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FUNCTION SPACES FOR THE HOMOTOPY CATEGORY OF CW COMPLEXES

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ABSTRACT. We shall construct an explicit internal mapping functor for the category of pointed CW complexes and homotopy classes of maps, without the use of Brown's theorem.

Let HCW be the category of pointed CW complexes and homotopy classes of pointed continuous maps. It is well-known that HCW is a symmetric monoidal category [**2**], pp 472, 512. The tensor product is the usual smash product $\wedge$; on objects, $X \wedge Y = X \times Y/X \vee Y$. The internal mapping functor MAP to be described below, together with the above structure, will yield a symmetric monoidal closed (smc) category [**2**], pp 475, 512. This means that, additionally, the exponential law

$$\mathrm{MAP}(X \wedge Y, Z) \simeq \mathrm{MAP}(X, \mathrm{MAP}(Y, Z)),$$

and certain coherency conditions hold.

Let Map(,) denote the internal mapping functor on the category CG of pointed, compactly generated spaces [**8**]. Note $CW \subset CG$. Unless otherwise stated, space shall mean pointed CW complex.

DEFINITION. MAP(,) = R Sin Map(,) where R and Sin are Milnor's realization and singular functors [**6**], respectively.

Then MAP is a bifunctor on HCW since R, Sin, and Map preserve homotopy of maps.

THEOREM. $\mathrm{MAP}(X \wedge Y, Z) \simeq \mathrm{MAP}(X, \mathrm{MAP}(Y, Z))$.

PROOF. By the exponential law in CG [**8**], Theorem 5.6, $R \operatorname{Sin} \mathrm{Map}(X \wedge Y, Z) \cong R \operatorname{Sin} \mathrm{Map}(X, \mathrm{Map}(Y, Z))$. Let $p : R \operatorname{Sin} \to 1_{CW}$ be the natural homotopy equivalence [**6**] and let Map_{SS} be the internal mapping functor on pointed simplicial sets [**5**]. Then

$$\begin{aligned} & R \operatorname{Sin} \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \xrightarrow{p^*, \simeq} R \operatorname{Sin} \mathrm{Map}(R \operatorname{Sin} X, \mathrm{Map}(Y, Z)) \\ (*) \qquad & \cong R\, \mathrm{Map}_{SS}(\operatorname{Sin} X, \operatorname{Sin} \mathrm{Map}(Y, Z)) \\ & \xleftarrow{p^*, \simeq} R\, \mathrm{Map}_{SS}(\operatorname{Sin} R \operatorname{Sin} X, \operatorname{Sin} \mathrm{Map}(Y, Z)) \end{aligned}$$

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$$(*) \qquad \cong R\,\mathrm{Sin\,Map}(R\,\mathrm{Sin}\,X,\ R\,\mathrm{Sin\,Map}(Y,Z))$$
$$\xleftarrow{p*,\,\simeq} R\,\mathrm{Sin\,Map}(X,\ R\,\mathrm{Sin\,Map}(Y,Z)),$$

as required.

We thank Professor J. C. Moore for pointing out the isomorphisms (*).

The remaining criteria for a smc category [2], p. 495, Theorem 5.10 are satisfied as follows:

(a) $\pi_0\mathrm{MAP}(\ ,\ ) = HCW(\ ,\ )$,

(b) p induces natural homotopy equivalences $\mathrm{MAP}(S^0, X) \to X$,

(c) The coherency condition MCC3, MCC4 of [2], loc. cit., follow from MC3 (coherent associativity of $\wedge$). (see [2], p. 472).

REMARKS. (a) Milnor proved that $\mathrm{Map}(X, Y)$ has the homotopy type of a CW complex if X is finite [6]. In this case $\mathrm{Map}(X, Y) \simeq \mathrm{MAP}(X, Y)$.

(b) Brown's theorem [1] does not immediately yield the exponential law (enriched adjunction), but this follows from alternate readily verified conditions [2], p. 491, Theorem 5.5 for a smc category.

(c) We used a similar technique in [4] to define an internal mapping function on the homotopy category of *CW spectra* (Boardman's category), see also [3].

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