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TRACTIONS IN CRITICAL POINT THEORY

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Introduction. The application of critical point theory to fields of mathematics as divergent as the theory of functions on a differentiable or topological manifold or global variational analysis, requires divergent preparatory analysis and alterations of the data that present the problem in modified form, but leave invariant the critical elements (equilibrium points, extremals, areas, etc.) whose discovery was the goal. See references [1] to [17].

In sharp contrast with this preparatory analysis is the essence of the critical point method, namely the discovery of the homological changes in the sublevel sets of a real-valued function or integral as the level increases or decreases through a critical value. These two aspects of the critical point theory will be termed *preparatory* and *critical*, respectively. We have found the following to be true.

In general problems preparation and critical operations must alternate.

It is for this reason that we shall attempt to characterize in *one* theorem the essential homological changes in sublevel sets as an isolated critical level is passed. The hypotheses of this theorem summarize preparatory analysis of very general character. In particular this theorem will greatly simplify global variational analysis.

1. **Hypotheses and principal theorem.** Let X be a metric space which is the domain of a continuous real-valued mapping

$$p \to F(p) : X \to R\,, \tag{1.1}$$

with values which are bounded above by a constant β. The theorems of this paper can be shown to be true for a topological space. A metric space simplifies some of the theorems.

We shall make use of the singular homology theory introduced by Eilenberg in reference [18]. See also reference [4]. Homology will be taken over an arbitrary commutative field $\mathcal{K}$. Homology groups will then be vector spaces over $\mathcal{K}$.

For each value $\alpha \in R$ we shall set

$$F_\alpha = \{p \in X \mid F(p) \leqq \alpha\}\,, \tag{1.2}$$

and term F_α the α-sublevel set of X. A fundamental problem is to know how F_β differs homologically from F_α when $\alpha < \beta$. We shall

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consider the case where there is just one value a on the interval $\alpha < c < \beta$ at which F_c changes its homological character. This value a will be termed *critical*. We shall give special conditions on F under which a critical level a exists together with a point σ at the F-level a near which the level sets of F behave topologically as if σ were an ordinary nondegenerate critical point. These conditions are both local and global. The global condition is in terms of a special deformation of X, termed an *F-traction*, which we shall now define. We begin by recalling other more familiar terms in deformation theory.

Deformations. Let t be a real variable termed the *time*. Let $I =$ $[0, 1]$ denote an interval for t. With us a deformation of a subspace A of a topological space χ is a continuous mapping

$$(p, t) \to D(p, t) : A \times I \to \chi, \tag{1.3}$$

such that $D(p, 0) \equiv p$ for $p \in A$. We term $D(p, 1)$ the *final image* $D_1(p)$ of p under D. For p fixed in A, the partial mapping

$$t \to D(p, t) : I \to \chi \tag{1.4}$$

will be called the *trajectory* of p under D.

Retracting Deformations. A deformation D of A onto a set $B \subset A$ is said to be a deformation *retracting* A onto B, if D deforms A on A onto B and leaves each point of B fixed. That is $D(p, t) = p$ for $p \in B$ and $0 \leqq t \leqq 1$. See reference [17].

Retracting deformations need to be supplemented. There is a larger related class of deformations which we now define.

Definition 1.1. *Tractions.* A deformation D of a subspace A of χ will be termed a *traction* of A into a subspace B of A if D deforms A on A into B and deforms B on B.

Each deformation retracting A onto B is a traction of A into B, but a traction of A into B is not in general a deformation retracting A onto B. However, a traction of A into B shares with a deformation retracting A onto B a fundamental property. There exists an isomorphic mapping of $H_q(A)$ onto $H_q(B)$ for each q. See § 2.

Definition 1.2. *An F-deformation of A on X.* A deformation D of A on X is called an *F-deformation* if $F(D(p, 0)) \geqq F(D(p, t))$, $((p, t) \in A \times I)$. An F-deformation which is a retraction or traction is called an *F-retraction* or *F-traction*, respectively.

A traction induced critical point σ of F will now be defined. Let an integer $\mu \geqq 1$ be given and fixed. We refer to points $\mathbf{x} = (x_1, \cdots, x_\mu) \in R^\mu$ and introduce the origin-centered μ-ball

$$D_e = \{\mathbf{x} \in R^\mu \mid \|\mathbf{x}\| < e\} \tag{1.5}$$

in R^μ of radius e.

DEFINITION 1.3. *A traction induced critical point σ of F.* Let k be an integer on the range $0, 1, \cdots, \mu$. A point $\sigma \in X$ will be called a *traction induced critical point* of F of *index* k if the following two conditions are satisfied. (See the Note at the end of the paper.)

LOCAL CONDITION I. For some sufficiently small positive constant e there exists an injective homeomorphism

$$\mathbf{x} \to \Phi_\sigma(\mathbf{x}) : D_e \to F_\beta, \qquad (F_\beta = X) \tag{1.6$'$}$$

onto a topological μ-ball $\Lambda_\sigma = \Phi_\sigma(D_e) \subset X$, containing the point $\sigma = \Phi_\sigma(0)$, while for $\mathbf{x} \in D_e$ and $\mathbf{p} = \Phi_\sigma(x)$,

$$\begin{aligned} F(\mathbf{p}) - F(\sigma) \equiv -x_1^2 - \cdots - x_k^2 + x^2_{k+1} \\ + \cdots + x_\mu^2 \equiv Q_k(\mathbf{x}), \qquad (\mu > 0) \end{aligned} \tag{1.6$''$}$$

introducing the quadratic form Q_k. The topological μ-ball Λ_σ may be a neighborhood of σ, relative to X, but is not *required* to be a neighborhood of σ.

GLOBAL CONDITION II. Set $F(\sigma) = a$. For some value $c \in R$ such that $a > c > a - e^2$ there exists an F-traction Δ of F_β into $\Lambda_\sigma \cup F_c$.

Theorem 1.1 makes use of the following two definitions.

DEFINITION 1.4. *The critical $(k-1)$-cycle w_c^{k-1} of σ on F_c.* Set

$$\rho^2 = a - c, \qquad (a > c > a - e^2) \tag{1.7}$$

and for $0 < k \leqq \mu$ introduce the $(k-1)$-sphere

$$s_{k-1} = \{\mathbf{x} \in D_e \mid x_1^2 + \cdots + x_k^2 = \rho^2; x_{k+1} = \cdots = x_\mu = 0\} \tag{1.8}$$

of radius ρ and denote $\Phi_\sigma(s_{k-1})$ by π_{k-1}. Then π_{k-1} is a topological $(k-1)$-sphere on F_c, as (1.6)$''$ shows. Let w_c^{k-1} be any singular $(k-1)$-cycle on π_{k-1} whose homology class on π_{k-1} is a base for the homology group $H_{k-1}(\pi_{k-1})$ over $\mathcal{K}$. We term w_c^{k-1} a *critical* $(k-1)$-*cycle* of σ.

DEFINITION 1.5. *Linking or non-linking critical cycles w_c^{k-1}.* The cycle w_c^{k-1} and σ will be said to be of *linking* or *non-linking* type, respectively, according as w_c^{k-1} is or is not homologous to 0 on F_c.

THEOREM 1.1 states the principal result of this paper.

THEOREM 1.1. *If the index k of the traction induced critical point σ is positive, and the homology groups over $\mathcal{K}$ of F_c are finitely generated, the connectivities R_q^{β} of F_{β} are finite and equal the respective connectivities R_q^c of F_c, except at most when $q = k - 1$ or k. When $q = k - 1$ or k, respectively,*

$$R_{k-1}^{\beta} - R_{k-1}^{c} = 0 \text{ or } -1, \tag{1.9)'$$

$$R_k^{\beta} - R_k^{c} = 1 \text{ or } 0, \tag{1.9)''$$

according as w_c^{k-1} is of linking or non-linking type.

The proof of Theorem 1.1 will lead to the following corollary.

COROLLARY 1.1. *Under the hypotheses of Theorem 1.1, if b is such that $F(\sigma) > b > c$, F_b admits an F-traction into F_c.*

In proving Theorem 1.1, a number of *propositions* and lemmas will be established. Our lemmas are regarded as secondary in importance to the propositions.

The proof of Theorem 1.1 will be completed in § 4. At the end of this section a *program* for the proof of Theorem 1.1 will be outlined. The formulation of this program requires the definition of a "*k-saddle* of F *hung* at σ" and a basic F-deformation d.

DEFINITION 1.6. *A k-saddle Ω_k^{ρ} of F, hung at σ.* We refer to the origin-centered μ-ball D_e of (1.5) and to the difference $\rho^2 = a - c < e^2$ where $a = F(\sigma)$. The euclidean k-disc

$$\omega_k^{\rho} = \{\mathbf{x} \in D_e \mid x_1^2 + \cdots + x_k^2 \leqq \rho^2;\ x_{k+1} = \cdots = x_{\mu} = 0\} \qquad (k > 0) \tag{1.10}$$

of radius ρ, has the $(k - 1)$-sphere s_{k-1} of (1.8) as boundary. Set

$$\Omega_k^{\rho} = \Phi_{\sigma}(\omega_k^{\rho}), \tag{1.11)'$$

and note that the topological $(k - 1)$-sphere

$$\pi_{k-1} = \Phi_{\sigma}(s_{k-1}) \tag{1.11)''$$

is the boundary of the topological k-disc Ω_k^{ρ}. Observe that

$$\Omega_k^{\rho} \subset F_a;\ \Omega_k^{\rho} \cap F_c = \pi_{k-1}. \tag{1.12}$$

The points of Ω_k^{ρ} are below the F-level $F(\sigma)$ except for σ. Ω_k^{ρ} is called a *k-saddle* of F, *hung* at σ. Its boundary π_{k-1} is the carrier of the critical $(k - 1)$-cycle w_c^{k-1}.

The following lemma concerns the subspace $\Lambda_{\sigma} \cup F_c$ of X introduced in Global Condition II.

LEMMA 1.1(i). *When* $k = \mu$, $\Lambda_\sigma \cup F_c = \Omega_\mu{}^\rho \cup F_c$.

(ii) *When* $0 < k < \mu$, *there exists an F-deformation d of* $\Lambda_\sigma \cup F_c$ *retracting* $\Lambda_\sigma \cup F_c$ *onto* $\Omega_k{}^\rho \cup F_c$.

PROOF OF (i). Recall that $\rho < e$ and $c = a - \rho^2$. From the definitions of Λ_σ and $\Omega_\mu{}^\rho$ we infer that *When* $k = \mu$, $\Omega_\mu{}^\rho = \{p \in \Lambda_\sigma \mid a \geqq F(p) \geqq c\}$ from which (i) follows.

PROOF OF (ii). Use will be made of Q_k-deformations defined on subspaces of R^μ, as F-deformations were defined on subspaces of X. When $0 < k < \mu$, we shall define a Q_k-deformation δ, retracting D_e on D_e onto a subset B of D_e of the form

$$(1.13) \qquad B = \{\mathbf{x} \in D_e \mid Q_k(\mathbf{x}) \leqq -\rho^2\} \cup \omega_k{}^\rho, \qquad (\rho^2 = a - c).$$

DEFINITION OF δ. Let $\mathbf{b} = (b_1, \cdots, b_\mu)$ be a point in $D_e - B$. On $D_e - B$, $Q_k(\mathbf{x}) > -\rho^2$. Let g_b be the sensed straight arc in D_e which joins the point $\mathbf{b}$ to the point $\mathbf{b}^* = (b_1, \cdots, b_k, 0, \cdots, 0)$ in D_e. As the point $\mathbf{x}$ moves along g_b in the positive sense, the coordinates $x_{k+1}, \cdots, x_\mu$ alone vary, with $x_{k+1}^2 + \cdots + x_\mu{}^2$ decreasing to 0. There are two cases to be considered, distinguished by the nature of the initial point $\mathbf{b}$.

Case I. $b_1{}^2 + \cdots + b_k{}^2 > \rho^2$.
Case II. $b_1{}^2 + \cdots + b_k{}^2 \leqq \rho^2$.

In both cases $Q_k(\mathbf{b}) > -\rho^2$. In Case I, $Q_k(\mathbf{b}^*) < -\rho^2$ so there is a subarc γ_b of g_b which joins $\mathbf{b}$ to a point $\mathbf{x}$ on g_b at which $Q_k(\mathbf{x}) = -\rho^2$. In Case II, set $\gamma_b = g_b$. The arcs γ_b clearly vary continuously with $\mathbf{b} \in D_e - B$. To define the deformation δ, one moves a point $\mathbf{x}$, given as $\mathbf{b}$ when $t = 0$, along γ_b at a velocity with respect to t equal to the length of γ_b. Under δ points on B are held fast. So defined, δ is a Q_k-deformation retracting D_e on D_e onto B.

DEFINITION OF d. Recall that $\Lambda_\sigma = \Phi_\sigma(D_e)$. The "trajectories" under d of points $\mathbf{x}$ initially in Λ_σ when $t = 0$, are taken as the images under Φ_σ of the trajectories of the corresponding points in D_e under the Q_k-deformation δ. Under d points $p \in F_c$ are held fast.

Since δ is a Q_k-deformation, retracting D_e onto B, the restriction of d to Λ_σ is, by (1.6)″, an F-deformation retracting Λ_σ onto $\Phi_\sigma(B) = \{p \in \Lambda_\sigma \mid F(p) \leqq c\} \cup \Phi_\sigma(\omega_k{}^\rho) = (\Lambda_\sigma \cap F_c) \cup \Omega_k{}^\rho$, $(0 < k < \mu)$. Since d leaves points of F_c fixed, (ii) of Lemma 1.1 follows.

PROGRAM FOR PROOF OF THEOREM 1.1. We shall denote $\Omega_k{}^\rho - \sigma$ by $\dot{\Omega}_k{}^\rho$ and consider the sequence of sets,

$$(1.14) \qquad F_\beta, \Lambda_\sigma \cup F_c, \Omega_k{}^\rho \cup F_c, \dot{\Omega}_k{}^\rho \cup F_c, F_c \qquad (k > 0).$$

Each set in (1.14) is a proper subset of its predecessor. We shall show that the q-th singular homology groups over $\mathcal{K}$ of successive sets in (1.14) are isomorphic, except for the successive sets,

$$\Omega_k{}^\rho \cup F_c, \; \dot{\Omega}_k{}^\rho \cup F_c \;, \quad (k > 0). \tag{1.15}$$

The connectivities over $\mathcal{K}$ of the two sets (1.15) will be seen in § 3 to have the same differences as do the connectivities of F_β and F_c of Theorem 1.1.

Under Condition II there is an F-traction of the first set in (1.14) into the second. By Lemma 1.1 there is an F-deformation retracting the second set in (1.14) onto the third. There is clearly an F-deformation retracting $\dot{\Omega}_k{}^\rho$ onto its geometric boundary π_{k-1} at the F-level c. We infer the following when $k > 0$.

LEMMA 1.2. *There exists an F-deformation retracting $\dot{\Omega}_k{}^\rho \cup F_c$ onto F_c.*

In § 2 we shall recall theorems by virtue of which the connectivities of successive sets in (1.14) are equal, excepting the two sets (1.15). In § 3 we shall compare the connectivities of the sets (1.15). We are led thereby to a proof of Theroem 1.1.

APPLICATIONS OF THEOREMS 1.1 AND 5.1. One of the simplest applications is to the case of a topologically nondegenerate function F defined on a compact topological manifold M_n. It is shown in reference [3] that F can be replaced by a similar mapping $\hat{F}$ differing infinitesimally from F, each of whose critical points is of *singleton* type, that is, has a critical value $a = \hat{F}(\sigma)$ assumed by just one critical point σ. The index of σ equals that of the critical point it replaces. Let β, a, c be values in R such that a is a critical value of $\hat{F}$ and the interval (β, c) contains no critical value of $\hat{F}$ other than a. It is shown in reference [3] that there is a "traction" that induces σ as a critical point. Theorems 1.1 and 5.1 can be applied to such critical points. The resultant global critical point-homology relations have the same form, in this topological case, that they have in the analytic nondegenerate case.

2. **F-Deformations and homologies.** We shall be concerned with singular homology theory (Eilenberg) on a topological space χ. No triangulations of χ are presupposed. Given χ and the field $\mathcal{K}$, singular q-cells on χ are combined linearly with coefficients in $\mathcal{K}$ to define a vector space denoted by $C_q(\chi, \mathcal{K})$. The singular q-cells on χ form a basis for $C_q(\chi, \mathcal{K})$. The elements of $C_q(\chi, \mathcal{K})$ are termed *q-chains*. The homology groups $H_q(\chi)$ over $\mathcal{K}$ are well-defined for each rational integer q. They are trivial if $q < 0$.

The following notation helps remove ambiguity.

DEFINITION 2.1. *The homology class* $((u^q, \chi))$. If u^q is a q-cycle in $C_q(\chi, \mathcal{K})$, then $((u^q, \chi))$ shall denote the subset of q-cycles in $C_q(\chi, \mathcal{K})$ which are homologous to u^q on χ. One can regard $((u^q, \chi)$ as an element in the homology group $H_q(\chi)$ over $\mathcal{K}$. T shall abbreviate topological.

THE CHAIN TRANSFORMATION $\hat{\varphi}$ (Eilenberg). Let there be given a continuous mapping $\varphi : \chi \to \chi'$ of a T-space χ into a T-space χ'. A singular q-cell σ^q on χ is defined by the class of equivalent mappings τ of vertex-ordered euclidean simplices into χ. In a chain transformation,

$$\hat{\varphi} : C_q(\chi, \mathcal{K}) \to C_q(\chi', \mathcal{K}), \quad (q = 0, 1, 2, \cdots) , \tag{2.1}$$

the image $\hat{\varphi}\sigma^q$ on χ' of a q-cell σ^q on χ is defined by compositions $\varphi \circ \tau$ with φ of the equivalent mappings τ into χ which define σ^q. The mappings $\hat{\varphi}$, so defined for cells σ^q on χ, are extended linearly over $\mathcal{K}$ to define the mappings (2.1). Eilenberg shows that $\hat{\varphi}$ is permutable with the boundary operator ∂. "Natural" homomorphisms

$$(\varphi_*)_q : H_q(\chi) \to H_q(\chi') \qquad (\text{over } \mathcal{K}) \tag{2.2}$$

are induced by $\hat{\varphi}$, in which, for each q-cycle z on χ, $((z, \chi))$ goes into $((\hat{\varphi}z, \chi'))$.

Let z be a q-cycle over $\mathcal{K}$ on χ and d a deformation of χ on χ. If d_1 is the terminal mapping of d, the homology $z \sim \hat{d}_1 z$ is valid on the image under d of any carrier $|z|$ of z. This is a classical theorem. See § 26 and § 27 of reference [4] for details.

The F-tractions, defined in § 1, induce isomorphisms as follows.

LEMMA 2.1. *Let χ and χ' be T-spaces with $\chi' \subset \chi$. If d is a traction of χ into χ', an isomorphism*

$$(i_*)_q : H_q(\chi') \rightleftarrows H_q(\chi) , \quad (q = 0, 1, \cdots) , \tag{2.3}$$

is induced by the inclusion mapping $i : \chi' \to \chi$.

The inclusion induced mapping (2.3) is a homomorphism. We affirm that if d is a traction of χ into χ' it is an isomorphism. This is true if the following is true:

(a) Each q-cycle on χ is homologous on χ to a q-cycle on χ';

(b) Each q-cycle on χ' which is bounding on χ is bounding on χ'.

In fact $(i_*)_q$ is surjective if (a) holds and has a null kernel if (b) holds. Isomorphisms are understood to be surjective in this paper.

PROOF OF (a). Since d deforms χ on χ into χ', (a) is clearly true.

PROOF OF (b). Let d_1 be the terminal mapping of d. As is well known, one can associate with d homomorphisms (see § 27 of reference [4])

$$(2.4) \qquad \mathbf{d} : C_q(\chi, \mathcal{K}) \to C_{q+1}(\chi, \mathcal{K}), \qquad (q = 0, 1, \cdots),$$

such that for each q-chain $z \in C_q(\chi, \mathcal{K})$

$$(2.5) \qquad \partial \mathbf{d} z = \hat{d}_1 z - z - \mathbf{d} \partial z.$$

Moreover the definition of $\mathbf{d}$ in reference [4], page 236, is such that $|\mathbf{d}z| \subset d|z|$, where $d|z|$ is the union of the trajectories of points of $|z|$.

In (b) a q-cycle $u_{\chi'}$ is given on χ' such that

$$(2.6) \qquad u_{\chi'} = \partial y_\chi ,$$

for some $(q + 1)$-chain y_χ on χ. By virtue of (2.5)

$$(2.7)' \qquad \partial d y_x = \hat{d}_1 y_\chi - y_\chi - \mathbf{d} \partial y_\chi .$$

On applying ∂ to the members of (2.7)′ and making use of (2.6) we find that

$$(2.7)'' \qquad 0 = \partial \hat{d}_1 y_\chi - u_{\chi'} - \partial \mathbf{d} u_{\chi'}.$$

Since d is a traction into χ' both $\hat{d}_1 y_\chi$ and $\mathbf{d} u_{\chi'}$ are on χ', so that $u_{\chi'} \sim 0$ on χ', confirming (b).

Lemma 2.1 follows from (a) and (b).

RELATIVE HOMOLOGIES OVER $\mathcal{K}$. Relative homologies were introduced by Lefschetz. Given a T-space χ a subspace A of χ is taken as a "modulus" and the pair (χ, A) termed "admissible." The q-th relative homology group of χ mod A is denoted by $H_q(\chi, A)$. See § 28 of reference [4]. We state a classical lemma.

LEMMA 2.2. *Let (χ, A) and (χ', A') be admissible set pairs with $\chi' \subset \chi$ and $A' \subset A$. The existence of a deformation d retracting χ onto χ' and A onto A', implies that the homomorphism*

$$(2.8)' \qquad (i_*)_q : H_q(\chi', A') \to H_q(\chi, A), \qquad (q = 0, 1, \cdots),$$

induced by the inclusion mapping

$$(2.8)'' \qquad i : (\chi', A') \to (\chi, A)$$

is an isomorphism.

Moreover, under the inverse of the mapping (2.8)′, the relative *homology class on* χ *of a q-cycle* z_χ *on* χ mod A, *goes into the relative homology class on* χ' mod A' *of* $\hat{d}_1 z_\chi$.

This lemma is established in slightly different form as Theorem 28.4 on page 251 of reference [4]. The proof in reference [4] suffices.

EXCISION. Among the axioms of Eilenberg and Steenrod, formulated on page 11 of reference [19], is the so-called Excision Axiom. Our next theorem formulates a simplified version of this axiom.

LEMMA 2.3. *Excision. Let* χ *be a metric space, A a proper subspace of* χ *and* A^* *a subspace of A such that for some positive e*

(2.9) $$(\chi - A)_e \subset \chi - A^*,$$

where $(\chi - A)_e$ *is the open e-neighborhood of* $\chi - A$, *relative to* χ.
There then exist isomorphisms,

(2.1) $$(i_*)_q : H_q(\chi - A^*, A - A^*) \approx H_q(\chi, A) , \qquad (q = 0, 1, \cdots) ,$$

over $\mathcal{K}$ *induced by the inclusion mapping*

(2.11) $$i : (\chi - A^*, A - A^*) \to (\chi, A).$$

This theorem is established as Theroem 28.3 of reference [3], page 249, in slightly different form.

We shall make use of a special theorem given by (2.14) below.

AN EXACTNESS RELATION. In reference [19] on page 11 one finds an Exactness Axiom that consists of a sequence of exactness relations. We shall have need for one of these relations in concrete form. Two group homomorphisms are involved, a first and second. The exactness relation affirms that the image of the first group under the first homomorphism is the kernel of the second. We shall define the two homomorphisms involved in the exactness relation.

There is given a T- space χ and a subspace $A \subset \chi$. The second homomorphism has the form

(2.12) $$(i_*)_{q-1} \ : \ H_{q-1}(A) \to H_{q-1}(\chi), \qquad (q = 1, 2, \cdots) ,$$

and is induced by the inclusion map $i : A \to \chi$.

The first homomorphism has the form

(2.13) $$\Theta_\partial \ : H_q(\chi, A) \to H_{q-1}(A)$$

and is defined as follows. If u^q is a q-chain on χ which is a q-cycle mod A, the homology class of u^q on χ mod A shall go into the homology class $((\partial u^q, A))$ of $H_{q-1}(A)$. One verifies the fact that if

$v^q \sim u^q$ on χ mod A, then $((\partial u^q, A)) = ((\partial v^q, A))$, so that a mapping (2.13) is well-defined.

The exactness theorem affirms that

$$\Theta_\partial (H_q(\chi, A)) = \ker(i_*)_{q-1}, \qquad (q = 1, 2, \cdots). \tag{2.14}$$

The proof of (2.14) is trivial. One can use the definitions found in § 28 of reference [4] on Relative Homologies to verify (2.14).

3. **Comparison of the homology groups of $\Omega_k^\rho \cup F_c$, *and* $\dot{\Omega}_k^\rho \cup F_c$.** In this section we shall be concerned with the k-saddle Ω_k^ρ of σ and the sublevel set F_c defined in § 1. For simplicity of notation we shall set

$$\Omega = \Omega_k^\rho, Y = \Omega_k^\rho \cup F_c. \tag{3.1}$$

We also set

$$\dot{\Omega} = \Omega - \sigma, \ \dot{Y} = Y - \sigma. \tag{3.2}$$

A first lemma follows.

Lemma 3.1. *The homomorphisms*

$$(i_*)_q : H_q(\Omega, \dot{\Omega}) \approx H_q(Y, \dot{Y}), \qquad (q = 0, 1, \cdots), \tag{3.3}$$

induced by the inclusion mapping

$$i : (\Omega, \dot{\Omega}) \to (Y, \dot{Y}) \tag{3.4}$$

are isomorphisms.

Use will be made of Excision Lemma 2.3. The set pair (χ, A) and set A^* of Lemma 2.3 will here be defined by setting

$$(\chi, A) = (Y, \dot{Y}); \ A^* = F_c - \pi_{k-1}, \tag{3.5}$$

so that in Lemma 2.3, $A^* \subset A$ and

$$\chi - A^* = \Omega; \ A - A^* = \dot{\Omega}. \tag{3.6}$$

The excision condition (2.9) is satisfied, since a sufficiently small neighborhood of $\chi - A = \sigma$, relative to Y, is included in $\chi - A^* = \Omega$.

The isomorphisms (3.3) follow from (2.10).

Bases of the homology groups in (3.3). The homology groups in (3.3) are over $\mathcal{K}$ and hence free. We shall establish the relations

$$\dim H_q(Y, \dot{Y}) = \dim H_q(\Omega, \dot{\Omega}) = \delta_k^q, \quad (q = 0, 1, \cdots). \tag{3.7}$$

Proof of (3.7). For $k > 0$ let Δ_k denote the origin-centered, closed k-disc of unit radius in R^k. Let $\dot{\Delta}_k$ denote Δ_k with the origin deleted.

The T-saddle $\Omega = \Omega_k{}^\rho$ is, by Definition 1.6, a topological k-disc with the critical point σ in its interior. There accordingly exists a surjective homeomorphism

$$(3.8)' \qquad \Theta_k : \Omega \to \Delta_k , \quad (k > 0) ,$$

with $\Theta_k(\sigma) = \mathbf{0}$. Under Θ_k, $\dot{\Omega}$ is mapped homeomorphically onto $\dot{\Delta}_k$. The chain transformation $\hat{\Theta}_k$ accordingly induces isomorphisms

$$(3.8)'' \qquad H_q(\Omega, \dot{\Omega}) \rightleftarrows H_q(\Delta_k, \dot{\Delta}_k), \quad (k > 0; q = 0, 1, \cdots).$$

The dimension R_q of the homology group $H_q(\Delta_k, \dot{\Delta}_k)$ is $\delta_k{}^q$, as is well known. Hence the second equality in (3.7) is true. The first equality in (3.7) is a consequence of (3.3).

We shall obtain bases for the relative homology groups in (3.8)$''$. To that end a definition is required.

Definition 3.1. *A prebase for a homology group.* Let $H_q(\chi, A)$ be a relative homology group over $\mathcal{K}$ with a finite base. A set of nontrivial relative q-cycles, one from each relative homology class in a base for $H_q(\chi, A)$, will be called a *prebase* for $H_q(\chi, A)$. If a base for $H_q(\chi, A)$ is empty, each prebase is empty.

We seek a prebase for $H_k(\Omega, \dot{\Omega})$. By virtue of (3.7) such a prebase will consist of a single relative k-cycle on Ω mod $\dot{\Omega}$. We shall define a prebase which is given by a singular k-cell on Ω mod $\dot{\Omega}$ and is *simply carried* by Ω in the sense of the following definition.

Definition 3.2. *Simply-carried singular q-cells.* A singular q-cell on χ is defined (Eilenberg) by an equivalence class of continuous mappings $\tau : s \to \chi$ of vertex-ordered euclidean q-simplices s into χ. If the mappings τ are homeomorphic mappings of their domains s onto their images $\tau(s)$, the resultant singular q-cell on χ will be said to be *simply-carried.*

We apply this definition as follows.

Definition 3.3. *A saddle k-cell $\kappa_\sigma{}^\rho$ of σ.* There exists a k-cell $\kappa_\sigma{}^\rho$ whose carrier is the k-saddle Ω of σ and which is simply-carried by Ω. We term $\kappa_\sigma{}^\rho$ a *saddle* k-cell of σ. Taken mod $\dot{\Omega}$, $\kappa_\sigma{}^\rho$ will be termed a relative *saddle k-cycle* on Ω mod $\dot{\Omega}$.

We shall verify a basic proposition.

Proposition 3.1. *With Ω and Y defined as in* (3.1), *the saddle k-cell $\kappa_\sigma{}^\rho$ of σ has the following properties*:

(i) *Taken mod $\dot{\Omega}$, it is a prebase for $H_k(\Omega, \dot{\Omega})$*;

(ii) *Taken mod $\dot{Y}$ it is a prebase for $H_k(Y, \dot{Y})$.*

Proof of (i). We begin by proving the following lemma.

Lemma 3.2. *If y^k is a singular k-cell whose carried is Δ_k and which is simply-carried, then, taken* mod $\dot{\Delta}_k$, *y^k is a k-cycle on Δ_k* mod $\dot{\Delta}_k$ *which is a prebase of $H_k(\Delta_k, \dot{\Delta}_k)$.*

Note that $|\partial y^k|$ is a topological $(k-1)$-sphere which is the geometric boundary of Δ_k. The remainder of the proof is left to the reader. Lemma 29.0 of reference [4] is relevant to the proof.

Granting the truth of Lemma 3.2, (i) of Proposition 3.1 follows from the isomorphism (3.8)$''$, as we now verify. The isomorphism (3.8)$''$ is induced by the homeomorphism Θ_k of $(\Omega, \dot{\Omega})$ onto $(\Delta_k, \dot{\Delta}_k)$. As a k-cycle on Ω, mod $\dot{\Omega}$, $\kappa_\sigma{}^\rho$ of Definition 3.3 is the image under the chain transformation $\hat{\Theta}_k$ of a singular k-cell y^k, characterized as in Lemma 3.2. Taken mod $\dot{\Omega}$, $\kappa_\sigma{}^\rho$ is a prebase of $H_k(\Omega, \dot{\Omega})$, since y^k, taken mod $\dot{\Delta}_k$, is a prebase of $H_k(\Delta_k, \dot{\Delta}_k)$. Thus (i) of Proposition 3.1 is true.

Proof of (ii). If i is the inclusion mapping (3.3) and $\kappa_\sigma{}^\rho$ the above saddle k-cell on Ω, then, $\kappa_\sigma{}^\rho$, taken mod $\dot{\Omega}$, is a prebase of $H_k(\Omega, \dot{\Omega})$, as we have just seen. Hence, by Lemma 3.1, $\hat{\imath}\kappa_\sigma{}^\rho$ will be a prebase for $H_k(Y, \dot{Y})$. Statement (ii) of Proposition 3.1 follows.

The decisive role of the "critical" $(k-1)$-cycle $w_c{}^{k-1}$ in Theorem 1.1 prompts the following lemma.

Lemma 3.3 (α). *The conditions of Definition* 1.4 *on a critical* $(k-1)$-*cycle $w_c{}^{k-1}$ of σ are satisfied by $\partial\kappa_\sigma{}^\rho$, where $\kappa_\sigma{}^\rho$ is the saddle k-cell of Definition* 3.3.

(β) *The* $(k-1)$-*cycles $w_c{}^{k-1}$ and $\partial\kappa_\sigma{}^\rho$ are on F_c and both bounding or nonbounding on F_c, or equivalently on $\dot{Y}$.*

Proof of (α). The $(k-1)$-cycle $\partial\kappa_\sigma{}^\rho$ is on the topological $(k-1)$-sphere π_{k-1} which gives the geometric boundary of Ω. Taken mod $\dot{\Omega}$, $\kappa_\sigma{}^\rho$ is a prebase of $H_k(\Omega, \dot{\Omega})$, by (i) of Proposition 3.1. It follows that $\partial\kappa_\sigma{}^\rho$ is a prebase of $H_{k-1}(\pi_{k-1})$ as one readily verifies. The reader will be aided in this proof by turning to Lemma 29.0 of reference [4]. Reference to Definition 1.4 of $w_c{}^{k-1}$ shows that $\partial\kappa_\sigma{}^\rho$ can serve in place of $w_c{}^{k-1}$. Thus (α) is true.

Proof of (β). That $w_c{}^{k-1}$ and $\partial\kappa_\sigma{}^\rho$ are both bounding or nonbounding on F_c is immediate. That this is equivalently true if F_c is replaced by $\dot{Y}$, follows from the existence of a deformation retracting $\dot{Y}$ onto F_c.

Thus Lemma 3.3 is true.

The following definition confirms and extends Definition 1.5.

DEFINITION 3.4. *A linking k-cycle* λ_σ^k. In case the critical $(k-1)$-cycle $\partial\kappa_\sigma^\rho$ bounds a k-cycle, say u^k on F_c, the difference,

$$\lambda_\sigma^k = \kappa_\sigma^\rho - u^k \tag{3.9}$$

is a k-cycle on Y and is called a *linking* k-cycle associated with σ. The $(k-1)$-cycle $\partial\kappa_\sigma^\rho$, as well as σ, is *then* said to be of linking type.

Note that λ_σ^k, like κ_σ^ρ, is a prebase of $H_k(Y, \dot{Y})$. With this understood we prove a basic proposition. We continue with the positive index k.

PROPOSITION 3.2. *Let e^k be an arbitrary k-cycle on Y. Two cases arise.*

Case I: *$\partial\kappa_\sigma^\rho$ of linking type. In this case for some element $K \in \mathcal{K}$*

$$e^k \sim K\lambda_\sigma^k\ , \quad (\textit{on } Y \bmod \dot{Y}). \tag{3.10}$$

Case II: *$\partial\kappa_\sigma^\rho$ of non-linking type. In this case*

$$e^k \sim 0\ , \quad (\textit{on } Y \bmod \dot{Y}). \tag{3.11}$$

In both cases Proposition 3.1 (ii) implies that for some element $K \in \mathcal{K}$

$$e^k = K\kappa_\sigma^\rho + \partial u^{k+1}, \quad (\text{on } Y \bmod \dot{Y}), \tag{3.12}$$

where u^{k+1} is a $(k+1)$-chain on Y. In both cases (3.12) implies that

$$K\partial\kappa_\sigma^\rho \sim 0\ , \quad (\text{on } \dot{Y}). \tag{3.13}$$

Case I: In this case (3.10) follows from (3.12), since $\kappa_\sigma^\rho = \lambda_\sigma^k$ on $Y \bmod \dot{Y}$, in accord with (3.9).

Case II: In this case $K = 0$ in (3.12) and (3.13). Otherwise

$$\partial\kappa_\sigma^\rho \sim 0\ , \quad (\text{on } \dot{Y}). \tag{3.14}$$

However (3.14) is impossible in Case II, as we now show.

There exists an F-deformation d retracting $\dot{Y}$ onto F_c. If then (3.14) held, the existence of d would imply, by Lemma 2.2, that $\partial\kappa_\sigma^\rho \sim 0$ on F_c contrary to the hypothesis in Case II. Hence $K = 0$ in (3.12), implying (3.11) in Case II.

Thus Proposition 3.2 is true.

We introduce the inclusion mapping

$$i : \dot{Y} \to Y \tag{3.15}$$

and the induced homomorphisms

$$(3.16)\qquad (i_*)_q : H_q(\dot{Y}) \to H_q(Y)\,, \quad (q = 0, 1, 2, \cdots).$$

The following lemma concerns $(i_*)_q$.

LEMMA 3.4. *The homomorphisms* $(i_*)_q$ *of* (3.16) *have the following properties*:

(a_1) *When* $q \neq k - 1$, $\mathrm{Ker}(i_*)_q = 0$;

(a_2) *When* $q = k - 1$, $\mathrm{Ker}(i_*)_q = 0$ *or has the non-null base* $((\partial\kappa_\sigma{}^\rho, \dot{Y}))$, *according as* σ *is or is not of linking type*;

(a_3) *When* $q \neq k$, $(i_*)_q$ *is surjective;*

(a_4) *When* q *is neither* k *nor* $k - 1$, $(i_*)_q$ *is an isomorphism over* $\mathcal{K}$ *of* $H_q(\dot{Y})$ *onto* $H_q(Y)$.

PROOF OF (a_1). Recall the exactness relation,

$$(3.17)\qquad \Theta_\partial(H_{q+1}(Y, \dot{Y})) \equiv \mathrm{Ker}(i_*)_q\,, \quad (q = 0, 1, 2, \cdots),$$

of which a general form is given in (2.14). It follows from (3.7) that $H_{q+1}(Y, \dot{Y}) = 0$ when $q + 1 \neq k$. Statement (a_1) can then be inferred from (3.17).

PROOF OF (a_2). According to (ii) of Proposition 3.1, $\kappa_\sigma{}^\rho$, taken as a k-cycle on $Y \bmod \dot{Y}$, is a prebase for $H_k(Y, \dot{Y})$. From the definition of the homomorphism, Θ_∂ of (2.13), we infer that a base for the image of $H_k(Y, \dot{Y})$ under Θ_∂ is 0 or the non-null homology class $((\partial\kappa_\sigma{}^\rho, \dot{Y}))$, according as $\partial\kappa_\sigma{}^\rho$ is bounding or non-bounding on $\dot{Y}$. Statement (a_2) now follows from (3.17).

PROOF OF (a_3). It is sufficient to show that if $c_+{}^q$ is a q-cycle on Y, and if $q \neq k$, then for some q-cycle $c_-{}^q$ on $\dot{Y}$

$$(3.18)\qquad c_+{}^q \sim c_-{}^q\,, \quad (\text{on } Y).$$

To verify (3.18) we infer from (3.7) that if $q \neq k$

$$(3.19)\qquad c_+{}^q = \partial e_+{}^{q+1} + e_-{}^q\,,$$

for suitable chains $e_+{}^{q+1}$ on Y and $e_-{}^q$ on $\dot{Y}$. An application of ∂ to the members of (3.19) shows that $e_-{}^q$ is a q-cycle on $\dot{Y}$, so that (3.18) holds with $c_-{}^q = e_-{}^q$. Thus (a_3) is true.

PROOF OF (a_4). Statement (a_4) is a consequence of (a_1) and (a_3). Thus Lemma 3.4 is true.

DEFINITION 3.5. *A prebase* $\mathbf{B}_q(\chi)$ *of* $H_q(\chi)$. Let χ be a topological space and q any rational integer. If the connectivity $R_q(\chi)$ of χ over $\mathcal{K}$ is finite, we shall denote by $\mathbf{B}_q(\chi)$ a *prebase* of $H_q(\chi)$. See Definition 3.1. The number of q-cycles in $\mathbf{B}_q(X)$ equals $R_q(X)$. This number may be 0. $\mathbf{B}_q(X)$ is then empty.

The connectivities $R_q(F_c)$ are finite by hypothesis of Theorem 1.1. It follows then from Lemma 1.2 that each connectivity $R_q(\dot{Y})$ is also finite. Hence a finite prebase $\mathbf{B}_q(\dot{Y})$ exists for each q.

Homological independence over $\mathcal{K}$ of q-cycles on X. A finite set S (possibly empty) of q-cycles on a topological space X will be said to be *homologically independent*, over $\mathcal{K}$ on X, if the corresponding homology classes in $H_q(X)$ are linearly independent over $\mathcal{K}$. An empty set S of q-cycles is understood to be homologically independent.

The following lemma is needed in the proof of Proposition 3.3.

LEMMA 3.5 (α). *If the index k of σ is positive and if σ is of non-linking type, there exists a prebase $\mathbf{B}_{k-1}(\dot{Y})$ of $H_{k-1}(\dot{Y})$ which contains the critical $(k-1)$-cycle w_c^{k-1} of Theorem* 1.1.

(β). *Moreover the $(k-1)$-cycles of the set*

$$\mathbf{B}_{k-1}(\dot{Y}) - w_c^{k-1}$$

are then homologically independent over $\mathcal{K}$ on Y.

PROOF OF (α). When $k > 0$ and σ is of non-linking type, the $(k-1)$-cycle $\partial \kappa_\sigma^\rho \not\sim 0$ on $\dot{Y}$. Equivalently by Lemma 3.3, the critical $(k-1)$-cycle $w_c^{k-1} \not\sim 0$ on $\dot{Y}$. There then exists a prebase $\mathbf{B}_{k-1}(\dot{Y})$ for $H_{K-1}(\dot{Y})$ that contains w_c^{k-1}.

PROOF OF (β). Suppose, contrary to (β), that there exists a nontrivial linear combination u^{k-1} over $\mathcal{K}$ of $(k-1)$-cycles in the set of β such that $u^{k-1} \sim 0$ on Y. The $(k-1)$-cycle u^{k-1} is then in $\operatorname{Ker}(i_*)_{k-1}$ and hence, by (a_2) of Lemma 3.4, $u^{k-1} \sim K w_c^{k-1}$ on $\dot{Y}$, with $K \in \mathcal{K}$. Equivalently the $(k-1)$-cycles in $\mathbf{B}_{k-1}(\dot{Y})$ are homologically dependent over $\mathcal{K}$ on $\dot{Y}$, contrary to the nature of a prebase of $H_{k-1}(\dot{Y})$.

Thus Lemma 3.5 is true.

NOTATION FOR PROPOSITION 3.3. The index k of σ is positive, as previously. For each q we seek a prebase for $H_q(Y)$ in terms of a prebase $\mathbf{B}_q(\dot{Y})$ for $H_q(\dot{Y})$. Prop. 3.3 will refer to the following exhaustive case division:

Case 1. $q \neq k$ or $k - 1$,
Case 2. $q = k, \sigma$ of non-linking type,
Case 3. $q = k - 1; \sigma$ of linking type,
Case 4. $q = k, \sigma$ of linking type,
Case 5. $q = k - 1, \sigma$ of non-linking type.
In Case 4, λ_σ^k denotes the linking k-cycle of Definition 3.4.

PROPOSITION 3.3 (I). *In cases* 1, 2, 3, *a prebase for* $H_q(\dot{Y})$ *is a prebase for* $H_q(Y)$.

(II). *In Case* 4, *if* $\mathbf{B}_k(\dot{Y})$ *is a prebase for* $H_k(\dot{Y})$, *then*

$$\mathbf{B}_k(\dot{Y}) \cup \lambda_\sigma^{K} \tag{3.19$'$}$$

is a prebase for $H_k(Y)$.

(III). *In Case* 5, *if* $B_{k-1}(\dot{Y})$ *is a prebase for* $H_{k-1}(\dot{Y})$ *that contains* w_c^{k-1}, *then*

$$\mathbf{B}_{k-1}(\dot{Y}) - w_c^{k-1} \tag{3.20}$$

is a prebase for $H_{k-1}(Y)$.

PROOF OF (I). In Case 1, the inclusion induced homomorphism $(i_*)_q$ of (3.16) is an isomorphism by (a_4) of Lemma 3.4, so that (I) is true in Case 1.

In case 2, Case II of proposition 3.2 implies that an arbitrary k-cycle on Y is homologous on Y to a linear combination, over $\mathcal{K}$, of k-cycles in $\mathbf{B}_k(\dot{Y})$. According to (a_1) of Lemma 3.4, $\operatorname{Ker}(i_*)_k = 0$. Hence the k-cycles in $\mathbf{B}_k(\dot{Y})$ are homologically independent over $\mathcal{K}$ on Y and so form a prebase for $H_k(Y)$ in Case 2.

In Case 3, $q = k - 1$ so that (3.7) implies that an arbitrary $(k-1)$-cycle on Y is homologous on Y to a linear combination over $\mathcal{K}$ of $(k-1)$-cycles in $\mathbf{B}_{k-1}(\dot{Y})$. Moreover, by ($a_2$) of Lemma 3.4, $\operatorname{Ker}(i_*)_{k-1} = 0$ when σ is of linking type. Hence $\mathbf{B}_{k-1}(\dot{Y})$ is a prebase for $H_{k-1}(Y)$ in Case 3.

PROOF OF (II). In Case 4, $w_c^{k-1} \sim 0$ on $\dot{Y}$. By virtue then of (3.10) of Proposition 3.2, an arbitrary k-cycle on Y is homologous on Y in Case 4, to a linear combination over $\mathcal{K}$ of k-cycles in the set (3.19)$'$. Moreover, the k-cycles in $B_k(\dot{Y})$ are homologically independent over $\mathcal{K}$ on Y; for $\operatorname{Ker}(i_*)_k = 0$ by (a_1) of Lemma 3.4, while λ_σ^k is homologous on Y over $\mathcal{K}$ to no linear combination, over $\mathcal{K}$, of k-cycles in $\mathbf{B}_k(\dot{Y})$. This follows from Proposition 3.1 (ii) and the definition (3.9) of λ_σ^k. Thus (II) is true.

PROOF OF (III). By (3.7), $\dim H_{k-1}(Y, \dot{Y}) = 0$. Hence each $(k-1)$-cycle on Y is homologous, over $\mathcal{K}$ on Y, to a linear combination, over $\mathcal{K}$, of $(k-1)$-cycles *in* $\mathbf{B}_{k-1}(\dot{Y})$ or, equivalently, *in* the set (3.20). Moreover, in Case 5, the $(k-1)$-cycles in the set (3.20) are homologically independent, over $\mathcal{K}$, on Y, by (β) of Lemma 3.5. Thus (III) is true.

4. **Verification of Theorem 1.1 and its corollary.** Theorem 1.1 follows from Proposition 3.3 as we shall see. Cf. "Program" at end of § 1.

THEOREM 1.1 compares connectivities of the spaces F_β and F_c. Proposition 3.3 indirectly compares the connectivities of $Y = \Omega_k{}^\rho \cup F_c$ and $\dot{Y} = \dot{\Omega}_k{}^\rho \cup F_c$ by comparing the prebases of the homology groups of Y and $\dot{Y}$. However, for each q, $H_q(F_\beta)$ and $H_q(Y)$ are isomorphic, as are $H_q(\dot{Y})$ and $Y_q(F_c)$. This is implied by our initial F-traction Δ and subsequent deformations. With this understood Proposition 3.3 implies Theorem 1.1.

VERIFICATION OF COROLLARY 1.1. An F-traction D of a topological subspace A of X into a subspace B of A has the property that if $A \supset F_b \supset B$ for some $b \in R$, then the restriction $D \mid F_b$ of D is an F-traction of F_b on F_b into B. A traction of A into B which is not an F-traction does not in general have this property.

Set $A = \{p \in F_a \mid F(p) < a\}$. When $k > 0$ let Δ_1 be the final image of F_β under the F-traction Δ of F_β into $\Lambda_\sigma \cup F_c$. (Global Condition II.) To Δ_1 one can apply the F-deformation d retracting $\Lambda_\sigma \cup F_c$ onto $\Omega_k{}^\rho \cup F_c$. (Lemma 1.1(ii).) We infer the existence of an F-traction δ of F_β into $\Omega_k{}^\rho \cup F_c$. It follows that $\delta \mid A$ is an F-traction of A into $\dot{\Omega}_k{}^\rho \cup F_c$. There is accordingly an F-traction D of A into F_c. Since $A \supset F_b \supset F_c$, Corollary 1.1 follows.

NOTE. Theorem 1.1 is true if the F-traction Δ of Global Condition II is merely a traction of X into $\Lambda_\sigma \cap F_c$. Our proofs show this to be the case. However with Δ merely a traction, a valid formulation of Corollary 1.1 is an open question.

It is a consequence of Corollary 1.1 that the inclusion map of F_c into F_b induces an isomorphism of $H_q(F_c)$ onto $H_q(F_b)$ for each b such that $c < b < F(\sigma)$.

5. **The case $k = 0$.** The case of a traction induced critical point σ of index $k = 0$ is included in Definition 1.3. In this case (1.6)″ takes the form

$$F(p) - F(\sigma) = x_1{}^2 + x_2{}^2 + \cdots + x_\mu{}^2 = Q_0(\mathbf{x}) \quad (x \in D_e). \tag{5.1}$$

The theorem which bears on this case has hypotheses which differ but little from the hypotheses when $k > 0$. As previously we set $F(\sigma) = a$ and $\Lambda_\sigma = \Phi_\sigma(D_e)$. When $k = 0$, Λ_σ is a topological μ-disc on which $F \mid \Lambda_\sigma$ has an absolute minimum value a. If a is also an absolute minimum value of F on X, then for any value $c \in R$ such that $c < a$, F_c is an empty set and, by convention, has null connectivities. In general, F_c is not empty for $c < a$.

The basic theorem when $k = 0$ follows.

THEOREM 5.1. *If the index k of a traction induced critical point σ is 0 and if the homology groups of F_c are finitely generated, the respective connectivities $R_q{}^\beta$ of F_β are finite and equal the connectivities $R_q{}^c$ of F_c when $q > 0$. When $q = 0$*

$$R_0{}^\beta - R_0{}^c = 1. \tag{5.2}$$

When F_c is empty Theorem 5.1 implies that $R_q{}^\beta = 0$ when $q > 0$ and that $R_0{}^\beta = 1$.

The proof of Theorem 5.1 is left to the reader. Theorem 5.1 has a corollary similar to Corollary 1.1 of Theorem 1.1.

Theorems 1.1 and 5.1 have the following useful corollary.

COROLLARY 5.1. *Let k be the index of a traction induced critical point σ of F as defined on $F_\beta = X$ and suppose that the homology groups of F_c of Global Condition* II *are finitely generated.*

The connectivities $R_q{}^\beta$ of F_β are then finite and equal the respective connectivities $R_q{}^c$ of F_c, except when $q = k - 1$ or k. Moreover either

$$R_k{}^\beta = R_k{}^c + 1 \text{ and } R_{k-1}^\beta = R_{k-1}^c, \tag{5.3}$$

or

$$R_{k-1}^\beta = R_{k-1}^c - 1 \text{ and } R_k{}^\beta = R_k{}^c. \tag{5.4}$$

In the case $k = 0$, only the first alternative occurs.

DEFINITION 5.1. The traction induced critical point σ of Corollary 5.1 will be said to be of "increasing" or "decreasing type" according as the alternative (5.3) or (5.4) occurs.

6. **An auxiliary lemma.** The following lemma is useful in applications of Theorem 5.1 in establishing the relevant traction.

Let $D_{\hat{e}}$ be defined by (1.5) with e replaced by $\hat{e} < e$. Corresponding to the homeomorphism Φ_σ of (1.6)$'$, set $\Phi_\sigma(D_{\hat{e}}) = \hat{\Lambda}_\sigma$ and recall that $\Phi_\sigma(D_e) = \Lambda_\sigma$.

LEMMA 6.1. *If $c < a$ and $a - c$ is sufficiently small, there exists an F-traction of $\Lambda_\sigma \cup F_c$ into $\hat{\Lambda}_\sigma \cup F_c$.*

If $k = 0$, the lemma is trivial. If $0 < k < \mu$ the traction is given by d of Lemma 1.1 on observing that d of Lemma 1.1 deforms $\hat{\Lambda}_\sigma \cup F_c$ on itself. When $k = \mu$, d properly defined moves no point.

It should be noted that the proof of Lemma 6.1 makes no use of the Global Condition II on σ.

7. **A global consequence of Theorem 5.3.** Suppose that there exists a finite sequence

(7.1) $$a_1 < a_2 < \cdots < a_r$$

of values of F assumed at the respective points,

(7.2) $$\sigma_1, \sigma_2, \cdots, \sigma_r$$

in X, with σ_1 affording an absolute minimum a_1 to F on X. Here σ_r and a_r equal σ and a respectively of § 1. Suppose, moreover, that there exist values in R,

(7.3) $$c_0 < c_1 < \cdots < c_{r-1} < c_r, \qquad (\text{with } c_r = \beta \text{ of § 1}),$$

such that

(7.4) $$c_0 < a_1 < c_1 < a_2 < \cdots < a_{r-1} < c_{r-1} < a_r < c_r,$$

and for $i = 1, \cdots, r$, that σ_i is a traction-induced critical point of index k_i, induced by an F-traction Δ_i of F_{c_i} into $\Lambda_{\sigma_i} \cup F_{c_{i-1}}$ (as in Global Condition II with Λ_{σ_i} defined as was Λ_σ in Local Condition I).

Let κ be the maximum of the indices $k_1, \cdots, k_r$. For $q = 0, 1, 2, \cdots$ let $m_q{}^+$ and $m_q{}^-$ be the number of points in the set (7.2) with index q which are of "increasing" or "decreasing" type, respectively, in the sense of Definition 5.1. The number of points (7.2) with index q is then

(7.5) $$m_q = m_q{}^+ + m_q{}^-, \qquad (q = 0, 1, 2, \cdots).$$

We term m_q the q-th type number of the set (7.2). Note that $m_0 = m_0{}^+$ and that for $q > \kappa$, $m_q{}^+ = m_q{}^- = 0$.

The set F_{c_0} is empty and so has null connectivities. Let R_q be the q-th connectivity of X, the domain of F. The change in the q-th connectivity of F_c as F_c changes from $F_{c_{i-1}}$ to F_{c_i} is given by Corollary 5.1. The resultant change in connectivities as F_c changes from F_{c_0} to F_{c_r} is the connectivity R_q of F_{c_r}. Corollary 5.1 implies that

(7.6) $$\begin{aligned} R_0 &= m_0^+ - m_1{}^-, \\ R_1 &= m_1^+ - m_2{}^-, \\ &\cdots \\ R_{\kappa-1} &= m_{\kappa-1}^+ - m_\kappa{}^-, \\ R_\kappa &= m_\kappa{}^+, \end{aligned}$$

and that $R_q = 0$ for $q > \kappa$. For q on the range $0, 1, \cdots, \kappa$, (7.6) implies that

(7.7) $$m_q - m_{q-1} + \cdots (-1)^q m_0 - m_{q+1}^- = R_q - R_{q-1} + \cdots (-1)^q R_0,$$

and hence that

(7.8) $m_q - m_{q-1} + \cdots (-1)^q m_0 \geqq R_q - R_{q-1} + \cdots (-1)^q R_0.$

We infer the following.

GLOBAL THEOREM 7.1. *Suppose that the mapping F of § 1 has traction-induced critical points* (7.2) *as characterized in the first paragraph of this section, with type numbers $m_0, m_1, \cdots, m_\kappa$. For $q > \kappa$, $m_q = 0$. Moreover, if R_q is the q-th connectivity of X, the domain of F, over $\mathcal{K}$, then $R_q = 0$ for $q > \kappa$ and*

$$
\begin{aligned}
m_0 &\geqq R_0\,, \\
m_1 - m_0 &\geqq R_1 - R_0\,, \\
m_2 - m_1 + m_0 &\geqq R_2 - R_1 + R_0\,, \qquad (7.9) \\
- \quad - \quad - &\quad - \quad - \quad - \\
m_\kappa - m_{\kappa-1} + \cdots (-1)^\kappa m_0 &= R_\kappa - R_{\kappa-1} + \cdots (-1)^\kappa R_0.
\end{aligned}
$$

As a consequence $m_q \geqq R_q$ for each $q \geqq 0$.

HISTORICAL NOTE. The methods of establishing the "tractions" inducing the critical points of Theorem 7.1 will vary with the case at hand. For the case of topologically nondegenerate functions on a topological manifold see [3]. In [3] the critical points are isolated. In the general case the critical points are not isolated. This happens in the case of the functions F associated with global variational theory.

NOTE ON LOCAL CONDITIONS I AND GLOBAL CONDITION II. Earlier treatments of the critical points of functions on differentiable manifolds have required that both the manifolds and the functions studied be continuously differentiable. When one applies the theory to variational analysis, the derivation of fundamental relations, such as those of Theorem 7.1, without setting the problem up in classical differentiable form, greatly simplifies the exposition. Another very essential innovation is that the topological μ-ball Λ_σ is not required to be a neighborhood of σ.

The derivation of the global relations of Theorem 7.1 are shown to be valid even when there is no underlying global differential structure. Kervaire, Eells and Kuiper, Milnor and many others have shown the great variety of topological and combinatorial manifolds which have no representative with a global differential structure. See, for example, [**20**]. Our theorems apply in examples, similar to these, and possibly more importantly when one is not *a priori* certain that there is a global differential structure.

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