similarly $(X_B{}^A, Y_B{}^A)$. We shall stick to the first version.

Consider now, for a fixed $W = W_B{}^A$, the functor $T_W = -\otimes W_B{}^A$: $\mathrm{Ens}^A \to \mathrm{Ens}^B$. By Lemma 5.1, we have the natural isomorphism

$$(X_A \otimes W_B{}^A, Y_B) \approx (X_A, (W_B{}^A, Y_B)),$$

in this case an isomorphism between hom-sets. This says that the functor $S^W = (W_B{}^A, -)$: $\mathrm{Ens}^B \to \mathrm{Ens}^A$ is right-adjoint to the functor T_W. The unit $1 \to T_W S^W$ and counit $S^W T_W \to 1$ are given by the section maps $X_A \to (W_B{}^A, X_A \otimes W_B{}^A)$ and evaluation maps $(W_B{}^A, Y_B) \otimes W_B{}^A \to Y_B$.

We show now that all adjunctions between the two categories Ens^A and Ens^B are of the above form. More precisely, consider the category $\mathrm{Adj}_A{}^B$ of adjunctions from Ens^A to Ens^B. This is equivalent to the category of functors $\mathrm{Ens}^A \to \mathrm{Ens}^B$ that have right adjoints, which is the same (since Ens^A has a small set of generators — the models) as the category of functors $\mathrm{Ens}^A \to \mathrm{Ens}^B$ that are cocontinuous. The notation $\mathrm{Adj}_A{}^B$ will consequently also be used to mean this category of functors. We have just described an assignment $W \mapsto T_W$ of objects in $\mathrm{Ens}_A{}^B$ to objects in $\mathrm{Adj}_A{}^B$. Similarly, an A-B-map $f : W \to W'$ determines a functor morphism $T_f = -\otimes f$; and we have thereby defined a covariant functor $T : \mathrm{Ens}_A{}^B \to \mathrm{Adj}_A{}^B$. On the other hand, any functor $F \in \mathrm{Adj}_A{}^B$, being cocontinuous, is determined up to isomorphism by its restriction $F \mid M^o(A)$ to the subcategory of models. This restriction amounts to a contravariant functor from A to Ens^B, i.e., it amounts to an A-B-biset W; and the isomorphism $A \otimes W \approx W$ implies that the functors T_W and F agree on $M^o(A)$. Thus every functor $F \in \mathrm{Adj}_A{}^B$ is isomorphic to a functor of the form T_W. Finally, for any W and W', an arbitrary morphism between the two cocontinuous functors T_W and $T_{W'}$ is determined by its values on the models. This implies that every such morphism is of the form T_f; and the representation is unique, because $A \otimes f \approx f$. We conclude that the functor T determines an equivalence between the categories $\mathrm{Ens}_A{}^B$ and $\mathrm{Adj}_A{}^B$.

Furthermore, if C is a third operator category, the relevant equivalences induce an equivalence between the multiplications

$$\mathrm{Adj}_A{}^B \times \mathrm{Adj}_B{}^C \to \mathrm{Adj}_A{}^C$$

and

$$\mathrm{Ens}_A{}^B \times \mathrm{Ens}_B{}^C \to \mathrm{Ens}_A{}^C.$$

This is because: $(X \otimes W_B{}^A) \otimes Z_C{}^B \approx X \otimes (W_B{}^A \otimes Z_C{}^B)$; if $f : W \to W'$ and $g : Z \to Z'$, then $(x \otimes f) \otimes g \approx (X \otimes f) \otimes g$; and $(f \otimes 1_Z) \cdot (1_{W'} \otimes g) = (1_W \otimes g) \cdot (f \otimes 1_{Z'}) = f \otimes g$. In particular, the monoidal categories $\mathrm{Adj}_{\mathrm{A}}{}^{\mathrm{A}}$ and $\mathrm{Ens}_{\mathrm{A}}{}^{\mathrm{A}}$ are equivalent. As a consequence, the condition for equivalence of the categories $\mathrm{Ens}^{\mathrm{A}}$ and Ens^{B} is the existence of a pair of mutually inverse bisets $W_B{}^{\mathrm{A}}$ and $Z_{\mathrm{A}}{}^{B}$. We can sum up by the following Eilenberg-Gabriel-Watts theorem for categories of actions.

5.2. THEOREM. *The bicategory whose objects are the categories* $\mathrm{Ens}^{\mathrm{A}}$, *and whose "hom-sets" are the categories* $\mathrm{Adj}_{\mathrm{A}}{}^{B}$, *is equivalent to the bicategory whose "hom-sets" are the categories* $\mathrm{Ens}_{\mathrm{A}}{}^{B}$ (*with tensor product as composition*). *In particular,* $\mathrm{A} \simeq_M B$ *if and only if there exist bisets* $W_B{}^{\mathrm{A}}$ *and* $Z_{\mathrm{A}}{}^{B}$ *such that*

$$
\begin{aligned}
& W_B{}^{\mathrm{A}} \otimes Z_{\mathrm{A}}{}^{B} \approx \mathrm{A} \quad \text{and} \\
& Z_{\mathrm{A}}{}^{B} \otimes W_B{}^{\mathrm{A}} \approx B.
\end{aligned} \tag{5.1}
$$

Condition (5.1) can be made more precise. Since it asserts that the functor T_W is an equivalence, it implies that the unit and counit for the adjunction $T_W \dashv S^W$, already specified by the evaluation and section maps, are isomorphisms:

$$X \approx (W, X \otimes W),$$

$$(W, Y) \otimes W \approx Y,$$

natural in X and Y, respectively. Similarly, for Z, we have isomorphisms $Y \approx (Z, Y \otimes Z)$ and $(Z, X) \otimes Z \approx X$, natural in Y and X. In particular, taking for X a model A^a and for Y a model B^b, observing that $A^a \otimes W = W = W^a$ and $B^b \otimes Z = Z^b$, and using naturality, we obtain isomorphisms

$$
\begin{aligned}
\mathrm{A} &\approx (W, W), \\
(W, B) \otimes W &\approx B, \\
B &\approx (Z, Z), \\
(Z, \mathrm{A}) \otimes Z &\approx \mathrm{A}.
\end{aligned}
$$

On the other hand, condition (5.1) asserts not only that T_W and T_Z are separately equivalences, but that they form an equivalence pair; so that the two functors

$$T_W, S^Z : \mathrm{Ens}^{\mathrm{A}} \to \mathrm{Ens}^{B}$$

are isomorphic, as are also the two functors

$$S^W, T_Z : \mathrm{Ens}^{B} \to \mathrm{Ens}^{\mathrm{A}}.$$

Of the four adjunctions

$$T_W \dashv S^W, S^Z \dashv T_Z, T_W \dashv T_Z, S^Z \dashv S^W,$$

the unit and counit have been specified only for the first two. Pick a definite isomorphism between, say, T_W and S^Z. This determines uniquely a conjugate isomorphism between S^W and T^Z; it also determines units and counits for the last two adjunctions, together with compatible isomorphisms between any two of the four adjunctions, it being understood that the isomorphism between, say, the first and the third is the identity on T_W. This gives rise, then, to the following (partly redundant) additional list of compatible isomorphisms, natural in X and Y:

$$\begin{aligned} X \otimes W &\approx (Z, X), \\ (W, Y) &\approx Y \otimes Z, \\ X &\approx X \otimes W \otimes Z, \\ Y \otimes Z \otimes W &\approx Y, \\ X &\approx (W, (Z, X)), \\ (Z, (W, Y)) &\approx Y. \end{aligned}$$

As before, taking X and Y to be models and using naturality, we obtain

$$\begin{aligned} W &\approx (Z, A), \\ (W, B) &\approx Z, \\ A &\approx W \otimes Z, \\ Z \otimes W &\approx B, \\ A &\approx (W, (Z, A)), \\ (Z, (W, B)) &\approx B. \end{aligned}$$

Notice also that because we have now selected a unit and counit for the adjunction $T_W \dashv T_Z$, the isomorphisms (5.1) are compatible, in that, e.g., the two isomorphisms

$$W \otimes Z \otimes W \approx (W \otimes Z) \otimes W \approx A \otimes W \approx W$$

and

$$W \otimes Z \otimes W \approx W \otimes (Z \otimes W) \approx W \otimes B \approx W$$

coincide. This is because the two corresponding isomorphisms between the functors $T_W\, T_Z\, T_W$ and T_W coincide. Similarly, there is no ambiguity for the isomorphism $Z \otimes W \otimes Z \approx Z$.

6. **Some re-examinations.** Let us now rederive, directly from condition (5.1), the result of § 3 concerning Morita equivalence. An A-B-biset W determines a category $C(W)$, in the following way. Regard A and B as disjoint; then an object in $C(W)$ is any object in either A or B; a morphism in $C(W)$ is either a morphism in A or B, or an element $w \in W$, with $w : a \to b$ if $w \in W_b{}^a$; and composition is given either by composition in A or B, or by left or right operation by A or B, whichever is relevant. There is an obvious functor from $C(W)$ onto the category $\mathbf{2}$; and in particular, for the A-A-biset A, the category $C(A)$ is isomorphic to the category $\mathbf{2} \times A$.

On the other hand, let f and g be any two functors from A and B, respectively, to a third category C. Then f and g determine an A-B-biset $[f, g]$: an element of $[f, g]_b^a$ is any triple (a, γ, b), where $\gamma : af \to bg$ is a morphism in C; and if $\alpha : a' \to a$, $\beta : b \to b'$ are morphisms in A and B, then $\alpha(a, \gamma, b) = (a', \alpha f \cdot \gamma \cdot \beta g, b')$.

Next, consider a tensor product $W_B{}^A \otimes Z_C{}^B$. Each of the categories $C(W)$ and $C(Z)$ contains a copy of the category B, and we can therefore form the amalgamation $C(W) *_B C(Z)$, i.e., the pushout (in the category of categories) of the diagram $C(W) \leftarrow B \to C(Z)$. It contains, as full subcategories, disjoint copies of A, B and C. The mapping of $C(W \otimes Z)$ into $C(W) *_B C(Z)$ that takes A identically onto A, C identically onto C, and the morphism (w, z) into the morphism $w \cdot z$, is well-defined and is an isomorphism with the full subcateogry generated by A and C in $C(W) *_B C(Z)$. We can therefore identify it with this subcategory. If the two categories A and C are the same, then the category $C(W) *_B C(Z)$ contains two copies of A, and we can perform on it a "self-amalgamation," i.e., take the colimit $K(W, Z)$ of the diagram $A \rightrightarrows C(W) *_B C(Z)$, where the arrows are the two inclusions. This is the same as the colimit of the diagram

$$\begin{tikzcd} & C(W) & \\ A \arrow[ur] \arrow[dr] & & B \arrow[ul] \arrow[dl] \\ & C(Z) & \end{tikzcd} .$$

The category $K(W, Z)(= K(Z, W))$ contains a copy of A and a copy of B; the two are disjoint, and their objects together are all the objects of $K(W, Z)$.

Now suppose condition (5.1) is satisfied. We assume that the isomorphisms (5.1) have been adjusted as described in § 5; in particular, we want the consistency condition mentioned in the last paragraph of § 5. The effect of this condition is that there is defined a single isomorphism between, e.g., a product $W \otimes Z \otimes \cdots \otimes W \otimes Z$ and the biset A, independent of parentheses.

The composition $C(W \otimes Z) \xrightarrow{\approx} C(A) \xrightarrow{\approx} \mathbf{2} \times A \to A$, where the last functor is the projection, determines a 1-1 correspondence between morphisms of the form $(w, z) : a \to a'$ in $C(W \otimes Z)$ and morphisms $\alpha : a \to a'$ in A. Write $\theta(w, z) = \alpha$. The function θ satisfies the identities

$$(6.1) \qquad \begin{aligned} \theta(w\beta, z) &= \theta(w, \beta z), \\ \alpha\,\theta(w, z) &= \theta(\alpha w, z), \ \ \theta(w, z)\alpha = \theta(w, z\alpha). \end{aligned}$$

Similarly, for $C(Z \otimes W)$, write $\theta(z, w)$ for the corresponding morphism in B.

Now reduce the category $K(W, Z)$ by the equivalence relation $\sim$ on morphisms generated by putting $w \cdot z \sim \theta(w, z)$ and $z \cdot w \sim \theta(z, w)$. Consider the effect of this reduction on the full subcategory $[A]$ generated by A in $K(W, Z)$. A morphism μ in $[A]$ is either a morphism in A, or of the form $\mu = w_1 \cdot z_1 \cdot w_2 \cdot z_2 \cdot \cdots \cdot w_n \cdot z_n$; the latter representation is unique up to replacements $w\beta \cdot z \leftrightarrow w \cdot \beta z$ and $z\alpha \cdot w \leftrightarrow z \cdot \alpha w$. to any morphism $\mu \in [A]$ assign the morphism $\bar{\mu} \in A$ as follows: if $\mu \in A$, then $\bar{\mu} = \mu$; if $\mu = w_1 \cdot z_1 \cdot \cdots \cdot w_n \cdot z_n$, then $\bar{\mu} = \theta(w_1, z_1) \cdots \theta(w_n, z_n)$. The identities (6.1) assure that $\bar{\mu}$ is well-defined. If now two morphisms μ and μ' differ by a replacement of the form $w \cdot z \leftrightarrow \theta(w, z)$ (with incorporation of $\theta(w, z)$ into an adjoining factor, if one exists, as an operator), then the identities (6.1) suffice to ensure $\bar{\mu} = \bar{\mu}'$. On the other hand, if μ and μ' differ by a replacement of the from $z \cdot w \leftrightarrow \theta(z, w)$, the equality $\bar{\mu} = \bar{\mu}'$ is ensured by the consistency condition. It follows that the effect of the equivalence relation $\sim$ on the full subcategory $[A]$ is to reduce it precisely to A, and therefore A appears as a full subcategory of the reduced category $\tilde{K} = K(W, Z)/\sim$. The same is true of B.

Since the functuin θ is surjective, for each object $a \in A$ there is a pair (w_a, z_a) such that $\theta(w_a, z_a) = 1_a$. Then in the category $\tilde{K}$, $w_a \cdot z_a = 1_a$. This mean that $z_a \cdot w_a$ is an idempotent in B, and a is its image. Thus, every object in A, and therefore every object in $\tilde{K}$, is the image of an idempotent in B. Similarly, every object in $\tilde{K}$ is the image of an idempotent in A. Hence (Lemma 3.8) A and B have equivalent idempotent completions.

Conversely, if A and B have equivalent idempotent completions, then by Lemma 3.8 they are both contained as full subcategories in a category C where every object is both the image of an idempotent in A and the image of an idempotent in B. Denoting by i_A and i_B the inclusions $A \subset C$ and $B \subset C$, consider the A-B-biset $W = [i_A, i_B]$ and the B-A-biset $Z = [i_B, i_A]$, and form the tensor product $W \otimes Z$. The assignment

$$\varphi : ((a, \gamma, b), (b, \gamma', a')) \mapsto \gamma\gamma'$$

determines a well-defined A- A-map from $W \otimes Z$ to A. We assert that φ is an isomorphism. To see this, select for each object $a \in A$ an object $b_a \in B$ and morphisms

$$b_a \underset{\pi_a}{\rightarrow} a \underset{\mu_a}{\rightarrow} b_a$$

such that $\mu_a \pi_a = 1_a$. For any morphism $\alpha : a \rightarrow a'$ in A, we have

$$\varphi((a, \mu_a, b_a), (b_a, \pi_a \alpha, a')) = \alpha.$$

Now suppose $\varphi((a, \gamma, b), (b, \gamma', a')) = \alpha$. Then the commutative diagram

$$\begin{tikzcd}
& b_a \arrow[dr, "\pi_a"] \arrow[dd, "\pi_a \gamma"] & \\
a \arrow[ur, "\mu_a"] \arrow[dr, "\gamma"'] & & a' \\
& b \arrow[ur, "\gamma'"'] &
\end{tikzcd}$$

shows that $((a, \gamma, b), (b, \gamma', a'))$ and $((a, \mu_a, b_a), (b_a, \pi_a, a'))$ are the same element of $W \otimes Z$. Thus φ is a bijection, and therefore an isomorphism. Similarly, $Z \otimes W \approx B$. This completes the rederivation of the result of § 3.

As a second look backward, let us replace the description in Theorem 3.9, of those categories B Morita-equivalent to a given category A, by one analogous to that in, e.g., [2] for the case of modules over a ring. We proceed again from condition (5.1).

The fact that $T_Z : \mathrm{Ens}^B \rightarrow \mathrm{Ens}^A$ determines an equivalence of categories implies the following. The image $B^b \otimes Z = Z^b$ of each model B^b is (as already observed) an indecomposable projective in Ens^A. The family $\{Z^b \mid b \in B\}$ is a generating set in Ens^A (i.e., for any two distinct A-maps $f, g : X \to X'$, there exists a member Z^b of the family and an A-map $\varphi : Z^b \to X$ such that $\varphi f \neq \varphi g$), since the same is true of the family $\{B^b\}$ in Ens^B. Finally, the full subcategory determined by the family $\{Z^b\}$ is equivalent with B^o, since it corresponds under T_Z to the full subcategory $M^o(B)$. Thus, if $A \simeq_M B$, then the category Ens^A contains a generating set consisting of indecomposable projectives, such that the full subcategory determined by them is equivalent to the category B^o.

Conversely, consider any full subcategory of Ens^A determined by a generating set consisting of indecomposable projectives. Denote by B the opposite of this subcategory, and by Z^b, $b \in |B|$, the members of the generating set. For the set $\{Z^b\}$ to be a generating set, the following property is necessary (and sufficient): for any object $a \in A$, there exists an epimorphism $Z^b \to A^a$ for some b. For, let Φ be the family of all A-maps $\varphi : Z_\varphi \to A^a$ from any member Z_φ of the generating set, and consider the canonical map $p : \coprod_{\varphi \in \Phi} Z \to A^a$. The definition of generating set implies that p is an epimorphism; hence, since A^a is an indecomposable projective, it follows as in the proof of Proposition 2.3 that there exist maps $\varphi \in \Phi$ and $j : A^a \to Z_\varphi$ such that $j\varphi = 1$. Thus, φ is epic; in fact, we have shown that A^a is an idempotent image of Z_φ. Hence, any object in $I^o(A)$, being by definition the idempotent image of a model, is the idempotent image of some Z^b. Furthermore, the assumption that the A-sets Z^b are indecomposable projectives means that the category B^o determined by them is a full subcategory of $I^o(A)$. It follows, by Lemma 3.8, Theorem 3.6′ and Corollary 3.7, that $A \simeq_M B$. Thus:

6.1. THEOREM. $A \simeq_M B$ *if and only if* B^o *is equivalent to a full subcategory of* $I^o(A)$ *whose objects constitute a generating set for* Ens^A.

This result also follows, essentially, from the characterization of functor categories in [3].

7. **Examples.** Let A be a monoid — i.e., a category with a single object a. The condition (Theorem 3.9) that a second monoid B be Morita-equivalent to A amounts to this: B must be isomorphic to a full subcategory B' of $\bar{A}$ (B' is then the full subcategory generated by a single object ϵ in $\bar{A}$), such that the object 1_a in $\bar{A}$ is the image of an idempotent in B'. In other words, there must exist, in addition to the idempotent ϵ, morphisms π and μ in A such that $\mu\pi = 1_a$ and the diagram

$$
\begin{tikzcd}
{} \arrow[r, "\pi"] \arrow[d, "\epsilon"'] & {} \arrow[r, "\mu"] \arrow[d, "1_a"] & {} \arrow[d, "\epsilon"] \\
{} \arrow[r, "\pi"'] & {} \arrow[r, "\mu"'] & {}
\end{tikzcd}
$$

is commutative; and the monoid $B' \approx B$ amounts to the monoid $\epsilon A \epsilon$ (for which ϵ is the identity morphism). This is Proposition 4 of [1] and equivalent to Theorem 6.1 of [7].

This example raises the question: when is a category A Morita-equivalent to a monoid? The answer is again provided by Theorem 3.9: when A contains an idempotent ϵ such that for every object $a \in A$ there exist morphisms π_a and μ_a for which $\mu_a \pi_a = 1_a$ and the diagram

$$
\begin{tikzcd}
{} \arrow[r, "\pi_a"] \arrow[d, "\epsilon"'] & {} \arrow[r, "\mu_a"] \arrow[d, "1_a"] & {} \arrow[d, "\epsilon"] \\
{} \arrow[r, "\pi_a"'] & {} \arrow[r, "\mu_a"'] & {}
\end{tikzcd}
$$

is commutative. The monoid in question then consists of all morphisms of the form $\epsilon \alpha \epsilon$, and these are all the monoids Morita-equivalent to A.

As a second example, consider the two well-known parallel versions of the definition of a graph (or better, of the category of graphs): by "arrows and objects," or by "arrows only". They are in essence as follows. Let A be the category whose objects are the category $\mathbb{1}$ and the category $\mathbb{2}$, and whose morphisms are all functors between them. Let B be the full subcategory of A generated by the object $\mathbb{2}$ alone. Then an "arrows-and-objects" graph is a left A-set, while an "arrows-only" graph is a left B-set. The fact that the two notions are equivalent is simply the fact that A and B are Morita-equivalent. Notice that this is a case of a category which is not equivalent to a monoid but is Morita-equivalent to a monoid.

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