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LINEAR COMBINATIONS OF CONVEX MAPPINGS

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1. **Introduction.** Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the unit disc $U : |z| < 1$. A function $f \in S$ is said to be *starlike of order* α, $(0 \leqq \alpha < 1)$, denoted $f \in S^*(\alpha)$, if

$$\operatorname{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > \alpha , \qquad (z \in U)$$

and is said to be *convex of order* α, denoted $f \in K(\alpha)$, if

$$\operatorname{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > \alpha , \qquad (z \in U).$$

It is well known that $f \in K(\alpha)$ if and only if $zf' \in S^*(\alpha)$.

In [1, p. 38], the following question is considered: Suppose $f, g \in K(0)$. For $0 < t < 1$, set

$$h = tf + (1 - t)g. \tag{1}$$

Is h starlike and univalent?

MacGregor answered this question in the negative. In [5], he showed that h need not be univalent in any disc $|z| < r$, $r > 1/\sqrt{2}$.

In this note, we investigate functions of the form (1) when $f, g \in K(\alpha)$. For $0 < \alpha < 1/2$, a radius of univalence is found. For $\alpha = 1/2$, we show that h is univalent and close-to-convex.

2. **Radius of univalence.** The development of this section will parallel that of MacGregor in [5], with the class $K(0)$ replaced by $K(\alpha)$. For $f(z) \in K(\alpha)$, consider the related function $g(z)$ defined by $g(z) = \epsilon \overline{f(\epsilon \bar{z})}$, $|\epsilon| = 1$. Note that

$$g'(z) = \overline{f'(\epsilon \bar{z})}, g''(z) = \bar{\epsilon} \overline{f''(\epsilon \bar{z})}. \tag{2}$$

From (2), we obtain

$$1 + z \frac{g''(z)}{g'(z)} = 1 + z\bar{\epsilon} \frac{\overline{f''(\epsilon \bar{z})}}{\overline{f'(\epsilon \bar{z})}},$$

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so that

$$\operatorname{Re}\left\{1 + z\frac{g''(z)}{g'(z)}\right\} = \operatorname{Re}\left\{1 + \epsilon\bar{z}\frac{f''(\epsilon\bar{z})}{f'(\epsilon\bar{z})}\right\},$$

Hence, the functions f and g are simultaneously in $K(\alpha)$. Suppose for some z_0, $|z_0| < 1$; we have $\operatorname{Re} f'(z_0) = 0$. Then choosing $\epsilon = z_0/\bar{z}_0 = e^{2i \arg z_0}$, it follows from (2) that $(1/2)f'(z_0) + (1/2)g'(z_0) = \operatorname{Re} f'(z_0) = 0$. Since the nonvanishing of the derivative is a necessary condition for univalence, the function $h(z) = (1/2)f(z) + (1/2)g(z)$ is not univalent in any disc $|z| < r$, $r > |z_0|$.

On the other hand, Kaplan [2] has shown that for any analytic function $h(z)$, the condition $\operatorname{Re} h'(z) > 0$, $|z| < r_0$, is a sufficient condition for $h(z)$ to be close-to-convex and consequently univalent in $|z| < r_0$. Note that for $f, g \in K(\alpha)$, the function $h = tf + (1-t)g$ satisfies $\operatorname{Re} h'(z) > 0$ at all points where $\operatorname{Re} f'(z) > 0$ and $\operatorname{Re} g'(z) > 0$. Hence, as MacGregor has pointed out for the class $K(0)$, the exact radius of univalence for functions of the form (1) ($f, g \in K(\alpha)$) is given by the supremum of the values of r for which $\operatorname{Re} f'(z) > 0$, $|z| < r$, where f varies over all functions in the class $K(\alpha)$.

THEOREM 1. *Suppose* $f, g \in K(\alpha)$, $0 \leqq \alpha \leqq 1/2$. *Then* $h = tf + (1-t)g$, $0 < t < 1$, *is univalent in the disc* $|z| < \sin \pi/4(1-\alpha)$. *This result is sharp.*

PROOF. It suffices to find the largest disc for which $\operatorname{Re} f'(z) > 0$ for all $f \in K(\alpha)$. By a theorem of Pinchuk [6], $|\arg f'(z)| \leqq 2(1-\alpha)\sin^{-1}|z|$, $f \in K(\alpha)$. But $\operatorname{Re} f'(z) > 0$ if and only if $|\arg f'(z)| < \pi/2$. The result follows upon solving $|\arg f'(z)| \leqq 2(1-\alpha)\sin^{-1}|z| < \pi/2$. For $\alpha = 1/2$, h is univalent and close-to-convex [2] in the disc U, whereas for $\alpha = 0$ the result of MacGregor is obtained.

The functions

$$F(z) = \frac{1}{(1-2\alpha)\epsilon}\left[\frac{1}{(1-\epsilon z)^{1-2\alpha}} - 1\right] \quad (|\epsilon| = 1, 0 \leqq \alpha < 1/2)$$

are in the class $K(\alpha)$. Setting

$$f(z) = \frac{1}{1-2\alpha}\left[\frac{1}{(1-z)^{1-2\alpha}} - 1\right],$$

a short computation shows that $\operatorname{Re} f'(z_0) = 0$ when

$$z_0 = \sin\frac{\pi}{4(1-\alpha)}\, e^{i\left(\frac{\pi}{4}\left(\frac{1-2\alpha}{1-\alpha}\right)\right)}.$$

Thus for $h(z) = (1/2)f(z) + (1/2)\epsilon\overline{f(\epsilon\bar{z})}$, $\epsilon = e^{i\left(\frac{\pi}{4}\left(\frac{1-2\alpha}{1-\alpha}\right)\right)}$, we have $h'(z_0) = 0$.

3. **Radius of convexity.** In the previous section, we showed that (1) was univalent for $f, g \in K(1/2)$ by showing that $\operatorname{Re} h'(z) > 0$ in U. We will improve upon this result in order to obtain a radius of convexity theorem. The following lemma is due to MacGregor [4].

LEMMA. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and satisfies $\operatorname{Re} f(z)/z > 1/2$ in U, then $f(z)$ is starlike in the disc $|z| < 1/\sqrt{2}$.*

THEOREM 2. *Suppose $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic with $\operatorname{Re} h'(z) > 1/2$ in U. Then $h(z)$ is convex in the disc $|z| < 1/\sqrt{2}$. This result is sharp.*

PROOF. By hypothesis, $\operatorname{Re} h'(z) = \operatorname{Re} zh'(z)/z > 1/2$. Then by the lemma, the function $zh'(z)$ is starlike in the disc $|z| < 1/\sqrt{2}$. Since h is convex if and only if $zh'(z)$ is starlike, the theorem is proved. To show sharpness, consider the function

$$h(z) = \int_0^z \frac{1 - (1/\sqrt{2})t}{1 - \sqrt{2}t + t^2}\, dt.$$

Setting $F(z) = zh'(z)$, we have $F'(1/\sqrt{2}) = 0$. Therefore, $F(z)$ is not starlike in any disc $|z| < r$, $r > 1/\sqrt{2}$. Consequently, $h(z)$ is not convex in any disc $|z| < r$, $r > 1/\sqrt{2}$.

THEOREM 3. *Suppose $f, g \in K(1/2)$. Then $h = tf + (1 - t)g$, $0 < t < 1$, is convex in the disc $|z| < 1/\sqrt{2}$.*

PROOF. Since $f, g \in K(1/2)$, the functions zf', $zg' \in S^*(1/2)$. A slight modification of an argument by Robinson [7, p. 32] shows that for $F \in S^*(1/2)$, we have $\operatorname{Re} F(z)/z > 1/2$. Hence, $\operatorname{Re} h'(z) = \operatorname{Re} zh'(z)/z = t \operatorname{Re} zf'(z)/z + (1 - t) \operatorname{Re} zg'(z)/z > 1/2$. The result now follows from Theorem 2.

REMARK. Although we have shown that h—defined by (1)—is univalent when $f, g \in K(1/2)$, we are unable to determine whether h must also be starlike. If so, this would answer in the affirmative the question posed by Hayman for functions in $K(1/2)$ instead of $K(0)$.

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