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FORMALLY SELF-ADJOINT QUASI-DIFFERENTIAL OPERATORS*

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Introduction. In the study of ordinary differential operators the class of formally self-adjoint differential expressions plays an important role. Such expressions generate symmetric operators in the L^2 spaces, and hence the well developed theory of symmetric and particularly self-adjoint operators in Hilbert space (or more generally in Banach spaces) can be applied to study the spectrum of such operators, among other things.

The classical definition of formal self-adjointness — see the book by Coddington-Levinson [3, p. 84] — is as follows: consider the nth order differential expression:

$$Ly = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y \ , \tag{1}$$

where

$$p_i \in C^i \text{ for } i = 0, 1, \cdots, n \ , \tag{2}$$

and the adjoint operator L^+ defined by

$$L^+ y = (-1)^n (\bar{p}_n y)^{(n)} + (-1)^{n-1} (\bar{p}_{n-1} y)^{(n-1)} + \cdots + \bar{p}_0 y. \tag{3}$$

The expression L is said to be formally self-adjoint if $L = L^+$.

It is well known — see Neumark [6] or Dunford and Schwartz [5, p. 1290] — that every formally self-adjoint differential expression L whose coefficients satisfy (2) is of the form

$$\sum_{j=0}^{[n/2]} (-1)^j (a_j y^{(j)})^{(j)} + \sum_{j=0}^{[(n-1)/2]} i[(b_j y^{(j)})^{j-1} + (b_j y^{j+1})^{(j)}] \ , \tag{4}$$

where a_j, b_j are real.

In particular every formally self-adjoint differential expression L with real coefficients satisfying (2) is of even order $n = 2m$ and has the form

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$$\sum_{j=0}^{m} (-1)^j (a_j y^{(j)})^{(j)} \tag{5}$$

with a_j real.

For $m = 1$, (5) reduces to the Sturm-Liouville operator

$$-(a_1 y')' + a_0 y.$$

On the other hand it can readily be shown, by "removing the parenthesis," that every expression of the form (5) with $a_i \in C^i$ is a formally self-adjoint expression.

It is interesting to note that the concept of formal self-adjointness can be defined in a broader sense, to be specified below, which yields — as a special case — that the expressions (4) [(5) in the real case] are formally self-adjoint even if no differentiability assumptions are made on the coefficients a_j, b_j — in this case, of course, the parenthesis in the expressions cannot be removed and so the form must be kept intact.

Many authors, e.g., Dunford and Schwartz [**5**, p. 1290], list (4) [or (5) in the real case] as the most general formally symmetric or self-adjoint differential expression.

Here we develop formally self-adjoint differential expressions much more general than (4) [or (5) in the real case]. Although these more general expressions were used by Shin in 1938 [7] they seem not to have been widely noticed. We give a description of formally self-adjoint differential expressions L in terms of the matrix F in the vector matrix representation $Y' = FY$ of the equation $Ly = 0$ which seems to us to be a natural one. Also, using the techniques of Akhiezer-Glazman [1] and Neumark [**6**], we obtain the characterization of all self-adjoint extensions of the symmetric operators in $L^2(I)$ — generated by these general real formally self-adjoint differential expressions — for I a compact interval $[a, b]$ a complete characterization is given in terms of two point boundary conditions at a and b — this characterization is completely independent of the coefficients of the differential expression. In the case of the interval $[0, \infty)$ our characterization depends on the operator — hence on the coefficients, except in the case when the deficiency index is half of the order of the differentiation, in that case the characterization is given directly in terms of boundary conditions specified at the point zero only. (For $I = (-\infty, \infty)$ the determination of the deficiency indices can be reduced to the half-line case $[0, \infty)$ by standard techniques — see [1, Theorem 3, p. 173] or [**6**, VIII, p. 184].)

In this paper we do not attempt to give a comprehensive account of the use of quasi-differential expressions in the literature nor do we try to give an historical view of their development. The use of various forms of ordinary quasi-differential expressions in the literature has been extensive. The interested reader is referred to the paper by J. H. Barrett [2] with its long list of references.

In section one general quasi-differential expressions and their adjoints are developed and the Lagrange identity and Green's formula are obtained. Section two reviews some basic properties of general quasi-differential equations. In section three basic properties of differential operators in the Hilbert space $L^2(I)$ which are defined in terms of differential expressions are developed for the case when I is a compact interval $[a, b]$. In section four the characterization of all self-adjoint extensions of the minimal operator is given for the compact interval case. Sections five and six treat the case when $I = [0, \infty)$.

1. **Quasi-Differential Expressions.** Just as the second order quasi-differential operator $(py')' + qy$ enjoys many advantages over the more classical $y'' + ry' + sy$ so one can formulate quasi-differential expressions of higher order to replace

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y.$$

Among the advantages of these over the classical ones are:

a. They are more general.
b. A formal adjoint can be defined which has the same "form" as the original.
c. Smoothness conditions on the coefficients are not needed in deriving the Lagrange identity—consequently the necessary and sufficient conditions for adjointness and in particular self-adjointness of two point boundary value problems can be stated very simply in terms of the matrices defining the boundary conditions only.

We now define these quasi-differential expressions. Let $\mathcal{I}$ be any non-degenerate interval and let k denote a positive integer > 1. Suppose $F = (f_{ij})$ is a $k \times k$ matrix of complex valued functions satisfying

(i) $f_{ij} = 0$ a.e. on $\mathcal{I}$ for $j > i + 1$

(ii) $f_{ij} \in L^1_{(\alpha,\beta)}$ for $j \leqq i + 1$, $\alpha, \beta \in \mathcal{I}$, and $f_{i,i+1} \neq 0$ a.e.

Define D_i, D_i^+, $\mathcal{L}$ and $\mathcal{L}^+$ by:

$$D_0y = y = D_0^+y\ ,$$

$$D_iy = 1/f_{i,i+1}[(D_{i-1}y)' - \sum_{j=1}^{i} f_{ij}D_{i-1}y]\ ,$$

$$D_i{}^+y = 1/f^*_{k-i,k+1-i}\left[(D^+_{i-1}y)' - \sum_{j=1}^{i}(-1)^{i+j+1}f^*_{k+1-j,k+1-i}D^+_{j-1}y\right],$$

for $i = 1, \cdots, k-1$,

$$\mathfrak{L}y = (D_{k-1}y)' - \sum_{i=1}^{k} f_{ki}D_{i-1}y \text{ , and}$$

$$\mathfrak{L}^+y = (D^+_{k-1}\,y)' - \sum_{i=1}^{k}(-1)^{k+i+1}f^*_{k+1-i,1}D^+_{i-1}y$$

for all y for which the respective right hand side exists a.e. in $\mathcal{I}$. Here we are using z^* to denote the complex conjugate of z.

The notation $\mathfrak{L}^+$ is justified by Theorem 1 below which generalizes the classical Lagrange identity and plays a fundamental role in the study of two point boundary value problems.

THEOREM 1. *For any u in the domain of $\mathfrak{L}$ and v in the domain of $\mathfrak{L}^+$*

$$v^*\mathfrak{L}u + (-1)^{k+1}u(\mathfrak{L}^+v)^* = (Zu, (-1)^kJZ^+v)' \text{ a.e. in } \mathcal{I}$$

where Zu, Z^+v denote the column vectors (D_iu), $(D_i{}^+v)$ $i = 0, \cdots, k-1$, respectively, and $(\,,)$ denotes the usual inner product in k dimensional Euclidean space and $J = ((-1)^i\delta_{i,k+1-j})$ where δ is the Kronecker delta.

PROOF. The proof consists in showing that

$$v^*\mathfrak{L}u = (D_2{}^+v)^*\left[(D_{k-3}u)' - \sum_{j=1}^{k-2}f_{k-2,j}D_{j-1}u\right]$$

$$+ (D_1{}^+v)^*\left[\sum_{j=1}^{k-2} f_{k-1,j}D_{j-1}u\right] - (D_0{}^+v)^*\left[\sum_{j=1}^{k-2} f_{k,j}D_{j-1}u\right]$$

$$+ [(D_0{}^+v)^*D_{k-1}u - (D_1{}^+v)^*D_{k-2}u]'.$$

Then we show that if, for $i \in \{2, \cdots, k-2\}$, the following identity holds, then it also holds with i replaced by $i+1$:

$$(\mathfrak{L}y)v^* = (-1)^i\cdot(D_i{}^+v)^*\cdot\left[(D_{k-i-1}u)' - \sum_{j=1}^{k-i} f_{k-i,j}D_{j-1}u\right]$$

$$+ (-1)^i\cdot(D^+_{i-1}v)^*\cdot\left[\sum_{j=1}^{k-i} f_{k+1-i,j}D_{j-1}u\right]$$

$$+ (-1)^{i-1}\cdot(D^+_{i-2}v)^*\cdot\left[\sum_{j=1}^{k-i} f_{k+2-i,j}D_{j-1}u\right]$$

$$+ (-1)^{i-2} \cdot (D^+_{i-3}v)^* \cdot \left[\sum_{k=1}^{k-i} f_{k+3-i,j} D_{j-1}u \right] + \cdots$$

$$+ (-1)^{i-i+1} \cdot (D^+_{i-i}v)^* \cdot \left[\sum_{j=1}^{k-i} f_{kj} D_{j-1}u \right]$$

$$+ [(D_0{}^+v)^* \cdot D_{k-1}u - (D_1{}^+v)^* \cdot D_{k-2}u + \cdots$$

$$+ (-1)^{i-1} \cdot (D^+_{i-1}v)^* \cdot D_{k-i}u]\,'.$$

Hence this identity holds for $i = k - 1$ and we obtain

$$\begin{aligned} v^*\mathfrak{L}u &= (-1)^{k-1}(D^+_{k-1}v)^*[(D_0u)' - f_{11}D_0u] \\ &\quad + (-1)^{k-1}(D^+_{k-2}v)^*(f_{21}D_0u) + (-1)^{k-2}(D^+_{k-3}v)^* f_{31}D_0u \\ &\quad + (-1)^{k-3}(D^+_{k-4}v)^* f_{41}D_0u \\ &\quad + \cdots + (-1)(D_0{}^+v)^* f_{k1}D_0u + [(D_0{}^+v)^* D_{k-1}u \\ &\qquad - (D_1{}^+v)^* D_{k-2}u + \cdots + (-1)^{k-2}(D^+_{k-2}v)^* D_1u]\,' \\ &= (-1)^k u(\mathfrak{L}^+v)^* + (Zu, (-1)^k JZ^+v)'. \end{aligned}$$

Note that for

$$F = \begin{bmatrix} 0 & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & 1 \\ -f_{k1} & -f_{k2} & \cdots & & -f_{kk} \end{bmatrix}$$

we have $\mathfrak{L}y = y^k + f_{kk}y^{k-1} + \cdots + f_{k1}y$ and $\mathfrak{L}^+$ is then the classical Lagrange adjoint of $\mathfrak{L}$, i.e., $\mathfrak{L}^+y = (-1)^k y^k + (-1)^{k-1}(f^*_{kk}y)^{k-1} + \cdots + (-1)(f^*_{k2}y)' + f_{k1}y$ so that theorem 1 in this case reduces to the classical Lagrange identity — except that we leave the bilinear form $(\,,)$ on the right hand side in terms of the quasi-derivatives D_iu, and $D_i{}^+v$ instead of u^i and v^i as is usually done. In the following we will refer to the result of theorem 1 as the Lagrange identity.

For convenience we introduce the notation

$$[u, v] = (Zu, (-1)^k JZ^+v) = \sum_{i=0}^{k-1} (-1)^{k+1+i} D_iu(D^+_{k-i-1}v)^*$$

for u in the domain of $\mathfrak{L}$ and v in the domain of $\mathfrak{L}^+$. Let D denote the set of all functions $y \in L_{\mathcal{I}}{}^2$ such that D_iy is absolutely continuous on every compact subinterval of $\mathcal{I}$ for $i = 0, \cdots, k - 1$ and $\mathfrak{L}y$ is in $L_{\mathcal{I}}{}^2$.

Let D^+ be defined similarly with D_i replaced by $D_i{}^+$. Using this notation we obtain as an immediate consequence of theorem 1

COROLLARY 1. *If $u \in D$, $v \in D^+$, $\alpha, \beta \in \mathcal{I}$, $\alpha < \beta$, then*

$$(\mathcal{L}u, v) + (-1)^{k+1}(u, \mathcal{L}^+v) = [u, v] \big|_\alpha^\beta .$$

Here the parentheses denote the $L^2_{(\alpha,\beta)}$ inner product and

$$[u, v] \big|_\alpha^\beta = [u, v](\beta) - [u, v](\alpha).$$

2. **Properties of Quasi-Differential Equations.** Given a function g, by a solution of

$$\mathcal{L}y = g \tag{1}$$

we mean a function y from $\mathcal{I}$ to the complex numbers $\mathcal{C}$ such that D_iy for $i = 0, \cdots, k-1$ is absolutely continuous on every closed finite subinterval and (1) is satisfied a.e. on $\mathcal{I}$.

Similarly, given a vector (matrix) function G we define a solution of

$$Y' = FY + G \tag{2}$$

to be a vector (matrix) function Y which is absolutely continuous and satisfies (2) a.e.

It follows from the definition of $\mathcal{L}$ in terms of the matrix F that (1) is equivalent to (2) where $Y = (D_iy)$ $i = 0, \cdots, k-1$, and G is the column vector $[0, \cdots, 0, g]$. By equivalence here is meant that, given a solution y of (1) if we form the vectors Y and G as indicated, then Y is a solution of (2) and conversely, for G of the indicated form if Y is a solution of (2) then its first component will be a solution of (1).

Consider the matrix equation

$$Y' = FY. \tag{3}$$

From the fundamental existence and uniqueness theorem it follows that for each $u \in \mathcal{I}$ (3) has a unique solution Y_u defined on $\mathcal{I}$ and satisfying $Y_u(u) = I$, where I denotes the $k \times k$ identity matrix. This theorem is easily proven by the Picard successive approximation technique. The reader not familiar with this is referred to [**6**, Satz 1, p. 165]. Let

$$M(t, u) = Y_u(t) \text{ for } t, u \in \mathcal{I}. \tag{4}$$

Let J be any $k \times k$ constant matrix satisfying

$$J^{-1}J^* = I \text{ or } J^{-1}J^* = -I, \tag{5}$$

and define

$$H = -J^{-1}F^*J. \tag{6}$$

Denote the unique solution of

$$X' = HX \tag{7}$$

satisfying $X(u) = I$ by X_u. Let

$$N(t, u) = X_u(t) \text{ for } t, u \in \mathcal{I}. \tag{8}$$

THEOREM 2. *For any $t, u \in \mathcal{I}$*

$$M(t, u) = J^{-1}N^*(u, t)J. \tag{9}$$

PROOF. For a given $u \in \mathcal{I}$ let $Z(t) = J^{-1*}M^*(t, u)J^*N(t, u)$. Then $Z'(t) = 0$ and $Z(u) = I$. Hence $Z(t) = I$ which is equivalent to theorem 2 in view of $N(t, u)N(u, t) = I$. This last result is a consequence of the fact that $N(t, u) = \phi(t)\phi^{-1}(u)$, where ϕ is any fundamental matrix of $X' = HX$.

COROLLARY 2. *For $J = ((-1)^i\delta_{i,k+1-j})$, where δ is the Kronecker δ,*

$$M_{ij}(t, u) = (-1)^{i+j}\bar{N}_{k+1-j,k+1-i}(u, t). \tag{10}$$

We remark that $\mathfrak{L}$ is related to F exactly as $\mathfrak{L}^+$ is related to H.

THEOREM 3. *Suppose the vector G is locally integrable and $x_0 \in \mathcal{I}$. If Y is a solution of* (2), *then*

$$Y(t) = M(t, x_0)Y(x_0) + \int_{x_0}^{t} M(t, u)G(u)\, du. \tag{11}$$

Moreover for any given value of $Y(x_0)$ formula (11) *defines a solution of* (2).

PROOF. Check by a direct computation that (11) defines a solution and use the uniqueness theorem. Formula (11) is known as the variation of constants or variation of parameters formula.

Corollaries 3 and 4 below are immediate consequences of formula (11) and the noted equivalence of (1) and (2).

COROLLARY 3. *For g locally integrable, the solution y of* (1) *satisfying $D_iy(x_0) = \alpha_{i+1}$ for $i = 0, \cdots, k-1$ is given by*

$$y(t) = \sum_{i=1}^{k} M_{1i}(t, x_0)\alpha_i + \int_{x_0}^{t} M_{1k}(t, u)g(u)\, du. \tag{12}$$

COROLLARY 4. *For g locally integrable the solution y of* (1) *satisfying* $D_i y(x_0) = 0$ *for* $i = 1, \cdots, k-1$ *is given by*

$$y(t) = \int_{x_0}^{t} M_{1k}(t, u) g(u)\, du. \tag{13}$$

Furthermore

$$D_i y(t) = \int_{x_0}^{t} M_{i+1,k}(t, u) g(u)\, du \text{ for } i = 0, \cdots, k-1. \tag{14}$$

Similarly the solution y of the adjoint equation

$$\mathfrak{L}^{+} y = g \tag{15}$$

satisfying $D_i^{+} y(x_0) = \alpha_{i+1}$ (*with g locally integrable*) *can be represented by*

$$y(t) = \sum_{i=1}^{k} N_{1i}(t, x_0)\alpha_i + \int_{x_0}^{t} N_{1k}(t, u) g(u)\, du. \tag{16}$$

For a function $P(t, u)$ of two variables we use the notation $P(\cdot, u)$ to denote the function whose value at t is $P(t, u)$ and, similarly, $P(t, \cdot)$ for the function whose value at u is $P(t, u)$. We note

Remark 1. For any $u \in$, $M_{1i}(\cdot, u)$ and $N_{1i}(\cdot, u)$, $i = 1, \cdots, k$, are bases for the solution spaces of $\mathfrak{L} y = 0$ and $\mathfrak{L}^{+} y = 0$, respectively.

Theorems 4 through 7 below are stated for the sake of completeness. The proofs of corresponding theorems given in [**5**] can be adapted readily into this context.

THEOREM 4. *Suppose* $f \in L'_{(\alpha,\beta)}$ *for every* $\alpha, \beta \in \mathcal{I}$, $x_0 \in \mathcal{I}$, $c_0, \cdots, c_{k-1} \in \mathcal{C}$. *Then* $\mathfrak{L} y = f$ *with* $D_i y(x_0) = c_i$, $i = 0, \cdots, k-1$, *has a unique solution. Furthermore, if* f, c_i *and all the coefficients of* $\mathfrak{L}$ *are real, then the* (*unique*) *solution is also real.*

DEFINITION. For any set of functions $y_1, \cdots, y_m$ for which $D_p y_i$, $p = 0, \cdots, m-1$, $i = 1, \cdots, m$ exist we define the Wronskian $W = W(y_1, \cdots, y_m)$ as follows

$$W = (w_{ij}) \text{ where } w_{ij} = D_{i-1} y_j, \quad i, j = 1, \cdots, m.$$

THEOREM 5. *Suppose* $y_1, \cdots, y_k$ *are solutions of* $\mathfrak{L} y = 0$. *If* $y_1, \cdots, y_k$ *are linearly dependent on* $\mathcal{I}$, *then* $W(x) = 0$ *for every* $x \in \mathcal{I}$. *If for some* $x_0 \in \mathcal{I}$, $W(x_0) = 0$, *then* $y_1, \cdots, y_k$ *are linearly dependent.*

THEOREM 6. *The set of all solutions of $\mathfrak{L}y = 0$ forms a k-dimensional vector space over $\mathcal{C}$. Furthermore, if all the coefficients of $\mathfrak{L}$ are real, then the set of real solutions forms a k-dimensional vector space over the reals.*

THEOREM 7. *Suppose $f \in L'_{(\alpha,\beta)}$ for every $\alpha, \beta \in \mathcal{I}$ and $y_1, \cdots, y_k$ are linearly independent solutions of $\mathfrak{L}y = 0$. Let $x_0 \in \mathcal{I}$, and let*

$$v_j = (-1)^{k+j} W(y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_k)/W(y_1, \cdots, y_k).$$

Then, if $\mathfrak{L}y = f$, there exist $\alpha_1, \cdots, \alpha_k \in \mathcal{C}$ such that

$$y(x) = \sum_{i=1}^{k} \alpha_i y_i(x) + \sum_{i=1}^{k} y_i(x) \int_{x_0}^{x} v_i(t) f(t)\, dt \,,$$

for each $x \in \mathcal{I}$. Moreover for any choice of the α_i, the above formula yields a solution of $\mathfrak{L}y = f$.

THEOREM 8. *Suppose $-\infty < a < b < \infty$ and $\mathcal{I} = [a, b]$. For any α_i, β_i $i = 0, \cdots, k-1$ in $\mathcal{C}$ there exists*

(a) *$w \in D$ such that $D_i w(a) = \alpha_i$, $D_i w(b) = \beta_i$, $i = 0, \cdots, k-1$*

(b) *$z \in D^+$ such that $D_i{}^+ z(a) = \alpha_i$, $D_i{}^+ z(b) = \beta_i$ $i = 0, \cdots, k-1$.*

Since theorem 8 might not qualify as a standard theorem we give the following proof which is adapted from [1].

PROOF (*part a*). Let z_i, $i = 0, \cdots, k-1$, denote the fundamental set of solutions of $\mathfrak{L}^+ y = 0$ such that $D_{k-i-1} z_i(a) = 1$ and all other quasi-derivatives are zero. Let $f \in L_{\mathcal{I}}{}^2$ be such that

$$(f, z_i) = \sum_{i=0}^{k-1} (-1)^{k+1+i} \beta_i D^+_{k-i-1} z_i(b).$$

(There exists such an f, in fact for any $\gamma_i \in \mathcal{C}$, $i = 0, \cdots, k-1$, let $f = \sum c_i Z_i$, then the determinant of the coefficients of the equations $(f, Z_i) = \gamma_i$ is the Gram determinant $|(Z_i, Z_j)|$ which is not zero.) Let y be the solution of $\mathfrak{L}y = f$ satisfying $D_i y(b) = \beta_i$. Then by Corollary 1

$$(\mathfrak{L}y, Z_j) = [y, Z_j] \Big|_a^b.$$

Hence $\sum_{i=0}^{k-1} (-1)^{k+i+1} D_i y(a)\, D^+_{k-i-1} Z_j(a) = 0$, and consequently $D_i y(a) = 0$, $i = 0, \cdots, k-1$. We have shown the existence of a $y \in D$ with the properties $D_i y(a) = 0$, $D_i y(b) = \beta_i$, $i = 0, \cdots, k-1$. Similarly we can show the existence of an $x \in D$ such that $D_i x(a) = \alpha_i$ and $D_i x(b) = 0$. The function $w = x + y$ has then property (a) of the theorem. The proof of *part* (*b*) is entirely similar.

3. **General Properties of Operators Defined by Quasi-Differential Expressions in the Regular Case.** By the regular case is meant that the interval $\mathcal{I}$ is closed and finite. Let $\mathcal{I} = [a, b]$. Let

$$D_0 = \{y \in D \mid D_i y(a) = 0 = D_i y(b), \quad i = 0, \cdots, k-1\},$$

$$D_0{}^+ = \{y \in D^+ \mid D_i{}^+ y(a) = 0 = D_i{}^+ y(b), \ i = 0, \cdots, k-1\}.$$

Let L_0 be $\mathfrak{L}$ restricted to D_0, and let $L_0{}^+$ be $\mathfrak{L}^+$ restricted to $D_0{}^+$. Our study of differential operators will take place in the context of the Hilbert space $\mathcal{H} = L^2_{[a,b]}$.

THEOREM 9. *Suppose $f \in \mathcal{H}$. Then $\mathfrak{L}y = f$ has a solution in D_0 if and only if f is orthogonal to every solution of $\mathfrak{L}^+ y = 0$.*

We will give two different proofs of theorem 9; the first is adapted from [1] and the second may be new.

FIRST PROOF. Denote the solution space of $\mathfrak{L}^+ y = 0$ by S^+. Let y be the unique solution of $\mathfrak{L}y = f$ satisfying $D_i y(a) = 0$, $i = 0, \cdots, k-1$. From theorem 1 it follows that for every $v \in S^+$

$$(f, v) = (\mathfrak{L}y, v) = [y, v](b) = \sum_{i=0}^{k-1} (-1)^{k+i+1} D_i y(b) (D^+_{k-i-1} v(b))^*.$$

Now if $D_i y(b) = 0$, $i = 0, \cdots, k-1$, then we have immediately that $(f, S^+) = 0$. On the other hand, if $(f, S^+) = 0$, then $[y, v](b) = 0$ for every $v \in S^+$ and hence $D_i y(b) = 0$, $i = 0, \cdots, k-1$ since the $D_j{}^+ v(b)$'s can be made arbitrary.

SECOND PROOF. Again let y be the solution of $\mathfrak{L}y = f$ satisfying $D_i y(a) = 0$, $i = 0, \cdots, k-1$. Then by Corollary 4

$$y(t) = \int_a^t M_{1k}(t, u) f(u)\, du.$$

If $D_i y(b) = 0$, then $\int_a^b M_{i+1,k}(b, u) f(u)\, du = 0$ and by Corollary 2, f is orthogonal to $N_{1k-i}(\,\cdot\,, b)$. Hence $(f, S^+) = 0$.

On the other hand, if $(f, S^+) = 0$, we simply reverse our steps above and conclude that

$$D_i y(b) = 0, \quad i = 0, \cdots, k-1.$$

From the second proof it follows immediately that $L_0{}^{-1}$ is a compact operator (from its domain S^{+1} onto D_0).

Denoting by R_0 and $R_0{}^+$ the ranges of L_0 and $L_0{}^+$, and by S and S^+ the solution space of $\mathfrak{L}y = 0$ and $\mathfrak{L}^+ y = 0$, respectively, we have

COROLLARY 5. *$\mathcal{H} = S^+ \oplus R_0$ and $\mathcal{H} = S \oplus R_0{}^+$.*

PROOF. The second part of the corollary is merely a restatement of theorem 6 and the first part is obtained by using $\mathfrak{L}^+$ and $D_0{}^+$ in place of $\mathfrak{L}$ and D_0 respectively.

THEOREM 10. *Each of D_0 and $D_0{}^+$ is dense in $\mathcal{H}$.*

PROOF. We prove only that D_0 is dense since the argument for $D_0{}^+$ is entirely similar. Suppose $h \in \mathcal{H}$ such that $(h, v) = 0$ for all $v \in D_0$. Let $\mathfrak{L}^+y = h$. Then as a consequence of theorem 1 we have

$$(\mathfrak{L}v, y) + (-1)^{k+1}(v, \mathfrak{L}^+y) = [v, y]\ |_a^b = [v, y](b) - [v, y](a).$$

Since $[v, y]\ |_a^b = 0$ for $v \in D_0$ we conclude that $(\mathfrak{L}v, y) = 0$. Hence, $h = \mathfrak{L}^+y = 0$, since $y \in S^+$ by the corollary to theorem 9.

Let L, L^+ denote the operators defined by the differential expressions $\mathfrak{L}, \mathfrak{L}^+$ restricted to the sets D, D^+ respectively.

THEOREM 11. (a) $L_0{}^* = (-1)^k L^+$,
(b) $L_0{}^{+*} = (-1)^k L$,
(c) $L_0 = (-1)^k L^{+*}$,
(d) $L_0{}^+ = (-1)^k L^*$,
*where * denotes the Hilbert space adjoint.*

PROOF. We will prove only part (a) since the other parts are entirely similar. From the Lagrange identity

$$(L_0x, y) = (x, (-1)^k L^+y) \text{ for every } x \in D_0, y \in D^+.$$

Hence $(-1)^k L^+ \subset L_0{}^*$.

Suppose $u \in D_{L_0{}^*}$. We want to show that $u \in D^+$ and $L_0{}^*u = (-1)^k L^+u$. Let $\mathfrak{L}^+v = (-1)^k L_0{}^*u$. By theorem 1

$$(L_0x, v) = (x, (-1)^k L^+v),$$

and by the definition of $L_0{}^*$

$$(L_0x, u) = (x, L_0{}^*u) = (x, (-1)^k L^+v) \text{ for all } x \in D_0.$$

So $(L_0x, u - v) = 0$, and hence, by Corollary 5, $u - v \in S^+ \subset D^+ \Rightarrow u \in D^+$. Also since $(L_0x, u) = (x, L_0{}^*u)$ and $(L_0x, u) = (x, (-1)^k L^+u)$ for $x \in D_0$, we have $L_0{}^*u = (-1)^k L^+u$. Since the adjoint of an operator is always closed it follows from theorem 8 that $L_0, L_0{}^+, L, L^+$ are all closed.

4. **Self-Adjoint Extensions on a Finite Interval.** For this section we assume, in addition to our basic assumptions (i) and (ii), that F is real, of even order and satisfies

$$(-1)^k JF^*J = F \text{ where } J = ((-1)^i \delta_{i,k+1-j}).$$

This is our real formal self-adjointness assumption, i.e., $\ell = \ell^+$ and the coefficients of ℓ are real. Under these assumptions L_0 is a symmetric operator with deficiency indices (k, k). The fact that L_0 is symmetric follows from the Lagrange identity and theorem 10, the deficiency indices being (k, k) is due to the fact that, on a finite interval $[a, b]$, all solutions of $\ell y - \lambda y = 0$ for all $\lambda \in \mathcal{C}$ are in $L^2_{(a,b)}$ and theorem 11.

Note also that, under these conditions, any self-adjoint extension L' of L_0 "is between" L_0 and L, i.e., $L_0 \subset L' \subset L$; so that L' is determined by its domain D'. This domain can be characterized by the following lemma which is a consequence of the Lagrange identity.

LEMMA 1. *Suppose $\rho \in D$ and let D' denote the domain of a self-adjoint extension L' of L_0. Then $\rho \in D'$ if and only if $[\rho, \psi]_a^b = 0$ for every $\psi \in D'$. Furthermore given a linear manifold D', $D_0 \subset D' \subset D$, having the property of the theorem, then D' is the domain of a self-adjoint extension of L_0.*

We wish to characterize D' in terms of boundary conditions. To do this we let $\eta_{\pm i}$ denote the eigenmanifolds of $L = L_0^*$ belonging to $\mp\, i$ and use

THEOREM 12. $D = D_0 \oplus \eta_i \oplus \eta_{-i}$.

This theorem is a special case of a well known result which characterizes the domain of the adjoint of any symmetric operator in abstract Hilbert space — see [6, p. 98].

The next theorem is also a special case of a well known fact from abstract Hilbert space theory — see [5, Satz 8, p. 152].

THEOREM 13. *Let U be a unitary transformation with domain η_i and range η_{-i}. Then*

$$D' = D_0 \oplus \{x + Ux \mid x \in \eta_i\}$$

is the domain of a self-adjoint extension of L_0. Furthermore all self-adjoint extensions of L_0 have their domains generated in this manner.

LEMMA 2. *Suppose D' is the domain of any self-adjoint extension of L_0. Then there exist w_i, $i = 1, \cdots, k$, in D' which are linearly independent* mod D_0 *and satisfy*

(17) $$[w_i, w_j]_a^b = 0 \textit{ for } i, j = 1, \cdots, k$$

such that D' consists of all φ in D with the property

(18) $$[\varphi, w_j]_a^b = 0 \textit{ for } j = 1, \cdots, k.$$

Moreover, given w_i, $i = 1, \cdots, k$, in D which are linearly independent mod D_0 *and satisfy* (17), *the set of all φ in D such that* (18) *holds is the domain of a self-adjoint extension of L_0.*

Proof. Let D' denote the domain of a self-adjoint extension of L_0. By theorem 13 there exist w_i in D', $i = 1, \cdots, k$, which are linearly independent modulo D_0 such that every $\varphi \in D'$ can be represented as $\varphi_0 + \sum_{i=1}^k \alpha_i w_i$ for $\varphi_0 \in D_0$ and $\alpha_i \in \mathcal{C}$. From lemma 1 we know that $[w_i, w_j]_a^b = 0$ for $i, j = 1, \cdots, k$. We now assert that D' consists of all $\varphi \in D$ which satisfy (18). Let $\varphi \in D$ such that $[\varphi, w_i]_a^b = 0$ for $i = 1, \cdots, k$. Then for any $\psi_0 \in D_0, \alpha_i \in \mathcal{C}$, $[\varphi, \psi_0 + \sum_{i=1}^k \alpha_i w_i]_a^b = 0$, hence $[\varphi, \psi]_a^b = 0$ for every $\psi \in D'$ and by lemma 1, $\varphi \vee D'$. On the other hand, if $\varphi \in D'$, then $[\varphi, w_i]_a^b = 0$ since $w_i \in D'$ for $i = 1, \cdots, k$.

Now suppose that $w_i \in D$ for $i = 1, \cdots, k$ satisfy (17) and are linearly independent mod D_0. Define D' by (18).

We conclude that $\hat{z}_p$ are $k + 1$ linearly independent solutions of $(A : B)X = 0$. (Here A, B are the $k \times k$ matrices with components a_{ij}, b_{ij} respectively and $(A : B)$ denotes the $k \times 2k$ matrix whose first k columns are those of A and whose second k columns are those of B.)

This contradiction establishes our claim. Next we note that $D_1 \subset D'$. Therefore $D' = D'_1$. We now complete the proof by showing that D' satisfies the conditions of lemma 1. Let $\phi, \psi \in D'$ with $\phi = \phi_0 + \sum_{i=1}^k \alpha_i w_i$, $\psi = \psi_0 + \sum_{i=1}^k \beta_i w_i$. Then

$$[\phi, \psi]_a^b = [\phi_0, \psi]_a^b + \sum_{i,j=1}^k \alpha_i \bar{\beta}_i [w_i, w_j]_a^b = 0.$$

Also suppose $\phi \in D$ such that $[\phi, \psi]_a^b = 0$ for every $\psi \in D'$. Then

$$0 = \left[\phi, \psi_0 + \sum_{i=1}^k \alpha_i w_i \right]_a^b = \sum_{i=1}^k \alpha_i [\phi, w_i]_a^b.$$

Since this holds for all $\alpha_i \in \mathcal{C}$, we must have $[\phi, w_i]_a^b = 0$, hence $\phi \in D'$.

We can now give the complete characterization of the domains of all self-adjoint extensions of L_0 in terms of two-point boundary conditions.

THEOREM 14. *If D' is the domain of a self-adjoint extension of L_0, then there exist $k \times k$ matrices A, B with the properties*

(19) *the rank of the $k \times 2k$ matrix $(A : B)$ is k*

(20) $AJA^*J = BJB^*J$ *where* $J = ((-1)^i \delta_{i,k+1-j})$

such that D' consists of all ϕ in D satisfying

(21) $$A\hat{\phi}(a) + B\hat{\phi}(b) = 0$$

with $\hat{\phi}$ being the vector $(D_i\phi)$, $i = 0, \cdots, k-1$. Furthermore, given matrices A and B of order k having properties (19) *and* (20), *then the* set of all ϕ *in D satisfying* (21) *is the domain of a self-adjoint extension of L_0.*

PROOF. Suppose D' is the domain of a self-adjoint extension of L_0. By lemma 2 there exist $w_i \in D'$ which are linearly independent mod D_0, satisfy (17) and are such that D' is characterized by (18). Let

(22) $$a_{ij} = (-1)^{j+1} D_{k-j}\bar{w}_i(a), \quad b_{ij} = (-1)^j D_{k-j}\bar{w}_i(b)\,, \quad \text{for } i,j = 1, \cdots, k\,,$$

and observe that conditions (18) become (21). Note also that

$$[w_i, w_j]\,_a^b = \sum_{p=1}^{k} a_{jp}D_{p-1}w_i(a) + \sum_{p=1}^{k} b_{jp}D_{p-1}w_i(b)$$

$$= \sum_{p+1}^{k} (-1)^p a_{jp}\bar{a}_{i,k+1-p} - \sum_{p=1}^{k} (-1)^p b_{jp}\bar{b}_{i,k+1-p}.$$

Hence conditions (17) reduce to (20).

We now show that conditions (21) are linearly independent by showing that the rank of the $k \times 2k$ matrix $(A : B)$ is k. Suppose that for some constants c_i

$$\sum_{i=1}^{k} c_i a_{ij} = 0 = \sum_{i=1}^{k} c_i b_{ij} \text{ for } j = 1, \cdots, k.$$

Then

$$0 = \sum_{i=1}^{k} c_i a_{ij} = \sum_{i=1}^{k} c_i(-1)^{k+1+j}(D_{k-j}w_i(a))^*$$

$$= \sum_{i=1}^{k} (-1)^{k+1+j}\, \bar{c}_i D_{k-j}w_i(a),$$

and similarly

$$0 = \sum_{i=1}^{k} (-1)^{k+j}\bar{c}_i D_{k-j} w_i(b).$$

This implies that $\sum_{i=1}^{k} \bar{c}_i w_i$ is in D_0, and therefore $\bar{c}_i = 0 = c_i$ for $i = 1, \cdots, k$.

We have shown that, given a self-adjoint extension of L_0, its domain consists of all functions $\varphi \in D$ which satisfy boundary conditions of type (21) having properties (19) and (20). It remains to show that the converse holds, i.e., given matrices A, B of order k satisfying conditions (19) and (20), if D' is the set of all φ's in D such that (21) holds, then D' is the domain of a self-adjoint extension of L_0.

Clearly D' is a linear manifold containing D_0 and contained in D. By theorem 8 we know that there exist $w_i \in D$, $i = 1, \cdots, k$ such that (22) holds. To see that the linear independence of equations (21) implies that the w_i's are linearly independent mod D_0 we need only to rearrange the steps in the argument for the converse statement above. Similarly it follows that the w_i's satisfy (17) and (18). We now know that D' consists of all φ's in D which satisfy (18) with respect to a set $w_1, \cdots, w_k$ in D satisfying (17) and which are linearly independent modulo D_0. From the proof of lemma 2 we also know that $\dim D' = k \bmod D_0$.

Consider the set D_1 of all functions ψ of the form

$$\psi = \psi_0 + \sum_{i=1}^{k} \alpha_i w_i ,$$

where $\psi_0 \in D_0$ and the α_i's are constants. Then $D_1 \subset D'$ and $\dim D_1 = k \bmod D_0$. Hence $D_1 = D'$. Now observe that for any $\varphi \in D'$, $[\varphi,\psi]_a^b = 0$ for all $\psi \in D'$, and if φ is in D such that $[\varphi, \psi]_a^b = 0$ for all $\psi \in D'$, then in particular φ satisfies (18) and hence $\varphi \in D'$, so that D' satisfies the conditions of lemma 1 and is therefore the domain of a self-adjoint extension of L_0.

We close this section with some remarks.

Remark 2. Conditions (20) are precisely those obtained in [3] by entirely different methods (and under an additional hypothesis on F) which characterize self-adjoint two point boundary value problems.

Remark 3. Our assumption that L_0 is generated by an even order differential expression can be removed if we replace L_0 by iL_0 in the odd order case.

Remark 4. In the case when L_0 is generated by the differential expression $(py')' + qy$ where p and q are real continuous functions the conditions (20) reduce to det $A =$ det B if A, B are real.

5. **The Case of One Regular and One Singular End Point.** In this case we may assume that our interval is $[0, \infty)$.

Let D_0' denote the set of all functions in D which vanish outside of a compact subinterval. (This compact subinterval may be different for different functions). Let L_0' denote the restriction of L to D_0'.

LEMMA 3. *For $y \in D_0'$, $z \in D$, we have $(L_0'y, z) = (y, (-1)^k L^+ z)$.*

PROOF. Let $[\alpha, \beta]$ be a subinterval outside of which y vanishes. It follows that $D_i y$, $i = 0, \cdots, k-1$, vanish outside $[\alpha, \beta]$ and at the endpoints α, β also. Hence by theorem 1 we have

$$\int_\alpha^\beta \mathfrak{L}(y)\bar{z} = (-1)^k \int_\alpha^\beta y\overline{\mathfrak{L}(z)}.$$

The lemma follows since y and $\mathfrak{L}y$ vanish outside $[\alpha, \beta]$.

LEMMA 4. *D_0' is dense in $\mathcal{H}$.*

PROOF. Suppose $h \in \mathcal{H}$ with $(h, D_0') = 0$.

Let $\mathfrak{L}^+ y = h$. We show that for any subinterval $\Delta = [\alpha, \beta]$ of $[0, \infty)$, $h = 0$ a.e. on Δ. Denote by $D_{0\Delta}$ all functions in D whose quasi-derivatives of orders 0 to $k-1$ vanish at α and β. By theorem 1, for any $x \in D_{0\Delta}$,

$$(y, L_0 x) = (-1)^k (x, L^+ y)$$

holds on Δ. But $(x, L^+ y) = (x, h) = 0$ since x can be considered defined on $[0, \infty)$ with values zero outside of Δ. Hence, by Corollary 1, y is in S^+ on Δ, and we have $h = \mathfrak{L}^+ y = 0$ a.e. on Δ. Since Δ is arbitrary, $h = 0$ a.e. and the lemma is established.

For the remainder of this section we again assume that the differential expression $\mathfrak{L}y$ is formally self-adjoint, of even order, and has real coefficients.

In this case it follows from the last two lemmas that L_0' is a symmetric operator. Consequently it has a closure which we denote by L_0.

THEOREM 15. $L_0^* = L$.

PROOF. By lemma 3, $L \subset L_0{}'^* = L_0{}^*$. We want to show that $L_0{}^* \subset L$, i.e., $D_{L_0{}^*} = D_L = D$ and $L_0{}^*\psi = L\psi$ for $\psi \in D$. Let $\psi \in D_{L_0{}^*}$. There exists $\psi_0 \in D$ such that $\mathfrak{L}\psi_0 = L_0{}^*\psi$. Let $a > 0$ and choose a $g \in L^2_{(0,\infty)}$ which vanishes outside of $[0, a]$ and which is orthogonal to the solution space of $\mathfrak{L}y = 0$ on $[0, a]$. By theorem 9 there exist φ such that $\mathfrak{L}\varphi = g$ and $D_i\varphi(0) = 0 = D_i\varphi(a)$, $i = 0, \cdots, k-1$. Extend φ by letting it be zero at all t where $t > a$. We have $\varphi \in D_{L_0{}'}$ and $(g, \psi) = (L_0\varphi, \psi) = (\varphi, L\psi) = (\varphi, L_0{}^*\psi)$. Also $(g, \psi_0) = (L_0\varphi, \psi_0) = (\varphi, L\psi_0) = (\varphi, L_0{}^*\psi)$. Hence $(g, \psi - \psi_0) = 0$, and, by Corollary 1, $\psi - \psi_0$ is in the solution space of $\mathfrak{L}y = 0$ on the interval $[0, a]$. Since this holds for all $a > 0$, $\psi - \psi_0$ is a solution on $[0, \infty)$. Hence $\psi \in D$ and $L\psi = L\psi_0 = L_0{}^*\psi$.

Remark 5. If $\varphi \in D_{L_0}$, then $D_i\varphi(0) = 0, i = 0, \cdots, k-1$.

PROOF. Let $a > 0$ and let $\psi \in D$ such that $\psi(t) = 0$ for $t \geqq a$. By the previous theorem $\psi \in D_{L_0{}^*}$, hence $(L_0\varphi, \psi) = (\varphi, L_0{}^*\psi)$. By theorem 1

$$(\mathfrak{L}\varphi, \psi) = (\varphi, \mathfrak{L}\psi) + [\varphi, \psi]_0^a.$$

Hence $[\varphi, \psi]_0^a = 0$ which implies $[\varphi, \psi](0) = 0$. Since the value of $D_i\psi$ can be chosen arbitrarily, the result follows.

Since the coefficients of $\mathfrak{L}$ are real, the deficiency numbers of L_0 are equal.

Furthermore we have

THEOREM 16. *The deficiency index m of L_0 satisfies $k/2 \leqq m \leqq k$.*

PROOF. The right inequality follows from the previous theorem.

For the proof of the left inequality we use theorem 12, i.e., that

$$D = D_{L_0{}^*} = D_0 \oplus \eta_i \oplus \eta_{-i}.$$

Since the deficiency spaces $\eta_{\pm i}$ are the eigenmanifolds of $L_0{}^* = L$ belonging to $\mp i$, it follows that

$$\dim \eta_i = \dim \eta_{-i} = m.$$

From the above representation of D we can infer that the maximum number of elements in D which are linearly independent mod $D_0 = D_{L_0}$ is $2m$. We now show that there exist k elements in D which are linearly independent mod D_0. Let $\varphi_i \in D$, $i = 1, \cdots, k$ such that $\det(D_{i-1}\varphi_j(0)) \neq 0$. One way to show that such φ_i's exist would be to choose $a > 0$, let $\Delta = [0, a]$. Then by theorem 8 we can choose functions φ_i in D_Δ such that we can assign arbitrary values to $D_i\varphi_j{}'(0)$, and we have $D_i\varphi_j(a) = 0$. Then let

$$\varphi_i(t) = \begin{cases} \varphi_i{}'(t) & \text{for } 0 \leqq t \leqq a \\ 0 & \text{for } t > a. \end{cases}$$

Suppose $\varphi = \sum_{i=1}^{k} c_i\varphi_i \in D_0$. Then by the remark following theorem 15, $\sum_{i=1}^{k} c_i D_j(0) = 0$, $j = 0, \cdots, k-1$. Hence $\det(D_{j-1}\varphi_i(0)) \neq 0$ implies $c_i = 0, i = 1, \cdots, k$. We have shown that $k \leqq 2m$, hence $k/2 \leqq m$.

6. **Self-Adjoint Extensions on the Half-Line.** Throughout this section we retain the assumption that our interval $\mathcal{I}$ is $[0, \infty)$ and that F is real, of even order k and satisfies

$$(-1)^k JF^*J = F \quad \text{for} \quad J = ((-1)^i \delta_{i,k+1-j}) ,$$

i.e., the differential expression $\mathfrak{L}y$ is of even order, formally self-adjoint, with real coefficients.

Since we are working on the interval $[0, \infty)$ we cannot claim that the deficiency spaces $\eta_{\pm i}$ have dimension k. However we do know that they have the same dimension (since the coefficients of $\mathfrak{L}$ are real), say m, and by theorem 16 we know that $k/2 \leqq m \leqq k$.

We now wish to characterize the domains of all self-adjoint extensions of L_0, as we did in section 4 in the case of a finite interval. First note that for $u, v \in D$, $\lim_{t\to\infty}[u, v]_0^t$ exists because the integrals $\int_0^\infty u(\mathfrak{L}v)$ and $\int_0^\infty v(\mathfrak{L}u)$ exist. Let $[u, v]_0^\infty = \lim_{t\to\infty}[u, v]_0^t$. Next note that the proof of lemma 2 is valid in our present context. We state the corresponding result as

LEMMA 5. *Let the deficiency indices of L_0 be (m, m). Suppose D' is the domain of a self-adjoint extension of L_0. Then there exist w_i, $i = 1, \cdots, m$ in D' which are linearly independent* mod D_0 and *satisfy*

$$[w_i, w_j]_0^\infty = 0, \ \textit{for} \quad i, j = 1, \cdots, m ,$$

such that D' consists of all $\varphi \in D$ with the property $[\varphi, w_i]_0^\infty = 0$ for *$i = 1, \cdots, m$. Conversely, given $w_i \in D$, $i = 1, \cdots, m$ which are linearly independent* mod D_0 *and satisfy $[w_i, w_j]_0^\infty = 0$ for $i, j = 1, \cdots, m$; if $D' = \{\varphi \in D \mid [\varphi, w_i]_0^\infty = 0 \text{ for } i = 1, \cdots, m\}$, then D' is the domain of a self-adjoint extension of L_0.*

The conditions $[\varphi, w_i]_0^\infty = 0$ for $i = 1, \cdots, m$ of lemma 5 can be considered abstract boundary conditions. They depend on the functions w_i which depend on L and hence on the coefficients of the differential expression $\mathfrak{L}y$. We can remove this dependence on the coefficients if the deficiency index $m = k/2$. First we state

LEMMA 6. *Suppose the deficiency indices of* L_0 *are* (m, m) *where* $m = k/2$. *Then for any* $\varphi, \psi \in D$ *we have* $[\varphi, \psi](\infty) = 0$.

PROOF. Just as in the proof of theorem 16 we choose $\psi_i \in D$ for $i = 1, \cdots, k$ such that the ψ_i's are linearly independent mod D_0 and satisfy, for some $a > 0$, $D_j\psi_i(a) = 0$ for $j = 0, \cdots, k-1, i = 1, \cdots, k$; and $\psi_i(t) = 0$ for $t > a$. From theorem 12 and the fact that $m = k/2$ we may conclude that

$$\dim D = k \bmod D_0.$$

Let $\varphi, \psi \in D$. There exists $\varphi_0 \in D_0$ and constants $\alpha_i \in \mathcal{C}$ such that $\varphi = \varphi_0 + \sum_{i=1}^{k} \alpha_i\psi_i$. Therefore

$$\begin{aligned}[\varphi, \psi](\infty) = [\varphi_0, \psi](\infty) + \sum_{i=1}^{k} \alpha_i[\psi_i, \psi](\infty) &= [\varphi_0\psi](\infty) \\ &= [\varphi_0, \psi]_0^\infty = 0.\end{aligned}$$

LEMMA 7. *Suppose the deficiency indices of* L_0 *are* (m, m) *with* $m = k/2$. *If* D' *is the domain of a self-adjoint extension of* L_0, *then there exist* $w_i \in D'$, $i = 1, \cdots, m$ *which are linearly independent* mod D_0 *and satisfy*

$$[w_i, w_j](0) = 0, \quad \textit{for } i, j = 1, \cdots, m, \tag{23}$$

such that D' *consists of all functions* $\varphi \in D$ *which satisfy*

$$[\varphi, w_i](0) = 0 \textit{ for } i = 1, \cdots, m. \tag{24}$$

Conversely, given $w_i \in D$, $i = 1, \cdots, m$, *which are linearly independent* mod D_0 *and satisfy* (23), *then the set of all* $\varphi \in D$ *such that* (24) *holds is the domain of a self-adjoint extension of* L_0.

PROOF. Lemma 7 is an immediate consequence of lemmas 5 and 6.

THEOREM 17. *Suppose the deficiency indices of* L_0 *are* (m, m) *with* $m = k/2$. *Let* A *be an* $m \times k$ *matrix of rank* m *such that*

$$AJ_kA^*J_m = 0, \quad [J_p = ((-1)^i\delta_{i,p+1-j})], \tag{25}$$

then the set of all $\varphi \in D$ *with the property*

$$A\hat{\varphi}(0) = 0, \textit{ for } \quad \hat{\varphi} = (D_i\varphi), \tag{26}$$

is the domain of a self-adjoint extension of L_0. *Furthermore, given such a self-adjoint extension, there exists a matrix* A *satisfying* (25), *having rank* m, *such that its domain consists of all* $\varphi \in D$ *satisfying* (26).

PROOF. We prove the furthermore statement first. Let D' denote the domain of a self-adjoint extension of L_0. By lemma 7 there exist $w_i, i = 1, \cdots, m$ in D which are linearly independent mod D_0, satisfy (23) such that D' is determined by (24). Let

$$(27)\quad a_{ij} = (-1)^{k+1+j}(D_{k-j}\bar{w}_i(0)), \quad i = 1, \cdots, m \quad j = 1, \cdots, k.$$

Then for $\varphi \in D$, $0 = [\varphi, w_j](0) =$

$$\sum_{i=0}^{k} (-1)^{k+i+1} D_i\varphi(0) D_{k-i-1}\bar{w}_j(0), \quad j = 1, \cdots, m\ ,$$

becomes $A\varphi(0) = 0$. In this terminology conditions (23) are

$$0 = [w_i, w_j](0) = \sum_{p=0}^{k-1} (-1)^{k+p+1} D_p w_i D_{k-p-1} \bar{w}_j$$

$$= \sum_{p=0}^{k-1} (-1)^{k+p+1}(-1)^{k+i+1}(-1)^{k+j+1} a_{j,p+1}\bar{a}_{i,k-p}$$

$$= \sum_{p=1}^{k-1} (-1)^p a_{jp}\bar{a}_{i,k-p+1} \quad \text{for } i, j = 1, \cdots, m.$$

These are exactly the conditions

$$AJ_k A^* J_m = 0.$$

The fact that the rank of A is m follows from the w_i's being linearly independent mod D_0 in exactly the same way as the corresponding result in section 4 followed.

On the other hand, suppose a matrix A is given with the properties of the theorem. We define D' as the set of all $\varphi \in D$ such that $A\hat{\varphi}(0) = 0$ and show that D' is the domain of a self-adjoint extension of L_0. To do this choose w_i, $i = 1, \cdots, m$ in D satisfying (27). Then (25) implies (23), and (26) becomes (24). The fact that the rank of A is m implies that w_i, $i = 1, \cdots, m$, are linearly independent mod D_0 can be established just as the corresponding result in connection with theorem 14 was. The conclusion follows from lemma 7.

Remark. The condition $(-1)^k JF^*J = F$ where $J = ((-1)^i\delta_{1,k+1-j})$ is the general formal self-adjointness criterion mentioned in the introduction. Any linear operator $\mathfrak{L}y$ which has a vector matrix representation $Y' = FY$ where the matrix F satisfies the above condition is formally self-adjoint. This self-adjointness condition on the matrix F

means that F is invariant under the composition of the following three operations: "flips" about the secondary diagonal, conjugation, multiplying f_{ij} by $(-1)^{i+j+1}$.

We now mention some examples.

1. Consider $k = 2$, $f_{11} = (ib_0)/a_1 + c$, $f_{12} = -1/a_1$, $f_{21} = (b_0^2 - a_0a_1)/a_1$, $f_{22} = (ib_0/a_1) - c$ where a_0, a_1, b_0 and c are real functions. The second order formally self-adjoint operator $\mathfrak{L}$ corresponding to this 2×2 matrix $F = (f_{ij})$ is:

$$\mathfrak{L}y = [-a_1y' + (ib_0 + a_1c)y]' + (ib_0 - a_1c)y' + (a_0 + a_1c^2)y.$$

For $b_0 = 0$ we get a second order formally self-adjoint operator with real coefficients: $\mathfrak{L}y = [-a_1y' + a_1cy]' - a_1cy' + (a_0 + a_1c^2)y$. If a_1 and c are differentiable this real operator reduces to:

$$\mathfrak{L}y = (-a_1y')' + [(a_1c)' + a_0 + a_1c^2]\,y.$$

Finally we note that $c = 0$ yields the Sturm-Liouville operator:

$$\mathfrak{L}y = -(a_1y')' + a_0y.$$

2. For $k = 3$ and $F = (f_{ij})$ with f_{ij} real the (anti) self-adjointness criteria are $f_{11} = -f_{33}$, $f_{12} = f_{23}$, $f_{21} = f_{32}$, $f_{22} = 0$, $f_{31} = 0$. The differential expression $\mathfrak{L}$ generated by this matrix F is given by

$$\begin{aligned}\mathfrak{L}y = {}& \{1/f_{12}[1/f_{12}(y' - f_{11}y)]' - f_{21}y\}' - f_{21}/f_{12}(y' - f_{11}y) \\ &+ f_{11}/f_{12}\{[1/f_{12}(y' - f_{11}y)]' - f_{21}y\}.\end{aligned}$$

The case $f_{11} = 0$ reduces to Barrett's "canonical form" for real third order formally self-adjoint operators [2, p. 435]. The term anti-self-adjoint is sometimes used in the odd order case. As previously mentioned in remark 3 the operator L_0 in the odd order case is not symmetric (it is anti-symmetric), but iL_0 is symmetric.

3. For $k = 4$, the form of the matrix F satisfying the self-adjointness criterion $(-1)^kJF^*J = F$, in the *real* case, is:

$$F = \begin{bmatrix} f_{11}, & f_{12}, & 0, & 0 \\ f_{21}, & f_{22}, & f_{23}, & 0 \\ f_{31}, & f_{32}, & f_{22}, & f_{12} \\ f_{41}, & -f_{31}, & f_{21}, & -f_{11} \end{bmatrix}$$

The case when $f_{ij} = 0$ for $i + j$ even reduces to Barrett's "canonical form" for the fourth order self-adjoint case [2, p. 475]. The special case with $f_{12} = 1$, $f_{23} = 1/r$, $f_{32} = p$, $f_{41} = -q$ and all other $f_{ij} = 0$ yields the operator

$$\ell y = [(ry'')' - py']' + qy$$

which is often listed as the most general fourth order formally self-adjoint differential operator.

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