ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 1, Winter 1976

EXTENSIONS OF UNIFORM AND PROXIMITY STRUCTURES

DON A. MATTSON

1. **Introduction.** Let (X, δ) be a proximity space, where X is dense in a completely regular space T. By $P(X)$ we denote the collection of real-valued proximity functions on X, and $P^*(X)$ is the algebra of bounded members of $P(X)$. Then $P^*(X)$ determines an admissible collection $\mathcal{P}^*$ of totally bounded pseudometrics for X. In [12] the equivalence of the following conditions is shown:

(A) T admits a compatible proximity relation δ_1 for which (X, δ) is a p-subspace of (T, δ_1).

(B) Every pseudometric in $\mathcal{P}^*$ has a continuous extension to T, and the collection of all such extensions is an admissible uniform structure for T.

We define (X, T, δ) to have the *real-extension property* if (A) holds and every member of $P(X)$ can be extended to a member of $P(T)$.

In this paper we characterize when (X, T, δ) has the R.E. property by means of (generalized) uniform structures for X and in terms of maximal round filters in (X, δ). Conditions equivalent to the property that X is C-embedded in T occur as a special case of Theorem 2.1, and Example 3.1 shows that the R.E. property is not coincidental with C-embedding.

Characterizations of when an arbitrary subset of a completely regular space is u^*-embedded [5] are also obtained by means of proximities and S.W. algebras.

2. **Characterizations of the R.E. property for dense subspaces.** Since the theory of uniform spaces may be approached by means of "entourages" or suitable collections of pseudometrics, we will distinguish between these by calling a (generalized) uniformity given by entourages a uniform structure, and its associated family of pseudometrics will be called a gage. (See [1], [9] and [15].) Every member f of $P(X)$ determines a pseudometric σ_f, compatible with δ, by $\sigma_f(x, y) = |f(x) - f(y)|$. Let $\mathcal{G}_P$ be the gage generated by the collection $\{\sigma_f : f \in P(X)\}$, and let $\mathcal{U}_P$ be the generalized uniform structure for X determined by $\mathcal{G}_P$ according to Leader's Theorem of [9].

Received by the editors on December 17, 1973, and in revised form on December 6, 1974.

Appropriate definitions and results concerning round filters may be found in [1] and [15]. By $v_\delta X$ we denote the realcompletion (see [14]) of (X, δ), and $\mathcal{V}_P$ is the generalized uniform structure determined by $P(v_\delta X)$. Notation concerning rings of continuous functions will follow that of Gillman and Jerison [6]. Let δX be the Smirnov compactification of (X, δ).

The proximity on the real numbers R will be that of the standard metric.

An arbitrary subset A of a completely regular space T is u-embedded (resp., u^*-embedded) in T (see [5]) if every admissible uniform structure for A generated by a subcollection of $C(A)$ (resp., $C^*(A)$) can be extended to an admissible uniform structure for T. Gantner has shown that if A is u-embedded in T, then A is C-embedded in T. A similar result holds for u^*-embedded subsets. (See Corollaries 3.12 and 3.14 of [5].) In Example 3.4 of [5], Gantner provides an example of a space X, dense and C-embedded in a compact space T, which has at least two distinct totally bounded admissible uniform structures. Thus, the implications of Corollaries 3.12 and 3.14 of [5] cannot, in general, be reversed. However, distinct totally bounded, admissible uniform structures for X are necessarily of different proximity class. This observation motivates the following definition: A proximity space (X, δ) is *u_p-embedded* in a completely regular space T if every (generalized) admissible uniform structure $\mathcal{U}_S$ for X determined by a family S satisfying $P^*(X) \subseteq S \subseteq P(X)$ has an extension to an admissible uniform structure for T. We note that all such $\mathcal{U}_S$ belong to the proximity class $\pi(\delta)$ of δ and, as O. Njåstad has shown in [14], that such uniform structures determine realcompletions of (X, δ).

THEOREM 2.1. *Let (X, δ) be a proximity space, where X is a dense (topological) subspace of a completely regular space T. Then the following are equivalent:*

(i) *Every point x of T is a limit point of a unique real maximal round filter $\mathcal{F}^x$ on (X, δ), and $\mathcal{F}^x$ is the unique maximal round filter on (X, δ) which clusters at x.*

(ii) *Every pseudometric σ in $\mathcal{G}_P$ has a unique continuous extension $\bar{\sigma}$ to T, and the collection $\mathcal{G} = \{\bar{\sigma} : \sigma \in \mathcal{G}_P\}$ is an admissible gage for T.*

(iii) *The canonical injection of (X, δ) into $v_\delta X$ has an extension to a homeomorphism τ of T into $v_\delta X$.*

(iv) *(X, T, δ) has the real-extension property.*

(v) *(X, δ) is u_p-embedded in T.*

PROOF. (i) implies (ii). By Theorem 3.2 (iii) of [11] every member σ of $\mathcal{G}_P$ has an extension to a continuous pseudometric $\bar{\sigma}$ for T. Now from the extension theorem of [12] and the fact that all extensions are continuous, it follows that $\mathcal{G}$ is admissible.

(ii) implies (iii). Let $\mathcal{U}$ be the generalized uniform structure associated with $\mathcal{G}$. Now $(v_\delta X, \mathcal{V}_P)$ is the completion of $(X, \mathcal{U}_P)$ (see Corollary 2.5 of [11]), and the canonical injection τ_0 of X into $v_\delta X$ is a uniform isomorphism of $(X, \mathcal{U}_P)$ into $(v_\delta X, \mathcal{V}_P)$. Thus, τ_0 has an extension to a uniformly continuous mapping τ of $(T, \mathcal{U})$ into $(v_\delta X, \mathcal{V}_P)$, hence into δX. By (iii) of [12], τ_0 also has an extension to a homeomorphism τ_1 of T into δX. Since such extensions must be unique, τ is a homeomorphism.

(iii) implies (iv). By (iv) of [12], T admits a compatible proximity δ_1 for which (X, δ) is a p-subspace of (T, δ_1). Take $f \in P(X)$ and let $\bar{f}$ be the extension of f to a member of $C(T)$ according to Theorem 3.2 (iv) of [11].

Let $A\delta_1 B$ in (T, δ_1) and suppose $\bar{f}[A]$ is remote from $\bar{f}[B]$ in R. Choose open remote sets U and V in R containing $\bar{f}[A]$ and $\bar{f}[B]$, respectively. Then $\bar{f}^{-1}(U)\delta_1\bar{f}^{-1}(V)$. If $\bar{f}^{-1}(U) \cap X$ is remote from $\bar{f}^{-1}(V) \cap X$ in (X, δ), then $\mathrm{Cl}_T(\bar{f}^{-1}(U) \cap X) = \mathrm{Cl}_T\bar{f}^{-1}(U)$ is remote from $\mathrm{Cl}_T(\bar{f}^{-1}(V) \cap X) = \mathrm{Cl}_T\bar{f}^{-1}(V)$ in (T, δ_1), which is a contradiction. Hence $(\bar{f}^{-1}(U) \cap X)\delta(\bar{f}^{-1}(V) \cap X)$. But f separates $\bar{f}^{-1}(U) \cap X$ and $\bar{f}^{-1}(V) \cap X$, contradicting the choice of f. Thus, $\bar{f} \in P(T)$, and since $P(T) \mid X \subseteq P(X)$ always, (X, T, δ) has the R.E. property.

(iv) implies (v). Let $\mathcal{U}_S$ be an admissible uniform structure for X, where S satisfies $P^*(X) \subseteq S \subseteq P(X)$. Now δ has an extension to δ_1 for T, and (X, T, δ) has the R.E. property. Thus S has an extension to S_1 where $P^*(T) \subseteq S_1 \subseteq P(T)$. Let $\mathcal{W}$ be the uniform structure for T determined by S_1. Evidently δ_1 is the proximity associated with $\mathcal{W}$, hence $\mathcal{W}$ is an admissible extension of $\mathcal{U}_S$ to T.

(v) implies (i). Since $\mathcal{U}_P$ is generated by $P(X)$, $\mathcal{U}_P$ has an extension to an admissible uniform structure $\mathcal{W}$ for T. If δ_1 is the proximity for T associated with $\mathcal{W}$, then (X, δ) is a dense p-subspace of (T, δ_1). Thus, T may be regarded as satisfying $X \subseteq T \subseteq \delta X$. Now each point x of T is a limit point of a unique maximal round filter $\mathcal{F}^x$ in (X, δ), which is the unique maximal round filter which clusters at x.

Since the restriction of $\mathcal{W}$ to $X \times X$ generates $\mathcal{U}_P$ and $\mathcal{G}_P$ is a gage for $\mathcal{U}_P$, it is clear that each $\mathcal{F}^x$, for $x \in T$, is a Cauchy filter relative to $\mathcal{U}_P$. By Theorem 2.4 of [11], $\mathcal{F}^x$ is real.

This completes the proof.

3. Examples and related results.

EXAMPLE 3.1. Let X be the unit ball in ℓ_2, the space of square summable real sequences, and let δ be the proximity for X induced by the standard metric. Take $T = \delta X$. Since $P(X) = P^*(X)$ here (cf. Ex. 2.6 of [11]), (X, T, δ) has the R.E. property. But X is not pseudocompact, hence X is not C-embedded in T. Thus (X, T, δ) can have the R.E. property when X is not C-embedded in T.

Let $\mathcal{C}(X)$ be the gage for X determined by $C(X)$. If we take $\delta = \beta$, the proximity for X associated with the Stone-Čech compactification βX of X, then Theorem 2.1 yields the following characterizations of C-embedding.

COROLLARY 3.2. *If X is dense in a completely regular space T, the following are equivalent:*

(i) *X is C-embedded in T.*
(ii) *$\mathcal{C}(X) = \mathcal{C}(T) \mid X \times X$.*
(iii) *For every family S satisfying $C^*(X) \subseteq S \subseteq C(X)$, $\mathcal{U}_S$ has an admissible extension to T.*
(iv) *Every point x of T is a cluster point of a unique real maximal round filter $\mathcal{F}^x$ in (X, β).*

The equivalence of (i) and (iv) supplements the results of J. W. Green [7].

R. Alò and H. Shapiro have shown in Theorem 3 of [4] that for an arbitrary subset A of a completely regular space T, A is C-embedded in T if and only if A is prerealcompact uniformly embedded in T. From Corollary 3.2 it follows that if X is dense and C-embedded in T, then those prerealcompact structures $\mathcal{U}_S$ where S satisfies $C^*(X) \subseteq S \subseteq C(X)$ have admissible extensions to T. Similar characterizations of C- and C^*-embedding may be found in [2], [3], [4] and [5].

COROLLARY 3.3. *The Hewitt realcompactification vX of X is that unique realcompact Hausdorff space containing X densely such that every $\mathcal{U}_s$, where $C^*(X) \subseteq S \subseteq C(X)$, has an admissible extension to vX.*

In [4] Alò and Shapiro have characterized vX as that unique realcompact Hausdorff space which contains X densely and such that every admissible prerealcompact uniformity has a continuous extension to vX. One may readily obtain examples of spaces X which have more than one admissible prerealcompact uniform structure which therefore may be continuously extended to vX, but where not all such extensions are admissible (cf. Example 3.4 of [5]).

Properties (i)–(iv) of Theorem 2.1 are distinct from those of the extension theorems of [11] and [12]. For, if (X, δ) is any proximity space satisfying $P(X) \neq P^*(X)$ and if $T = \delta X$, then T satisfies properties (i)–(iv) of the extension theorem of [12], but clearly (X, T, δ) cannot have the R.E. property. Theorem 3.2 of [11] provides a characterization of when every member of $P(X)$ may be continuously extended over T, but Example 3.3 of [11] shows that this property may occur in cases where δ cannot be extended to a compatible proximity for T.

If X^* is the set of points in δX which are close to local clusters in (X, δ), then $X \subseteq X^* \subseteq v_\delta X \subseteq \delta X$, and (X, X^*, δ) has the R.E. property. (See [10] and [11].) Moreover, if (X, T, δ) has the R.E. property then each point x in T is a cluster point of a unique cluster $\mathcal{C}_x$ in (X, δ) which contains small sets relative to $\mathcal{U}_P$. In Example 3.4 we show that the converse of this is false, even in case T has the property that each point of T is a cluster point of a unique local cluster from (X, δ).

Example 3.4. Let X be the positive integers with the discrete topology and define $f(x) = x^{-1}$ and $g(x) = 1$, if x is even, and $g(x) = 0$, if x is odd. Then f and g determine pseudometrics σ_f and σ_g, respectively, which generate an admissible uniformity $\mathcal{U}$ for X. Take $T = X \cup \{\alpha\}$, where $\alpha \notin X$, and such that each point of X is isolated. The basic neighborhoods of α are the sets $N_\alpha = \{\alpha\} \cup \{2n : n \geqq m\} \cup \{4n + 1 : n \geqq k\}$ where $m, k \in X$. Now X is dense in the completely regular space T, and if δ is the proximity relation associated with $\mathcal{U}$, there is no compatible proximity for T which renders (X, δ) a p-subspace. (See Ex. 2 of [12].) It is easily verified that any pseudometric compatible with δ must be totally bounded, hence (X, δ) is a precompact proximity space. By Theorem 12 of [10], every cluster from (X, δ) is local. Now each point of T is a cluster point of a unique local cluster in (X, δ), but (X, T, δ) does not have the R.E. property.

There are exactly two free maximal round filters in (X, δ) and by Theorem 12 of [10] and Theorem 2.2 of [11], both must be real. Now one of these converges to α and the other clusters at α but does not converge. Thus, we cannot replace (i) of Theorem 2.1 by the weaker condition that each point x in T is a limit point of a unique real maximal round filter in (X, δ).

4. **u^*-embedding and p-subspaces.** For completeness we include the following result of T. Gantner (Theorem 3.21 of [5]) with a new proof based on the result that each proximity class $\pi(\delta)$ contains a unique totally bounded uniform structure (cf. Theorem 21.28 of [15]).

Lemma 4.1 (Gantner). *Let S be a subset of a completely regular space X. If $\mathcal{U}$ is an admissible totally bounded uniform structure for S that has an admissible extension to X, then $\mathcal{U}$ has an admissible totally bounded extension to X.*

Proof. Let δ be the proximity for S associated with $\mathcal{U}$. If $\mathcal{U}_1$ is an admissible extension of $\mathcal{U}$ to X, then $\mathcal{U}_1$ determines a compatible proximity δ_1 for X. Let $\mathcal{U}^*$ be the unique totally bounded uniform structure in $\pi(\delta_1)$. Since the restriction of $\mathcal{U}^*$ to $S \times S$ is in $\pi(\delta)$ and is totally bounded, it follows that $\mathcal{U}^*$ is an extension of $\mathcal{U}$. This completes the proof.

An S. W. algebra A^* on X is a uniformly closed subalgebra of $C^*(X)$ which contains the constant functions and separates the points of X. (See [8], p. 449.) A^* is compatible if the collection $Z[A^*]$ of zero-sets of members of A^* is a base for the closed sets in X. Thus, a compatible S.W. algebra is just the algebra of bounded proximity functions for some compatible proximity δ for X.

If A^* is an S.W. algebra for a subspace S of X, we say that an S.W. algebra B^* on X is an *extension* of A^* if A^* is the collection $\{f \mid S : f \in B^*\}$, i.e. $A^* = B^* \mid S$.

Theorem 4.2. *For a subset S of a completely regular space X, the following are equivalent:*

(i) *S is u*-embedded in X.*

(ii) *For each compatible proximity δ for S, there exists a compatible proximity δ_1 for X such that (S, δ) is a p-subspace of (X, δ_1).*

(iii) *Every compatible S.W. algebra on S has an extension to a compatible S.W. algebra on X.*

Proof. (i) implies (ii). This follows from the proof of Lemma 4.1.

(ii) implies (iii). Given a compatible S.W. algebra A^* on S, there is a compatible proximity δ for X where $A^* = P^*(S)$. Let δ_1 be the extension of δ to X according to (ii). The closure of S in $\delta_1 X$ is the Smirnov compactification δS of S. Since every member of $P^*(S)$ has a continuous extension to δS and since δS has the real-extension property relative to $\delta_1 X$, it follows that $P^*(X) \mid S = P^*(S)$.

(iii) implies (i). Let $\mathcal{U}$ be an admissible, totally bounded uniform structure for S, and let δ be the proximity relation determined by $\mathcal{U}$. Then the S.W. algebra $P^*(S)$ generates $\mathcal{U}$ and has an extension to a compatible S.W. algebra A^* for X. Let $\mathcal{U}_1$ be the uniform structure for X generated by A^*. Since A^* is compatible, $\mathcal{U}_1$ is admissible and clearly $\mathcal{U}_1$ is totally bounded. Thus, $\mathcal{U}_1$ is the desired extension and the proof is complete.

We note that in case S is a closed subset of X, then from Theorem 3.16 of [5] it follows that each of the conditions of Theorem 4.2 is equivalent to the condition that S be C^*-embedded in X.

The author wishes to thank the referees of this paper for their valuable suggestions concerning the proofs and the exposition.

REFERENCES

1. E. M. Alfsen and O. Njåstad, *Proximity and generalized uniformity*, Fund. Math. **52** (1963), 235-252.

2. R. A. Alò, *Uniformities and embeddings*, Proceedings of the Kanpur Topological Conference: General Topology and its Relations to Modern Analysis and Algebra (1968), 17-32.

3. R. A. Alò and H. L. Shapiro, *Extensions of totally bounded pseudometrics*, Proc. Amer. Math. Soc. **19** (1968), 877-884.

4. ——, *Continuous uniformities*, Math. Annalen **185** (1970), 322-328.

5. T. Gantner, *Extensions of uniform structures*, Fund. Math. **66** (1970), 263-281.

6. L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Math., Princeton, N.J., 1960.

7. J. W. Green, *Filter characterizations of C- and C*-embeddings*, Proc. Amer. Math. Soc. **35** (1972), 574-580.

8. R. B. Kirk, *Algebras of bounded real-valued functions* I, Indag. Math. 34 (1972), 443-451.

9. S. Leader, *On pseudometrics for generalized uniform structures*, Proc. Amer. Math. Soc. **16** (1965), 493-495.

10. ——, *On completion of proximity spaces by local clusters*, Fund. Math. **48** (1960), 201-216.

11. D. Mattson, *Real maximal round filters in proximity spaces*, Fund. Math. **78** (1973), 183-188.

12. ——, *Dense p-subspaces of proximity spaces*, Math. Scand. **27** (1970), 223-226.

13. S. Meyers, *Functional uniformities*, Proc. Amer. Math. Soc. **2** (1951), 153-158.

14. O. Njåstad, *On real-valued proximity mappings*, Math. Annalen **154** (1964), 413-419.

15. W. Thron, *Topological structures*, Holt, Rinehart, and Winston, New York, 1966.

MOORHEAD STATE COLLEGE, MOORHEAD, MINNESOTA 56560