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QUASI-PURE PROJECTIVE AND INJECTIVE TORSION FREE ABELIAN GROUPS OF RANK 2

D. ARNOLD, C. I. VINSONHALER AND W. J. WICKLESS

An abelian group, G, is *quasi-pure projective* (q.p.p.) if for every pure subgroup, A, of G and every $f \in \mathrm{Hom}(G, G/A)$ there is $g \in \mathrm{Hom}(G, G)$ with $\pi_A g = f$, where $\pi_A \in \mathrm{Hom}(G, G/A)$ is the quotient map. Dually, G is *quasi-pure injective* (q.p.i.) if for every pure subgroup, A, of G and every $f \in \mathrm{Hom}(A, G)$ there is $g \in \mathrm{Hom}(G, G)$ with $gi_A = f$, where $i_A \in \mathrm{Hom}(A, G)$ is the inclusion map. This paper contains a characterization of q.p.p. and q.p.i. torsion free abelian groups of rank 2; a partial solution to Problem 17 of Fuchs [2].

A torsion free abelian group, G, is *homogeneous* if any two pure rank 1 subgroups of G are isomorphic and *strongly homogeneous* if for any two pure rank 1 subgroups of G there is an automorphism of G sending one onto the other.

THEOREM A: *If G is a homogeneous reduced torsion free abelian group of rank* 2, *then*

(1) *G is q.p.p. iff G is completely decomposable,*

(2) *G is q.p.i. iff G is strongly homogeneous.*

A strongly homogeneous group, G, is *special* if p-rank $G \leqq 1$ for all primes p, where *p-rank* G is the Z/pZ-dimension of G/pG. Special torsion free abelian groups of finite rank have been described by Richman [6]. The next theorem gives a characterization of rank 2 strongly homogeneous groups as well as a method for constructing strongly homogeneous rank 2 groups that are not special.

THEOREM B: *If G is a torsion free abelian group of rank* 2, *then G is strongly homogeneous iff either*

(1) *G is homogeneous completely decomposable or*

(2) (a) *$Q \otimes_Z \mathrm{End}(G) = Q(\sqrt{N})$ for some square free integer N*; (b) *$NG = G$*; (c) *p-rank $G \leqq 1$ for all primes $p \neq 2$ such that N is a quadratic residue* mod p; (d) *2-rank $G \leqq 1$ if N is a quadratic residue* mod 8; *and* (e) *if N is not a quadratic residue* mod 8, *and if $g \in G$, $\alpha \in Q$, with $\frac{1}{4}(g + \alpha\sqrt{N}g) \in G$, then $\frac{1}{2}g \in G$.*

A torsion free abelian group, G, is *$R(G)$-locally free* if p-rank $G = 0$ or rank G for all primes p.

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THEOREM C: *If G is a non-homogeneous reduced torsion free abelian group of rank 2, then*:

(1) *G is q.p.p. iff G is $R(G)$-locally free and any two independent elements of G have incomparable type,*

(2) *G is q.p.i. iff either $G = A \oplus B$ with* $\sup\{\text{type }(A), \text{type }(B)\} = \text{type }(Q)$ *or any two independent elements of G have incomparable type and G is p-reduced for all primes p with $pG \neq G$.*

Examples of groups satisfying the hypotheses of Theorem C.1 are given in Section 2. Furthermore, a reduced torsion free abelian group is both q.p.p. and q.p.i. iff either G is homogeneous completely decomposable or G satisfies the condition of Theorem C.1.

Fundamental references are Fuchs [2] and [3] and Reid [5]. Let G be a torsion free abelian group of finite rank and $0 \neq x \in G$. For a prime, p, the *p-height of x in G*, $h_p(x)$, is i if $x \in p^iG \backslash p^{i+1}G$ and ∞ if no such i exists; $H(x)$ is the sequence $(h_p(x))$ indexed by the primes; if $y \in G$ then $H(x)$ and $H(y)$ are *equivalent* if $h_p(x) = h_p(y)$ for all but a finite number of primes, q, with $h_q(x) < \infty$ and $h_q(y) < \infty$; the *type of x in G*, $T(x)$, is the equivalence class determined by $H(x)$; if $X = \langle x \rangle_*$, the pure subgroup of G generated by x, then $T(a) = T(b)$ for all $a, b \in X$, so that the *type of X*, $T(X)$, is well defined.

Two rank 1 groups A and B are isomorphic iff $T(A) = T(B)$. The *typeset of* G is $\{T(A) \mid A \text{ is a pure rank 1 subgroup of } G\}$. Thus G is homogeneous iff the typeset of G is a singleton. The *inner type of* G, $IT(G)$, is $\inf\{\tau \in \text{typeset } G\}$ where the order on the typeset of G is induced by the natural ordering of $\{H(x) \mid x \in G\}$.

If A and B are two pure subgroups of G with $a \in A$, $b \in B$, then $h_p(a + b) \geqq \min\{h_p(a), h_p(b)\}$ and equality holds if $h_p(a) < h_p(b)$ or $G = A \oplus B$.

A torsion free abelian group, G, is *completely decomposable* if G is the direct sum of rank 1 subgroups and *strongly indecomposable* if whenever $nG \subseteq A \oplus B \subseteq G$ for some non-zero integer n, then either $A = 0$ or $B = 0$. The *quasi-endomorphism ring* of G is $Q \otimes_Z \text{End}(G)$, where $\text{End}(G)$ is the endomorphism ring of G. If G is strongly indecomposable then every $0 \neq f \in \text{End}(G)$ is either a monomorphism or is nilpotent (Reid [5]).

Finally, if p is a prime, then $p^\omega G = \bigcap_{i=1}^{\infty} p^iG$ is the *p-divisible subgroup of* G. The group G is *p-reduced* if $p^\omega G = 0$.

§1. **$R(G)$-locally free and strongly homogeneous rank 2 groups.** Let G be a torsion free abelian group of rank n. For each maximal independent subset $\{x_1, \cdots, x_n\}$ of G define $Y_i = \{q_ix_i \mid q_1x_1 + \cdots + q_nx_n \in G \text{ for } q_j \in Q \text{ and } 1 \leqq j \leqq n\}$ and $X^i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\rangle_*$ (where $\langle S\rangle_*$ denotes the pure subgroup of G generated by S).

Then $G/X' \simeq Y_i$ for $1 \leqq i \leqq n$ and G is a subdirect sum of $Y_i, \cdots, Y_n$. (e.g., see Fuchs [2], p. 42.).

The following equivalences are known; their verification is routine.

LEMMA 1.1: *Let G be a torsion free abelian group of rank n and define X^i and Y_i as above for $1 \leqq i \leqq n$. The following are equivalent:*

(a) *G is $R(G)$-locally free;*

(b) *$Z_p \otimes_Z G = G_p$ is a free Z_p-module for all primes p with $pG \neq G$, where Z_p is the localization of Z at p;*

(c) *If p is a prime and $pY_i = Y_i$ for some i, then $pG = G$.*

A torsion free abelian group G of rank 2 is a *$\sqrt{N}$-group* if $Q \otimes_Z \mathrm{End}(G) = Q(\sqrt{N})$ for some square free integer N. Identify $\sqrt{N}$ with the quasi-endomorphism of G whose square is N. In this case every non-zero endomorphism of G is a monomorphism.

PROOF OF THEOREM B: Homogeneous completely decomposable groups are strongly homogeneous (since every pure subgroup is a summand, e.g., see Fuchs [3] p. 115). Thus we may assume that G is strongly indecomposable since if $G/(A \oplus B)$ is bounded and G is homogeneous then $G \simeq A \oplus B$ (Fuchs [3]).

($\rightarrow$) (a) Strongly homogeneous groups are $\sqrt{N}$-groups (Reid [5]).

(b) Choose $0 \neq x$ with $\sqrt{N}x \in G$. Since G is strongly homogeneous there is an automorphism $\alpha + \beta\sqrt{N}$ of G with $\alpha, \beta \in Q$ and $(\alpha + \beta\sqrt{N})(x) = \gamma(\sqrt{N}x)$ for $\gamma \in Q$. Now x and $\sqrt{N}x$ are independent so $\alpha = 0$ and $\beta\sqrt{N}$ is an automorphism of G. Write $\beta = c/d$ for relatively prime integers c and d. Then $G = \beta\sqrt{N}(G) = (c/d)\sqrt{N}(G)$ so that $G = (c^2/d^2)NG$, $dG = G$, $cG = G$, $\sqrt{N}G = G$, and $NG = G$. Note that $\sqrt{N}$ is, in fact, an automorphism of G.

(c) and (d) Assume that p-rank $G = 2$ and choose $0 \neq x \in G$. Define ℓ to be $\sup\{h_p(bx + \sqrt{N}x) \mid b$ is an integer relatively prime to $p\}$. If $\ell = \infty$ then the p-height of $\sqrt{N}x + \langle x\rangle_*$ in $G/\langle x\rangle_*$ is ∞ so that p-rank $G \leqq 1$, a contradiction. Thus ℓ is finite.

The hypotheses on p guarantee that N is a quadratic residue mod p^i for all $i \geqq 3$. Consequently, there is $b \in Z$ with $b^2 \equiv N (\mathrm{mod}\, p^{2\ell+1})$. Since $\sqrt{N}$ is an automorphism of G, $h_p(\sqrt{N}x) = h_p(x)$ and $h_p(x) \leqq h_p((b + \sqrt{N})(x)) \leqq \ell$ (it is sufficient to assume that b is relatively prime to p since otherwise p divides N and by (*b*), $pG = G$ a contradiction).

Since G is strongly homogeneous there is an automorphism $\alpha + \beta\sqrt{N}$ of G with $\alpha, \beta \in Q$; $(\alpha + \beta\sqrt{N})(b + \sqrt{N})(x) = m\sqrt{N}x$ for some $m \in Z$; and $p^{\ell+1} \nmid m$ (since $h_p(bx + \sqrt{N}x) \leqq \ell$). But G is a $\sqrt{N}$-group so that every non-zero endormorphism is monic. Thus $(\alpha + \beta\sqrt{N})(b + \sqrt{N}) = m\sqrt{N}$. Multiply both sides of the preceding equations by $b-\sqrt{N}$ to see that $\alpha = -mN/(b^2 - N)$ and $\beta = (mb)/(b^2 - N)$. Let $g = (\alpha + \beta\sqrt{N})(\sqrt{N}x) \in G$ so that $(b^2 - N)g = -mN(-bx + \sqrt{N}x)$, a contradiction (since $p^{2\ell+1} \mid b^2 - N$; $p \nmid N$; $p^{\ell+1} \nmid m$; and $h_p(-bx + \sqrt{N}x) \leqq \ell$).

(e) Assume N is not a quadratic residue mod 8. Let $g \in G, \alpha \in Q$ be such that $x = \frac{1}{4}(g + \alpha\sqrt{N}g) \in G$ but $\frac{1}{2}g \notin G$. By (b) $2 \nmid N$ and, since $\sqrt{N}$ is an automorphism of G, $h_2(\sqrt{N}g) = h_2(g) = 0$. It follows that $\alpha = c/d$, $c, d \in Z$, with d odd. Thus, by adding a suitable integral multiple of $\sqrt{N}g$ to x, we see that $\frac{1}{4}(g + a\sqrt{N}g) \in G$ with a an odd integer. Now define a map from $\langle y = g + (a-2)\sqrt{N}g\rangle_*$ into $\langle g\rangle_*$ sending y into $2mg$ for suitable odd $m \in Z$. Note that $h_2(y) = 1$. A short computation shows that no endomorphism of G lifts the above map.

($\leftarrow$) Since G is a $\sqrt{N}$-group, G is homogeneous (Reid [5]). Thus it is sufficient to prove that if $g_1, g_2 \in G$ with $h_p(g_1) = h_p(g_2)$ for all primes p then there is $\alpha + \beta\sqrt{N} \in \operatorname{End}(G)$ with $\alpha, \beta \in Q$ and $(\alpha + \beta\sqrt{N})(g_1) = g_2$ (to see that $\alpha + \beta\sqrt{N}$ is an automorphism construct the inverse sending $g_2 \to g_1$ and note that every non-zero endomorphism is a monomorphism).

We may assume that $g_1 = ax + b\sqrt{N}x$ and $g_2 = cx + d\sqrt{N}x$ for some $0 \neq x \in G, a, b, c, d \in Z$ (clear denominators if necessary). There is a unique $\alpha + \beta\sqrt{N} \in Q \otimes \operatorname{End}(G)$ with $(\alpha + \beta\sqrt{N})(g_1) = g_2$, i.e., $(\alpha + \beta\sqrt{N})(a + b\sqrt{N}) = c + d\sqrt{N}$. The only problem is in proving that $\alpha + \beta\sqrt{N} \in \operatorname{End}(G)$.

Regard G as a subgroup of $Q \otimes G$ with $G \subset G_p \subset Q \otimes G$ so that $G = \bigcap_p G_p$ (where $G_p = Z_p \otimes_Z G$, and Z_p is the localization of Z at the prime p). With this convention, it is enough to show that $(\alpha + \beta\sqrt{N})(G_p) \subseteq G_p$ for all primes p.

Multiplying the defining equation for $\alpha + \beta\sqrt{N}$ by $a - b\sqrt{N}$ shows that $\alpha = (ac - bdN)/(a^2 - b^2N)$ and $\beta = (ad - bc)/(a^2 - b^2N)$.

Let p be a prime divisor of $a^2 - b^2N$. If p divides N then $pG = G$, by (b) so that $(\alpha + \beta\sqrt{N})(G_p) \subseteq G_p$. Otherwise N is a quadratic residue mod p. If $p \neq 2$ or if $p = 2$ and N is a quadratic residue mod 8 then p-rank $G \leqq 1$. Suppose that p-rank $G = 1$.

Then G_p is homogeneous; reduced, hence strongly indecomposable; $qG_p = G_p$ for all primes $q \neq p$, and p-rank $G_p = 1$. Thus G_p is strongly homogeneous (e.g., see Murley [4]). It follows that $(\alpha + \beta\sqrt{N})(G_p) \subseteq G_p$ (e.g., regard g_1, g_2 as elements of G_p and use the uniqueness of $\alpha + \beta\sqrt{N}$).

Now assume that p is a prime and does not divide $a^2 - b^2N$. Then $a^2 - b^2N$ is a unit in G_p so that $(\alpha + \beta\sqrt{N})(G_p) \subseteq G_p$.

We are left with the case that $p = 2$, $2G \neq G$ (in particular, N is odd), N is not a quadratic residue mod 8, and $a^2 \equiv b^2N(\text{mod } 2)$. Choose $0 \neq g \in G$ such that $h_2(g) = 0 = h_2(\sqrt{N}g)$. Then by condition (e), any element x in G_2 can be written in the form $x = (u/2)\, g + (v/2)\, \sqrt{N}g$ where $u, v \in Z_2$. A direct computation shows $(\alpha + \beta\sqrt{N})x \in G_2$.

PROPOSITION 1.2: *Let G be a $\sqrt{N}$-group. If p is a prime such that $p \nmid N$ and if N is not a quadratic residue* mod *p, then p-rank* $G = 0$ *or* 2.

PROOF: Assume that p-rank $G = 1$ and that k is the least positive integer with $k\sqrt{N} \in \text{End}(G)$. The minimality of k guarantees the existence of $x \in G$ with $h_p(k\sqrt{N}x) = 0 = h_p(x)$. Now $x + pG$ generates $G/pG \simeq Z/pZ$ so there is $y \in G$ and an integer, c, relatively prime to p with $py = cx + k\sqrt{N}x$. Multiplying by $c - k\sqrt{N}$ yields $p(c - k\sqrt{N})(y) = (c^2 - k^2N)(x)$. Thus $c^2 \equiv k^2N(\text{mod } p)$ a contradiction to the assumption that N is not a quadratic residue mod p.

COROLLARY 1.3: *If G is a special qpi $\sqrt{N}$-group, then* $pG = G$ *for all primes p such that* $p \mid N$ *or N is not a quadratic residue* mod *p*.

COROLLARY 1.4: *Let G be a torsion free abelian group of rank* 2. *If G is R(G)-locally free and strongly homogeneous, then G is completely decomposable.*

PROOF: The only other possibility is that G is a strongly indecomposable $\sqrt{N}$-group (Reid [5]). But G is $R(G)$-locally free so by Theorem B, $pG = G$ for all primes p such that $p \mid N$ or N is a quadratic residue mod p.

Let A be a pure rank subgroup of G and $B = \sqrt{N}(A)$, a pure rank 1 subgroup of G since $\sqrt{N}$ is an automorphism of G (see proof of Theorem B). It is sufficient to prove that the p-component of $G/(A \oplus B)$ is zero for all primes p such that $p \nmid N$ and N is not a quadratic residue mod p.

Let $g \in G$ and $pg = a + b \in A \oplus B$. If $h_p(a) > 0$ or $h_p(b) > 0$ then $g \in A \oplus B$. So suppose that $h_p(a) = h_p(b) = 0$. Then $b = (m/n)\sqrt{N}a$ for some relatively prime integers m and n, i.e., $pg = (1 + m/n\sqrt{N})(a)$ and $png = (n + m\sqrt{N})(a)$. Multiplying by $n - m\sqrt{N}$ shows that $pn(n - m\sqrt{N})g = (n^2 - m^2N)(a)$. But $h_p(a) = 0$ so $n^2 \equiv m^2N(\text{mod } p)$; $nb = m\sqrt{N}(a)$; $h_p(b) = 0$; and g.c.d.$(m, n) = 1$ guarantees that $p \nmid m$. Also $p \nmid N$ so N is a quadratic residue mod p. It follows that G is completely decomposable, a contradiction.

REMARKS: (1) The hypothesis that rank $G = 2$ is not necessary for Corollary 1.4.(E. L. Lady, private communication).

(2) Theorem B can be used to construct examples of strongly homogeneous strongly indecomposable rank 2 groups that are not special. If $G \subset Q(\sqrt{N})$ is a rank 2 special group then $H = G \cap (\bigcap_{p \in S} H_p)$ is strongly homogeneous where S is a set of primes, H_p is a free Z_p-module and $2 \neq p \in S$ is a prime such that N is not a quadratic residue mod p or $p = 2$ and N is not a quadratic residue mod 8.

§2. Quasi-pure-projective groups of rank 2.

THEOREM 2.1: *Let G be a strongly indecomposable q.p.p. torsion free abelian group of finite rank. Then G is R(G)-locally free.*

PROOF: Assume that G is not $R(G)$-locally free and let p be a prime with $0 < p\text{-rank } G < \text{rank } G$. Choose a p-basic subgroup, B, of G (e.g., see Fuchs [2], Chapter VI) and let C be the pure subgroup of G generated by B. It follows that G/C is a non-zero p-divisible group and that $(1/p)\pi_C$ is a well defined map from G to G/C.

Since G is q.p.p. there is $g \in \text{End}(G)$ with $\pi_C g = (1/p)\pi_C$. Let $h = pg\text{-}1$ so that $h(G) \subseteq C$ and $\ker h \subseteq p^\omega G$. Since C is pure in G and $p^\omega C = 0$, $h : C \to C$ is a monomorphism. Choose a positive integer n with $nC \subseteq h(C) \subseteq C$ (since h is a monomorphism, h is a unit in $Q \otimes_Z \text{End}(C)$, Reid [5]). A short computation shows that $nG \subseteq p^\omega G \oplus C \subseteq G$, contradicting the assumption that G is strongly indecomposable.

LEMMA 2.2: *If G is a torsion free abelian group of rank 2 with pure rank 1 subgroups A and B, then $T(A) \leqq T(B)$ iff $T(G/B) \leqq T(G/A)$.*

PROOF: Suppose that $A \neq B$ and choose non-zero elements x_1 and x_2 in A and B respectively. Then $\{x_1, x_2\}$ is a maximal independent subset of G. Using the notation preceding Proposition 1.1, $X^1 = B$, $X^2 = A$, $G/X^1 \simeq Y_1 \supseteq A$; and $G/X^2 \simeq Y_2 \supseteq B$. But $Y_1/A \simeq Y_2/B$ (Fuchs [2], p. 42) so that $T(Y_1) + T(B) = T(Y_2) + T(A)$ with $T(A) \leqq T(Y_1)$ and $T(B) \leqq T(Y_2)$. Consequently, $T(A) \leqq T(B)$ iff $T(Y_1) \leqq T(Y_2)$.

LEMMA 2.3: *If G is a reduced q.p.p. torsion free abelian group of rank 2 with $IT(G) \in$ typeset G, then G is homogeneous.*

PROOF: Let A be a pure rank 1 subgroup of G with $T(A) = IT(G)$ and let B be another pure rank 1 subgroup of G. Now $T(A) \leqq T(B)$, so by Lemma 2.2, there is $0 \neq f: G/B \to G/A$. Since G is q.p.p. there is $g \in \text{Hom}(G, G)$ with $f\pi_B = \pi_A g$. But $g(B) \subseteq A$ so the proof is complete if $g(B) \neq 0$ (i.e., $T(B) \leqq T(A)$).

Suppose that $g(B) = 0$, i.e., $\ker g = B$ and that $T(A) < T(B)$. Then $g(G) \subseteq \langle g(A) \rangle_*$. Furthermore, assume that $\langle g(A) \rangle_* = B$. Since G is reduced there is a prime p with $pB \neq B$ and it is sufficient to assume that $g(G) \not\subseteq pB$. Choose $x \in G$ with $h_p(g(x)) = 0$, in particular $x \notin B$, and define $C = \langle px - g(x)\rangle_*$. It follows that $(1/p)\pi_C g$ is a well defined element of $\mathrm{Hom}(G, G/C)$. Since G is q.p.p. there is $h \in \mathrm{Hom}(G, G)$ with $\pi_C h = (1/p)\pi_C g$. Now $h(B) \subseteq C$ and B is fully invariant (since $T(A) < T(B)$) so that $h(B) \subseteq B \cap C = 0$. Moreover, $ph - g \in \mathrm{Hom}(G, G)$ and $h(G) \subseteq \langle h(A)\rangle_* \neq B$ (for if $h(G) \subseteq B$ then $(ph - g)(G) \subseteq B \cap C = 0$, a contradiction to the assumption that $h_p(g(x)) = 0$).

As a consequence of the preceding remarks, we need only consider the case that $g(B) = 0$, $T(A) < T(B)$ and $\langle g(A)\rangle_* \neq B$. But g is a non-zero endomorphism of G that is neither monic nor nilpotent so G cannot be strongly indecomposable (Reid [5]).

Choose pure rank 1 subgroups D and E and a non-zero integer n with $nG \subseteq D \oplus E \subseteq G$. Since G is not homogeneous we may assume that either $T(D) < T(E)$ or $T(D)$ and $T(E)$ are incomparable. Now G is reduced so there is a prime p with $pE \neq E$. Choose non-zero elements d and e in D and E, respectively, with $h_p(e) = 0$. Define $H = \langle p^{k+1} d + e\rangle_*$ where $n = p^k \ell$ and g.c.d.$(p, \ell) = 1$. It follows that the p-height of $x + H$ in G/H is $\geqq k + 1$ for all $x \in E$. Let $f' \in \mathrm{Hom}(G\, G/H)$ be the composite of $G \xrightarrow{n} D \oplus E \to E \to (E + H)/H \subseteq G/H$ and define $f = (1/p^{k+1})f'$. In particular, $f(e) = (n/p^{k+1})(e + H) = (\ell/p)(e + H)$.

Since G is q.p.p. there is $g \in \mathrm{Hom}(G, G)$ with $\pi_H g = f$. By the hypotheses, E is fully invariant. But $g(e) + H = f(e) = \ell/p(e + H)$ so that $(pg - \ell)(e) \in E \cap H = 0$. Thus $pg(e) = \ell(e)$ with g.c.d.$(p, \ell) = 1$, a contradiction to the assumption that $h_p(e) = 0$. The proof is now complete.

Proof of Theorem A.1: $(\leftarrow)$ Homogeneous completely decomposable groups are q.p.p. since every pure subgroup is a summand (Fuchs [3]).

$(\rightarrow)$ It is sufficient to show that if G is strongly indecomposable then a contradiction occurs (since if G is quasi-isomorphic to a homogeneous completely decomposable group then G is homogeneous completely decomposable Fuchs [3]).

Now G is $R(G)$-locally free (Theorem 2.1). Thus if G is strongly homogeneous, then, by Corollary 1.4, G is completely decomposable giving the desired contradiction.

Let A and B be two pure rank 1 subgroups of G with $A \cap B = 0$. By Lemma 2.2 there is an isomorphism $f: G/A \to G/B$. Since G is q.p.p. there are g, $h \in \mathrm{Hom}(G, G)$ with $\pi_B g = f\pi_A$ and $\pi_A h = f^{-1}\pi_B$. But $\pi_B gh = f\pi_A h = ff^{-1}\pi_B$ so that $gh - 1 \in \mathrm{Hom}\,(G, B)$. Since every non-zero endomorphism of G is a monomorphism $gh = 1$.

Similarly, $hg = 1$, so g is an automorphism of G with $g(A) = B$; i.e., G is strongly homogeneous.

PROOF OF THEOREM C.1: Let G be a torsion free abelian group of rank 2. Then $IT(G) \notin$ typeset (G) iff any two independent elements of G have incomparable type (use the fact that $T(A \oplus B) \geqq \min\{T(A), T(B)\}$).

$(\rightarrow)$ By Lemma 2.3, $IT(G) \notin$ typeset G. Thus G is strongly indecomposable and Theorem 2.1 applies.

$(\leftarrow)$ Let $f \in \mathrm{Hom}(G, G/A)$, where A is a pure rank 1 subgroup of G, and let $B = \ker f$. Now $T(G/B) \leqq T(G/A)$ so by Lemma 2.2, $T(A) \leqq T(B)$. If $A \cap B = 0$ then $T(A) = IT(G) \in$ typeset (G) (if $x \in G$, $mx \in A \oplus B$ for some $0 \neq m \in Z$ so that $T(x) \geqq \min\{T(A), T(B)\} = T(A)$), a contradiction. Thus $A = B$ and f induces $f' = c/d \in \mathrm{Hom}(G/A, G/A)$ for some relatively prime integers c and d. Now $d(G/A) = G/A$ so $dG = G$ by Proposition 1.1. Thus $c/d \in \mathrm{End}(G)$ and G is q.p.p.

EXAMPLE: There is a non-homogeneous q.p.p. reduced torsion free abelian group of rank 2. Let $V = Qx \oplus Qy$ be a Q-vector space of dimension 2 and $S = \{x, y\} \cup \{ax + by \mid a, b \in Z$ and g.c.d.$(a, b) = 1\}$. Write P, the set of primes of Z, as a disjoint union of countably many infinite subsets, say $P = \bigcup_{i=1}^{\infty} P_i$. Enumerate the elements of S and define G to be the subgroup of V generated by $\{s_i/p_i \mid s_i \in S, p_i \in P_i\}$.

If $g \in G$ then $mg = ns$ for some integers m and n and $s \in S$. Thus typeset $(G) = \{T(s) \mid s \in S\}$. It follows that $h_p(s_i) = 0$ if $p \notin P_i$ and 1 if $p \in P_i$. Thus $IT(G) \notin$ typeset G since $IT(G) = T(Z)$. Furthermore, G is $R(G)$-locally free so G is q.p.p. by Theorem C.1.

§3. Quasi-pure-injective groups of rank 2.

PROOF OF THEOREM A.2: $(\rightarrow)$ If G is homogeneous completely decomposable then every pure subgroup of G is a summand (Fuchs [3]) so that G is strongly homogeneous. Otherwise, $\mathrm{End}(G)$ is a subring of an algebraic number field (Beaumont-Pierce [1]) and every non-zero endomorphism of G is a monomorphism (Reid [5]).

Let A and B be two pure rank 1 subgroups of G. Since G is homogeneous there is an isomorphism $f: A \rightarrow B$. Furthermore, G is q.p.i. so there is $g \in \mathrm{Hom}(G, G)$ with $gi_A = f$ (where $i_A \in \mathrm{Hom}(A, G)$ is the inclusion map). Similarly, choose $h \in \mathrm{Hom}(G, G)$ with $hi_B = f^{-1}$. Then $hgi_A = hf = hi_Bf = 1_A$ and $hg = 1_G$ ($hg - 1$ is not a monomorphism so $hg - 1 = 0$). Similarly, $gh = 1$ so that g is an automorphism of G with $g(A) = B$.

($\leftarrow$) Let A be a pure rank 1 subgroup of G and $i_A \in \mathrm{Hom}(A,G)$ the inclusion map. If $f \in \mathrm{Hom}(A, G)$ then $B = \langle f(A)\rangle_*$ is a pure rank 1 subgroup of G. Now G is strongly homogeneous so choose an automorphism α of G with $\alpha(A) = B$. But αi_A and f are elements of the rank 1 group $\mathrm{Hom}(A, B)$, hence $c\alpha i_A = df$ for some relatively prime integers c and d. Consequently, $cB = c\alpha i_A(A) = df(A) = dB$ and $dB = B$. Since G is homogeneous, $dG = G$, $c(\alpha/d) \in \mathrm{End}(G)$ and $f = c(\alpha/d)i_A$, as desired.

Proposition 3.1: *Let G be a non-homogeneous reduced torsion free abelian group of rank 2 with pure rank 1 subgroups A and B such that $G/(A \oplus B)$ is bounded. Then G is q.p.i. iff $T(A)$ and $T(B)$ are incomparable and* $\max\{T(A), T(B)\} = T(Q)$. *In this case* $G \simeq A \oplus B$.

($\leftarrow$) Note that p-rank $G \leqq 1$ for all primes p so that $G \simeq A \oplus B$ (e.g., see Murley [4] or Beaumont-Pierce [1]). Assume that $G = A \oplus B$, and let C be a pure rank 1 subgroup of G and $f \in \mathrm{Hom}(C, G)$. By using the projection maps of G onto A and B one can verify that there is $g \in \mathrm{Hom}(G, G)$ with $gi_c = f$, i.e., G is q.p.i.

($\rightarrow$) Since G is non-homogeneous either $T(A) < T(B)$ or $T(A)$ and $T(B)$ are incomparable. In either case, B is fully invariant. Suppose that $\max\{T(A), T(B)\} \neq T(Q)$. Choose elements a and b of A and B, respectively, with $h_p(a) = h_p(b) = 0$. Then $h_p(pa + b) = h_p(b) = 0 \leqq h_p(a + b)$. Since $G/(A \oplus B)$ is bounded there is an integer k, relatively prime to p, and a homomorphism $f: \langle pa + b\rangle_* \rightarrow G$ with $f(pa + b) = k(a + b)$. Choose $g \in \mathrm{Hom}(G, G)$ with $gi_c = f$, where $C = \langle pa + b\rangle_*$. Then $g(a) = \alpha a + \beta b$ and $g(b) = \gamma b$ for some α, $\beta, \gamma \in Q$. Moreover, $g(pa + b) = p\alpha a + p\beta b + \gamma(b) = k(a + b)$ and $\alpha a = (k/p)(a) \in G$, contradicting the assumption that $h_p(a) = 0$. Thus $\max\{T(A), T(B)\} = T(Q)$.

Since G is reduced, $T(A) < T(B)$ is impossible.

Proposition 3.2: *Suppose that G is a non-homogeneous strongly indecomposable torsion free abelian group of rank 2. Then G is q.p.i. iff $IT(G) \notin$ typeset G and $p^\omega G = 0$ or G for all primes p.*

Proof: ($\leftarrow$) In this case any two distinct pure rank 1 subgroups have incomparable type. Thus if A is a pure rank 1 subgroup of G and $f: A \rightarrow G$ then f is multiplication by m/n, where m and n are relatively prime integers. Consequently, $nA = A$, $nG = G$, and $m/n \in \mathrm{End}(G)$.

($\rightarrow$) Suppose that there is a prime p with $p^\omega G = A$, a pure rank 1 subgroup of G. Then $1/p \in \mathrm{Hom}(A, G)$. Since G is q.p.i. there is $f \in \mathrm{Hom}(G, G)$ with $fi_A = 1/p$. But G is strongly indecomposable so f is a monomorphism (f is either a monomorphism or nilpotent; the latter is impossible). Furthermore, $(pf-1)(A)=0$ so that $pf-1$ is nilpotent. Thus $1+(pf-1)=pf$ is an automorphism of G so that $pG=G$, a contradiction. Consequently, $p^\omega G=0$ or G for all primes p.

Assume that $IT(G)\in$ typeset (G) and choose pure rank 1 subgroups A and B of G with $T(A)<T(B)$. There is $0\neq f\in \mathrm{Hom}(A,B)$ so (since G is q.p.i.) there is $0\neq g\in \mathrm{Hom}(G,G)$ with $gi_A=f$. But B is fully invariant so $g(G)\subseteq B$, i.e., g is not a monomorphism. Thus g is nilpotent since G is strongly indecomposable. By Reid [5], $Q\otimes_Z \mathrm{End}(G)$ $=Q\oplus Qg$.

Since $G\neq A\oplus B$ there is $0\neq x\in G$ and a prime p with $px=a+b\in A\oplus B$ and $h_p(a)=h_p(b)=0$ (otherwise, $A\oplus B$ is p-prime in G for all primes p, hence pure). Consequently, there is $f: A\rightarrow G$ with $f(a)=mx$ for some integer m relatively prime to p. But G is q.p.i. so choose $h\in \mathrm{Hom}(G,G)$ with $hi_A=f$. Now $h=\alpha+\beta g$ for some $\alpha,\beta\in Q$ and $h(a)=(\alpha+\beta g)(a)=(m/p)(a+b)\in G$. Since $g(G)\subseteq B$, $\alpha a=(m/p)(a)$ and $\alpha=m/p$. On the other hand, $h(b)=(\alpha+\beta g)(b)=\alpha(b)=(m/p)(b)\in G$ (since g is nilpotent and $g(G)\subseteq B$) a contradiction to the assumption that $h_p(b)=0$. The proof is now complete.

The proof of Theorem C.2 is now a consequence of the results of this section.

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New Mexico State University, Las Cruces, New Mexico 88003
University of Connecticut, Storrs, Connecticut 06268