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EIGENVALUE BIFURCATION FOR ODD GRADIENT OPERATORS

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1. **Introduction.** It is the purpose of this paper to derive some results concerning the set of solutions near the trivial solution $u = 0$ of equations of the form

$$\lambda u = Lu + T(u) + V(u), \qquad u \in H. \tag{1.1}$$

Here λ is a real parameter whose value is specified, and H is a real separable Hilbert space. L is a self-adjoint linear operator, T is homogeneous of odd degree k, and V is odd and of higher order than T. It is assumed that T and V are gradient operators, but not that the potentials are weakly continuous.

We will establish lower bounds for the number of solutions of (1.1) near $u = 0$ when λ is in a one-sided neighborhood of $\bar{\lambda}$, an isolated eigenvalue of L. Because both sides of (1.1) are odd, non-trivial solutions occur in pairs $\pm u$. These solutions are shown to be approximately of the form $|\lambda - \bar{\lambda}|^{1/(k-1)}v$, where v satisfies the finite dimensional problem $v = \operatorname{sgn}(\lambda - \bar{\lambda})\, PTv$, where P is the projection onto the subspace of $L - \bar{\lambda} I$.

The above hypotheses imply that the set of solutions of (1.1) coincides with the set of critical points of a certain even, real valued function $\phi(u)$. Our methods are based in part upon an appropriate modification of the Lusternik-Schnirelman theory.

In section 2 we obtain a result which is in part an analog and in part an extension of the main result of the Lusternik-Schnirelman theory for critical points of an even function on a manifold. It is an analog in that in the present case we consisder even functions on a ball in a finite dimensional space. The fact that this manifold has a boundary requires a modification of the theory. The extension of the theory lies in the circumstance that, if we wish, we may seek points which are only approximately critical. This is done in order to obtain a statement as to the spacing apart of the approximate critical points. The purpose of this spacing property is that when we use the theorem in conjunction with a Galerkin approximation scheme to obtain critical points of ϕ, we may be able to obtain more than one pair of critical points at the same critical level of ϕ, and thus overcome a limitation of the use of Lusternik-

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Schnirelman theory in conjunction with a Galerkin approximation scheme.

In section 3 we state our hypotheses for the terms in equation (1.1) and our main results on bifurcation; in section 4 we give the proofs.

The bifurcation problem for equations similar to (1.1) has been investigated by many authors. See, for example, [**1**, **2**, **5**, **6**, 7, 9, **10**] and the references cited in some of these. The Lusternik-Schnirelman theory has previously been applied to bifurcation theory in [**1**, **2**, **5**, 7, 9], however, our development enjoys two main advantages. First, we are able to treat λ as a parameter whose value is specified rather than to be determined. Secondly, we have eliminated the hypothesis that the potential of $T(u) + R(u)$ be weakly continuous. In [**6**, **10**] there are treatments of bifurcation theory for equations of the form (1.1) in which $T(u)$ is a homogeneous "polynomial" and is a gradient, though not necessarily odd, and $R(u)$ is of smaller order and is not necessarily a gradient. They consider the relative extrema of the functional $(T(u),u|_S$, where S is the unit sphere in the nullspace of $L - \bar{\lambda} I$, and they find solutions of (1.1) related to these extrema. We obtain solutions of (1.1) which, for $|\lambda - \bar{\lambda}|$ small, are also related to critical points of $(T(u), u)|_S$, but these critical points are not necessarily extrema.

2. **Preliminary results.** Throughout this section E will denote a real euclidean space, $B_r(S_r)$ the closed ball (the sphere) of radius r about the origin in E, and ψ a C^1 real valued function on a neighborhood of the origin in E. The results of this section could be generalized to the situation where E is an arbitrary real Banach space, but we have not done so because in any case we have to use a Galerkin approximation scheme and so we need these results only for finite dimensional spaces.

We shall say that ψ satisfies *condition* (I) *with parameters* a, b, r, h if ψ is defined in a neighborhood of B_r, $h \in (0, 1)$, and for all $u \in S_r$ such that $a \leqq \psi(u) \leqq b$ there holds $(\psi'(u), u) > -h\|\psi'(u)\| \|u\|$.

LEMMA 2.1. *Let ψ, satisfying condition* (I) *with parameters a, b, r, h, be even, and have a Lipschitz continuous derivative. Then there exists a positive constant d and an odd, Lipschitz continuous vector field $v : B_r \to E$ such that for all $u \in B_r$ such that $a - d \leqq \psi(u) \leqq b + d$ there holds*

$$\|\psi'(u)\| \geqq \|v(u)\|, \qquad (\psi'(u), v(u)) \geqq (1 - h^2)\|\psi'(u)\|^2, \tag{2.1}$$

and for all u in a neighborhood of S_r there holds

$$(v(u), u) \geqq 0. \tag{2.2}$$

PROOF. We define sets

$$\sigma = \{u \in S_r \mid a \leqq \psi(u) \leqq b\},\ \hat{\sigma} = \{u \in \sigma \mid (\psi'(u), u) \leqq 0\},$$

$$\sigma_\epsilon = \{u \in B_r \mid \operatorname{dist}(u, \sigma) \leqq \epsilon\},\ \hat{\sigma}_\epsilon = \{u \in B_r \mid \operatorname{dist}(u, \hat{\sigma}) \leqq \epsilon\}.$$

We write $\psi'(u) = \alpha(u) + \beta(u)$, where $\alpha(u) = (\psi'(u), u)\|u\|^{-2}u$ and hence $(\beta(u), u) = 0$. For $u \in B_r$ such that $\psi'(u) \neq 0$ we may define $\theta(u) = \cos^{-1}[(\psi'(u), u)\|\psi'(u)\|^{-1}\|u\|^{-1}]$. Evidently, for $u \in \sigma$ we have $\cos\theta(u) > -h$. Using the compactness of σ, we may choose a neighborhood of σ of the form σ_ϵ such that $\cos\theta(u) > -h$ for $u \in \sigma_\epsilon$. Since $\hat{\sigma}_\epsilon \subset \sigma_\epsilon$ and $\cos\theta(u) \leqq 0$ for $u \in \hat{\sigma}$, by choosing $\epsilon > 0$ smaller, if necessary, we may arrange that $-h < \cos\theta(u) < h$ for all $u \in \hat{\sigma}_\epsilon$. It follows that $\|\beta(u)\| > (1 - h^2)^{1/2}\|\psi'(u)\|$ for all $u \in \hat{\sigma}_\epsilon$.

Now we let

$$\eta(u) = \operatorname{dist}(u, \hat{\sigma}_{\epsilon/2})/(\operatorname{dist}(u, \hat{\sigma}_{\epsilon/2}) + \operatorname{dist}(u, B_r\backslash\hat{\sigma}_\epsilon)),\ u \in B_r.$$

Then η is even, Lipschitz continuous, $0 \leqq \eta \leqq 1$, $\eta = 0$ on $\hat{\sigma}_{\epsilon/2}$ and $\eta = 1$ on $B_r\backslash\hat{\sigma}_\epsilon$. We let $\bar{v} = \eta\alpha + \beta$. Then $\bar{v}$ is odd and Lipschitz continuous, and $\bar{v} = \psi'$ on $B_r\backslash\hat{\sigma}_\epsilon$. Also $(\psi'(u), \bar{v}(u)) = \eta(u)\|\alpha(u)\|^2 + \|\beta(u)\|^2 \geqq \|\beta(u)\|^2$, hence for all $u \in \hat{\sigma}_\epsilon$, $(\psi'(u), \bar{v}(u)) > (1 - h^2)\|\psi'(u)\|^2$. Combining cases,

$$(\psi'(u), \bar{v}(u)) \geqq (1 - h^2)\|\psi'(u)\|^2. \tag{2.3}$$

Also clearly

$$\|\bar{v}(u)\| \leqq \|\psi'(u)\|. \tag{2.4}$$

Since $(\bar{v}(u), u) = 0$ if $u \in \hat{\sigma}_{\epsilon/2}$, and by compactness of $\hat{\sigma}$, $(\alpha(u), u) \geqq \text{const} > 0$ for $u \in \sigma\backslash\hat{\sigma}_{\epsilon/2}$, it follows that $(\bar{v}(u), u) \geqq 0$ in a neighborhood of σ. Hence there exists $d > 0$ such that $(\bar{v}(u), u) \geqq 0$ in a neighborhood in B_r of $\{u \in S_r \mid a - d \leqq \psi(u) \leqq b + d\}$. In the same manner that η was constructed, we may construct $\zeta : B_r \to R$ which is even, Lipschitz continuous, such that $0 \leqq \zeta \leqq 1$, $\zeta = 0$ in a neighborhood in B_r of $\{u \in S_r \mid \psi(u) \leqq a - d$ or $\psi(u) \geqq b + d\}$, $\zeta = 1$ on $\{u \in B_r \mid a \leqq \psi(u) \leqq b\}$. We then let $v(u) = \zeta(u)\bar{v}(u)$. Clearly $v(u)$ is odd and Lipschitz continuous, and using (2.3) and (2.4), $\|v(u)\| \leqq \|\psi'(u)\|$, $(\psi'(u), v(u)) \geqq (1 - h^2)\|\psi'(u)\|^2$, and it is clear that $(v(u), u) \geqq 0$ in a neighborhood of S_r. This completes the proof.

We introduce the notations

$$K_{c,r} = \{u \in B_r \mid \psi(u) = c, \psi'(u) = 0\},\ \psi_{c,r} = \{u \in B_r \mid \psi(u) \leqq c\}.$$

LEMMA 2.2. *Let ψ satisfy the hypothesis of Lemma 2.1 with $a = b = c$. Let U be an open neighborhood in B_r of $K_{c,r}$. Then there exists $d > 0$ and a continuous function $\eta : [0, \infty) \times B_r \to B_r$ such that $\eta(t, -u) = -\eta(t, u)$ and $\eta(1, \psi_{c+d,r} \backslash U) \subset \psi_{c-d,r}$.*

PROOF. This theorem and its proof are analogous to theorem 4 of [3], and it should suffice to indicate the differences in the proofs. Let v be as in Lemma 2.2 with $a = b = c$, and consider the function $\eta(t, u)$ defined by the initial value problem

$$\eta_t(t, u) = -v \circ \eta(t, u), \quad \eta(0, u) = u. \tag{2.5}$$

Since v is Lipschitz continuous, (2.5) has a unique maximal solution. This maximal solution is defined for all $t \geqq 0$, since the solution curves of (2.5) do not terminate because of meeting the boundary of B_r. Also, the properties of $v(u)$ listed in Lemma 2.1 are similar to those of a pseudo-gradient vector field, and hence the proof of Theorem 4 of [3] is seen to carry through in the present case.

We next introduce the topological notion of genus. Let F be a real Banach space, and if $C \subset F$, let $\Sigma(C)$ denote the class of subsets of $C \backslash \{0\}$ closed and symmetric with respect to the origin. If $A \in \Sigma(F)$ let the *genus* of A, denoted by $\gamma(A)$, be the least integer k such that there exists an odd continuous map from A to $R^k \backslash \{0\}$. If no such integer exists, let $\gamma(A) = \infty$. We let $\gamma(\emptyset) = 0$.

LEMMA 2.3. *Let $A, B \in \Sigma(F)$.*

(a) *If there exists an odd continuous map $f: A \to B$, then $\gamma(A) \leqq \gamma(B)$.*

(b) *If $A \subset B$, then $\gamma(A) \leqq \gamma(B)$.*

(c) $\gamma(A \cup B) \leqq \gamma(A) + \gamma(B)$.

(d) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \backslash B}) \geqq \gamma(A) - \gamma(B)$.*

(e) *If A is compact, then $\gamma(A) < \infty$ and there is an open neighborhood U of A such that $\bar{U} \in \Sigma(F)$ and $\gamma(\bar{U}) = \gamma(A)$.*

(f) *If there exists an odd homeomorphism of A onto the n-sphere, then $\gamma(A) = n+1$.*

(g) *If F is a Hilbert space and G is an n-dimensional subspace of F and A is such that $A \cap G^\perp = \emptyset$, then $\gamma(A) \leqq n$.*

The proofs of these properties are elementary and can be found in [4], for example, except that for (g), which can be found in [3].

DEFINITION 2.4. Let $\psi : B_r \to R$. Then, for positive integers $m \leqq \dim E$, define

$$c_{m,r}(\psi) = \inf\{\sup\{\psi(u) \mid u \in A\} \mid A \in \Sigma(B_r), \gamma(A) \geqq m\}.$$

Clearly $c_{m,r}(\psi) < c_{n,r}(\psi)$ if $m < n$.

We come to the principal result of this section.

THEOREM 2.5. *Let ψ satisfy the hypotheses of Lemma* 2.1; *let* $\epsilon = b-a \geqq 0$; *let* $a \leqq c_{m,r}(\psi) \leqq c_{n,r}(\psi) \leqq b$, *where* $m < n$; *and let* $\psi(u) > b$ *for all* $u \in B_{\bar{r}}$, *where* $\bar{r} < r$. *Then there exist* $n - m + 1$ *distinct non-zero pairs* $\pm u_i$, $m \leqq i \leqq n$, *such that*

$$a \leqq \psi(u_i) \leqq b, \tag{2.6}$$

$$\|\psi'(u_i)\| \leqq \max\left(\sqrt{\epsilon/(1-h^2)}, 2\epsilon/((1-h^2)\bar{r})\right), \tag{2.7}$$

$$\bar{r} \leqq \|u_i\| \leqq r, \tag{2.8}$$

$$\|u_i - u_j\| \geqq \bar{r}, j \neq i. \tag{2.9}$$

In particular, if $b = a$ *then* $\psi'(u_i) = 0$.

REMARK. In the case $b = a$, one may prove the stronger result that $\gamma(K_{a,r}) \geqq n - m + 1$. This is an easy consequence of Lemmas 2.2 and 2.3, and the proof goes essentially like that for corresponding results in, for example, [11, p. 312] and [3, p. 69]. However, we shall not need this result for the present paper.

PROOF. By the continuity of ψ' and the compactness of B_r, since ψ satisfies condition (I) with parameters a, b, r, h, there exists $b_1 > b$, such that for all $b^* \in (b, b_1]$, ψ satisfies condition (I) with parameters a, b^*, r, h^*, where $h^* \to h$ as $b^* \to b$, and such that $\psi(u) > b_1$ for all $u \in B_{\bar{r}}$. Applying Lemma 2.1, let $v(b^*, u)$ be an odd, Lipschitz continuous vector field on B_r such that if $a \leqq \psi(b^*, u) \leqq b^*$, then

$$\begin{gathered}\|\psi'(u)\| \geqq \|v(b^*, u)\|, \\ (\psi'(u), v(b^*, u)) \geqq (1 - h^{*2})\|\psi'(u)\|^2,\end{gathered} \tag{2.10}$$

and such that for all u in a neighborhood of S_r there holds

$$(v(b^*, u), u) \geqq 0. \tag{2.11}$$

Until something further is said, it will be convenient to suppress the dependence of v on b^*. Then the initial value problem (2.5) defines a function $\eta : [0, \infty) \times B_r \to B_r$ which is odd in u and continuous.

We will show how to construct u_i, given that if $i > m$ then u_j, $m \leqq j \leqq i-1$, have been constructed. Let

$$\alpha_1 = \{u \in B_r \backslash B_{\bar{r}} \mid (u, u_j) = 0, m \leqq j \leqq i-1\},$$

$$\alpha_2 = \{u \in B_r \backslash B_{\bar{r}} \mid \operatorname{dist}(u, \alpha_1) < \bar{r}/2\}.$$

It is understood, of course, that if $i = m$ then $\alpha_1 = \alpha_2 = B_r \backslash B_{\bar{r}}$. Let $A \in \Sigma(B_r)$ be such that $\gamma(A) \geqq i$ and $\sup_A \psi \leqq b^*$. By Definition 2.4, $\sup_A \psi \geqq a$.

We claim that $\gamma(A \cap \alpha_1) \geqq m$. Suppose that $\gamma(A \cap \alpha_1) < m$ were the case. Then, using Lemma 2.3(e) we may write $A = A_1 \cup A_2$, where $A_1, A_2 \in \Sigma(B_r), \gamma(A_1) = \gamma(A \cap \alpha_1)$, and $A_2 \cap \alpha_1 = \emptyset$, hence by Lemma 2.3(g), $\gamma(A_2) \leqq i - m$. It follows from Lemma 2.3(c) that $\gamma(A) < i$, which is a contradiction. Hence $\gamma(A \cap \alpha_1) \geqq m$.

Let $B = \eta(1, A \cap \alpha_1)$. Clearly $B \subset B_r$. $\gamma(B) \geqq m$, by Lemma 2.3(a). Hence $\sup_B \psi \geqq a$. Let $v \in B$ be such that $\psi(v) = \sup_B \psi$, and let $\bar{u}$ be such that $\eta(1, \bar{u}) = v$. We introduce the function $f(t) = \psi \circ \eta(t, \bar{u})$. Then $f'(t) = (\psi' \circ \eta(t, \bar{u}), \eta_t(t, \bar{u}))$, so by (2.5) and (2.10)

$$f'(t) \leqq -(1 - h^{*2}) \|\psi' \circ \eta(t, \bar{u})\|^2, \tag{2.12}$$

hence $f(t)$ is decreasing, and since $\sup_A \psi \leqq b^*$,

$$0 \leqq \psi(\bar{u}) - \psi(v) \leqq b^* - a \equiv \epsilon^*. \tag{2.13}$$

On the other hand,

$$\begin{aligned} \psi(\bar{u}) - \psi(v) &= -\int_0^1 f'(t)\, dt \\ &\geqq (1 - h^{*2}) \int_0^1 \|\psi' \circ \eta(t, \bar{u})\|^2\, dt, \end{aligned} \tag{2.14}$$

using (2.12). It follows from (2.13) and (2.14) that for some $\bar{t} \in (0, 1)$, $\|\psi' \circ \eta(\bar{t}, \bar{u})\| \leqq (\epsilon^*/(1 - h^{*2}))^{1/2}$.

We now distinguish two cases. Suppose, for the first case, that for all $t \in [0, 1]$, $\eta(t, \bar{u}) \in \alpha_2$. Then $\eta(\bar{t}, \bar{u}) \in \alpha_2$, and if we let $u_i^* = \eta(\bar{t}, \bar{u})$, we have $\|\psi'(u_i^*)\| \leqq (\epsilon^*/(1 - h^{*2}))^{1/2}$ and $\|u_i^* - u_j\| \geqq \bar{r}$, $m \leqq j \leqq i - 1$. Suppose, for the second case, that $\eta(t, \bar{u}) \notin \alpha_2$ for some $t \in [0, 1]$. We may choose $\bar{t} \in [0, 1]$ so that $\eta(\bar{t}, \bar{u})$ lies on the boundary of α_2, and we let $w = \eta(\bar{t}, \bar{u})$. We introduce the arc length parameter s defined by $s(0) = 0$, $ds/dt = \|\psi' \circ \eta(t, \bar{u})\|$, and the function $g(s) = f(t)$. $ds/dt \neq 0$ since the trajectory starting at $\bar{u}$ cannot contain a critical point of ψ, hence, using (2.12), $g'(s) \leqq -(1 - h^{*2}) \|\psi' \circ \eta(t, \bar{u})\|$. Clearly $0 \leqq \psi(\bar{u}) - \psi(w) \leqq \epsilon^*$, hence

$$(1 - h^{*2}) \int_0^{s(\bar{t})} \|\psi' \circ \eta(t, \bar{u})\|\, ds \leqq \epsilon^*. \tag{2.15}$$

Since $\bar{u}$ lies in α_1 and w lies on the boundary of α_2 and since s is the arc length from $\bar{u}$ to w, $s(\bar{t}) \geqq \bar{r}/2$, hence from (2.15), there exists $t_1 \in [0, 1]$ such that $\|\psi' \circ \eta(t_1, \bar{u})\| \leqq 2\epsilon^*/((1 - h^{*2})\bar{r})$. We let $u_i^* = \eta(t_1, \bar{u})$. Combining the two cases, we have (2.6)–(2.9), for $m \leqq j \leqq i - 1$, with $u_i^*, b^*, \epsilon^*, h^*$ replacing u_i, b, ϵ, h respectively. We then consider a sequence of values for b^* approaching b, a corresponding sequence of values for h^* approaching h, and from the corresponding sequence for u_i^* we may select a convergent subsequence. If we take the limit of the subsequence as u_i then (2.6)–(2.9), for $m \leqq j \leqq i-1$, clearly hold. Thus, we construct $u_m, u_{m+1}, \cdots, u_n$ in succession, and this completes the proof of the theorem.

3. **Assumptions and statement of results.** We shall make the following assumptions for the terms on the right hand side of (1.1).

(A) $L:H \to H$ is a compact, self-adjoint, linear operator with an eigenvalue $\bar{\lambda}$ of multiplicity n. It follows that H may be written in the form $N \oplus F \oplus G$, where N is the nullspace of $L - \bar{\lambda} I$, $L : F \to F$ and $L : G \to G$, and where for some positive constants δ_1, δ_2

$$(Lu, u) \leqq (\lambda - \delta_1)\|u\|^2, \quad \text{if } u \in F, \tag{3.1}$$

$$(Lu, u) \geqq (\lambda + \delta_2)\|u\|^2, \quad \text{if } u \in G. \tag{3.2}$$

(B) $T : H \to H$ is homogeneous of degree k, an odd integer $\geqq 3$; T is the strong gradient of the functional $(k+1)^{-1}(T(u), u)$; $PT(u) \neq 0$ for $0 \neq u \in N$, where P is the orthogonal projection onto N; and T satisfies

$$\|T(u) - T(v)\| \leqq g(\|u\| + \|v\|)\|u - v\|, \tag{3.3}$$

where $g(t) \to 0$ as $t \to 0$. Hence $\|T(u)\| \leqq C\|u\|^k$ for some $C > 0$, and T is Lipschitz continuous.

(C) $V : D \to H$, where D is a neighborhood of the origin in H; V is odd and is the strong gradient of $\rho : D \to R$; $V(u) = o(\|u\|^k)$ as $\|u\| \to 0$, hence $\rho(u) = o(\|u\|^{k+1})$ as $\|u\| \to 0$;

$$\|V(u) - V(v)\| \leqq h(\|u\| + \|v\|)\|u - v\|, \tag{3.4}$$

where $h(t) \to 0$ as $t \to 0$. In particular, $V(u)$ is Lipschitz continuous.

We define the integers

$$\alpha = \gamma[\{u \in N \mid \|u\| = 1, (T(u), u) \leqq 0\}], \tag{3.5}$$

$$\beta = \gamma[\{u \in N \mid \|u\| = 1, (T(u), u) \geqq 0\}]. \tag{3.6}$$

The principal results on bifurcation are summarized in the following theorem.

THEOREM 3.1. *Assume that* (A), (B), (C) *hold. Then there exist* $\delta_3, \delta_4 > 0$ *such that for* $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_3]$ (*resp.* $\lambda \in [\bar{\lambda} - \delta_4, \bar{\lambda})$) *equation* (1.1) *has at least* $n-\alpha$ (*resp.* $n-\beta$) *distinct non-zero pairs* $\pm\, u$ *of solutions, each of the form* $|\lambda - \bar{\lambda}|^{1/(k-1)} u_1 + o(|\lambda - \bar{\lambda}|^{1/(k-1)})$ *as* $\lambda \to \bar{\lambda}$, *where* u_1 *satisfies* $u_1 = \operatorname{sgn}(\lambda - \bar{\lambda}) PT u_1$.

It seems a reasonable conjecture that there exist at least n pairs of solutions of (1.1) bifurcating from $\lambda=\bar{\lambda}, u=0$. This will be shown to be contingent upon another conjecture which seems quite reasonable, to the author at least.

CONJECTURE 3.2. *Let S be the unit sphere about the origin in R^n and let $A, B \in \Sigma(S)$ be disjoint. Then $\gamma(A) + \gamma(B) \leqq n$.*

COROLLARY 3.3. *Assuming the validity of Conjecture* 3.2, *and that* (A), (B), (C) *hold, then $\alpha + \beta \leqq n$ and hence there exist at least n pairs $\pm u$ of solutions of* (1.1) *bifurcating from $\lambda=\bar{\lambda}$, $u=0$, of the form stated in Theorem* 3.1.

4. **Proofs.** We shall use the results of Section 2 in combination with a Galerkin approximation scheme. Under hypotheses (A), (B), (C), the expression $-\lambda u + Lu + T(u) + V(u)$ is the strong gradient of the functional

$$\phi(u) = -\frac{\lambda}{2}\|u\|^2 + \frac{1}{2}(Lu, u) + \frac{1}{k+1}(T(u), u) + \rho(u), \tag{4.1}$$

hence the set of critical points of ϕ is the same as the set of solutions of (1.1).

(A) implies that H is spanned by the set of eigenvectors of L. Moreover, by considering subspaces of H spanned by a finite number of these eigenvectors we may obtain a sequence $\{H_j\}$ of finite dimensional subspaces of H such that, for all j, $H_j \subset H_{j+1}$, $H_j = N \oplus F_j \oplus G_j$, where $F_j \subset F$, $G_j \subset G$, and $L: F_j \to F_j$, $L: G_j \to G_j$, $L: H_j \to H_j$, and $\bigcup_{j=1}^{\infty} H_j$ is dense in H. We define $\sigma_j = \dim F_j$, $\phi_j = \phi \mid H_j \cap D$, and henceforth B_r shall denote the closed ball in H about the origin of radius r.

In order to find critical points of ϕ, we first find critical points of ϕ_j by considering the minimax levels $c_{m,r}(\phi_j)$ with r and m suitably chosen; then we show that certain subsequences of these critical points converge, as $j \to \infty$, to critical points of ϕ. This yields critical points of ϕ for the case mentioned in Theorem 3.1 where $\lambda > \bar{\lambda}$. We deal with the other case, where $\lambda < \bar{\lambda}$, by applying the procedure just outlined to $-\phi$ in place of ϕ.

We remark here that it might or might not be possible to obtain critical points of ϕ more directly by considering minimax levels $c_{m,r}(\phi)$ if the results of section 2 were generalized to the case of functionals on Banach spaces. This method would fail if $\dim F = \infty$, since then we would not be able to establish that $c_{m,r}(\phi)$ is a critical level for any values of m and r.

LEMMA 4.1. *If T satisfies hypotheses* (B), *then there exists $\ell > 0$ such that for all $a, b \in (-\ell, \ell)$ there holds*

$$\alpha = \gamma[\{u \in N \mid \|u\| = 1, (T(u), u) \leqq a\}], \tag{4.2}$$

$$\beta = \gamma[\{u \in N \mid \|u\| = 1, (T(u), u) \geqq b\}]. \tag{4.3}$$

PROOF. Let $\Theta(u)$ be the restriction of $(T(u), u)$ to N, and let $S = \{u \in N \mid \|u\| = 1\}$. We must show that there exists $\ell > 0$ such that for all $a, b \in (-\ell, \ell)$ there holds

$$\gamma[\{u \in S \mid \Theta(u) \leqq 0\}] = \gamma[\{u \in S \mid \Theta(u) \leqq a\}], \tag{4.4}$$

$$\gamma[\{u \in S \mid \Theta(u) \geqq 0\} = \gamma[\{u \in S \mid \Theta(u) \geqq b\}]. \tag{4.5}$$

Consider the application of the Lusternik-Schnirelman theory of critical points of functionals on manifolds to $\Theta(u)$ on S (See [**8**, **11**], for example). By definition, a critical point of Θ relative to S is a point $u_0 \in S$ such that $\Theta'(u_0)$ is a multiple of u_0. From (B) we have that $\Theta'(u) = (k+1)PT(u)$. By [8, p. 186], the set $\{u \in S \mid \Theta(u) = c_k\}$, $k = 1, 2, \cdots, n$, where

$$c_k = \sup\{c \in R \mid \gamma[\{u \in S \mid \Theta(u) \geqq c\}] \geqq k\},$$

contains critical points of Θ. We will show that no c_k equals zero. Suppose otherwise, that some $c_k = 0$. Then there exists $u_0 \in S$ such that $\Theta(u_0) = 0$ and $\Theta'(u_0) = (k+1)PT(u_0) = eu_0$ for some $e \in R$. But $\Theta(u_0) = (k+1)^{-1}(\Theta'(u_0), u_0) = (k+1)^{-1}e$ by the homogeneity of Θ, hence $e = 0$, and $PT(u_0) = 0$, which contradicts (B). Hence actually no c_k equals zero.

Next we observe that $\gamma[\{u \in S \mid \Theta(u) \geqq c\}]$ is a monotonic integer valued function of c, and if there were a jump at $c = 0$ then some c_k would equal zero. Since this is not the case, for some $\ell > 0$, (4.3) holds for $b \in (-\ell, \ell)$.

By carrying out the above procedure with $-\Theta$ in place of Θ, we obtain the assertion of (4.2).

We introduce the notation

$$b_{i,j,r} = c_{\sigma_j+n-i+1,r}(\phi_j). \tag{4.6}$$

LEMMA 4.2. *Under hypotheses* (A), (B), (C) *there exist positive constants K_1, K_2, δ_5, r_1 such that if $\lambda \in (\bar{\lambda}, \bar{\lambda}+\delta_5)$ and $r \in (0, r_1)$, then*

$$-K_1(\lambda-\bar{\lambda})^{(k+1)/(k-1)} < b_{i,j,r} < -K_2(\lambda-\bar{\lambda})^{(k+1)/(k-1)}, \tag{4.7}$$

for $i = 1, 2, \cdots, n-\alpha$.

PROOF. We introduce the homogeneous forms

$$(4.8)\qquad \begin{aligned} q(u) &= -\frac{\lambda}{2}\|u\|^2 + \frac{1}{2}(Lu, u), \\ t(u) &= \frac{1}{k+1}(T(u), u), \end{aligned}$$

in terms of which $\phi(u) = q(u) + t(u) + \rho(u)$. From (A) it follows that every $u \in H$ may be expressed in the form $u_1 + u_2 + u_3$, where $u_1 \in N$, $u_2 \in F$, $u_3 \in G$, and

$$(4.9)\qquad q(u) = q(u_1) + q(u_2) + q(u_3).$$

Let $s_{j,a} = \{u \in N \oplus F_j \mid \|u\| = a\}$. Then $\gamma(s_{j,a}) = \sigma_j + n$, hence from Definition 2.4 and (4.6),

$$b_{i,j,r} \leqq \inf\{\sup\{\phi_j(u) \mid u \in s_{j,a}\} \mid 0 < a < r\}.$$

From (3.1), $q(u) \leqq -(\lambda - \bar{\lambda})\|u\|^2/2$ for $u \in N \oplus F$, hence $q(u) \leqq -(\lambda - \bar{\lambda})a^2/2$ for $u \in s_{j,a}$. Also, from (B) and (C), $t(u) + \rho(u) \leqq C\|u\|^{k+1}$ for some constant C, if $\|u\|$ is sufficiently small. Hence $\phi_j(u) \leqq f(a) \equiv -(\lambda-\bar{\lambda})a^2/2 + C\,a^{k+1}$ if $u \in s_{j,a}$, and a is sufficiently small. It is found that the minimum of $f(a)$ is of the form $-K_2(\lambda-\bar{\lambda})^{(k+1)/(k-1)}$ taken at $a = const.\,(\lambda-\bar{\lambda})^{1/(k-1)}$. Hence r_1 may be chosen so that the upper bound of (4.7) holds for $r < r_1$.

To deal with the lower bound of (4.7), by Lemma 4.1 we may choose $a > 0$ such that if

$$(4.10)\qquad \tilde{U} = \{u \in N \mid \|u\| = 1, t(u) \leqq a\}\ ,$$

then $\gamma(\tilde{U}) = \alpha$. Let P be the orthogonal projection onto N; let

$$(4.11)\qquad U_j = \{u \in H_j \cap B_r \mid Pu \neq 0, Pu/\|Pu\| \in \tilde{U}\};$$

let Q_j be the orthogonal projection onto F_j; and let

$$(4.12)\qquad V_j = \{u \in H_j \cap B_r \mid Q_j u \neq 0\}.$$

Suppose that $A \in \Sigma(H_j \cap B_r)$ is such that $\gamma(A) \geqq \sigma_j + \alpha + 1$. We may show that $A\backslash(U_j \cup V_j) = \emptyset$. Suppose that, on the contrary, $A \subset U_j \cup V_j$. Then since A is compact, there exist $B, C \in \Sigma(H_j \cap B_r)$ such that $A \subset B \cup C$, $B \subset U_j$, $C \subset V_j$. Clearly there is an odd continuous map of B into $\tilde{U}$, hence by Lemma 2.3(a), $\gamma(B) \leqq \alpha$. By Lemma 2.3(g), $\gamma(C) \leqq \sigma_j$. Hence by Lemma 2.3(b,c), $\gamma(A) \leqq \sigma_j + \alpha$, a contradiction. So $A\backslash(U_j \cup V_j) \neq \emptyset$.

Now we let $u \in A\backslash(U_j \cup V_j)$ and derive a lower bound for $\phi_j(u)$. Clearly u is of the form $v+w$, where $v \in N$, $w \in G$, hence by (3.2) and (4.9),

$$q(u) \geqq -\tfrac{1}{2}(\lambda - \bar{\lambda})\|v\|^2 - \tfrac{1}{2}(\lambda - \bar{\lambda} - \delta_2)\|w\|^2. \tag{4.13}$$

By use of the mean value theorem and the appropriate chain rule,

$$t(u) = t(v) + (T(v + \Theta w), w), 0 < \Theta < 1. \tag{4.14}$$

since $v \notin U_j$, $t(v) \geqq a\|v\|^{k+1}$, hence from (4.14),

$$t(u) \geqq a\|v\|^{k+1} - C_1(\|v\|^k + \|w\|^k)\|w\|, \tag{4.15}$$

for some $C_1 > 0$. Since $\rho(u) = o(\|u\|^{k+1})$ as $\|u\| \to 0$, we have

$$\rho(u) \geqq -\tau(\|u\|)(\|v\|^{k+1} + \|w\|^{k+1}), \tag{4.16}$$

where $\tau(s) \to 0$ as $s \to 0$. Adding (4.13), (4.15), and (4.16),

$$\begin{aligned}\phi(u) \geqq & -\tfrac{1}{2}(\lambda - \bar{\lambda})\|v\|^2 - \tfrac{1}{2}(\lambda - \bar{\lambda} - \delta_2)\|w\|^2 + a\|v\|^{k+1} \\ & - C_1(\|v\|^k + \|w\|^k)\|w\| - \tau(\|u\|)(\|v\|^{k+1} + \|w\|^{k+1}).\end{aligned} \tag{4.17}$$

We suppose that r is sufficiently small so that, for $\|u\| \leqq r$, $|\tau(u)| \leqq a/4$, and we distinguish two cases. First we note that there is a positive constant C such that if $\|w\| < C\|v\|$ then

$$\phi(u) \geqq -\tfrac{1}{2}(\lambda - \bar{\lambda})\|v\|^2 - \tfrac{1}{2}(\lambda - \bar{\lambda} - \delta_2)\|w\|^2 + \tfrac{1}{2}a\|v\|^{k+1},$$

from which it follows that $\phi(u) \geqq \text{const}\,(\lambda - \bar{\lambda})^{(k+1)/(k-1)}$. Suppose now that $\|w\| > C\|v\|$. Then we choose $\delta_5 > 0$ so that when $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_5)$ there holds

$$-\tfrac{1}{2}(\lambda - \bar{\lambda})\|v\|^2 - \tfrac{1}{2}(\lambda - \bar{\lambda} - \delta_2)\|w\|^2 \geqq \tfrac{1}{4}\delta_2\|w\|^2.$$

Also, the latter three terms on the right hand side of (4.17) are bounded below by an expression of the form $-C_2\|w\|^{k+1}$, hence

$$\phi(u) \geqq \delta_2\|w\|^2 - C_2\|w\|^{k+1}. \tag{4.18}$$

The right hand side of (4.18) is non-negative for $\|w\| \leqq (\delta_2/(4C_2))^{1/(k-1)}$, hence if $r_1 \leqq (\delta_2/(4C_2))^{1/(k-1)}$ and $\|u\| \leqq r_1$ then again $\phi(u) \geqq \text{const}\,(\lambda - \bar{\lambda})^{(k+1)/(k-1)}$, hence

$$\sup\{\phi(u) | u \in A\} \geqq \text{const}(\lambda - \bar{\lambda})^{(k+1)/(k-1)},$$

and the lower bound in (4.7) follows. This completes the proof.

LEMMA 4.3. *Assume that* (A), (B), (C) *hold, and let* $e > 0$. *Then there exist positive constants* δ_6, r_2 *such that if* $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_6)$ *and* $0 < r < \min(r_2, e(\lambda - \bar{\lambda})^{1/(k-1)})$, *then there exists* $h \in (0, 1)$, *independent of* j, *such that all* ϕ_j *satisfy condition* (I) *with parameters* $-(1/12)(\lambda - \bar{\lambda})r^2, 0, r, h$.

PROOF. In view of the fact that ϕ_j is the restriction of ϕ to H_j, it suffices to show that there exist positive constants δ_6, r_2 such that if $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_6)$ and

$$0 < r < \min(r_2, e(\lambda - \bar{\lambda})^{1/(k-1)}), \tag{4.19}$$

there exists $h \in (0, 1)$ such that if $\|u\| = r$ and

$$-\frac{1}{12}(\lambda-\bar{\lambda})r^2 \leqq \phi(u) \leqq 0, \tag{4.20}$$

then

$$(\phi'(u), u) > -h\|\phi'(u)\|r. \tag{4.21}$$

Throughout, we assume that $\|u\| = r$ and that (4.20) holds. We denote by "const" any positive constant depending only on L, T, V. We assume that r_2 is sufficiently small so that $B_{r_2} \subset D$ the domain of V and hence that of ϕ.

We proceed by considering three cases such that always the hypotheses of at least one case hold.

As the first case we assume that

$$q(u) \leqq -\tfrac{1}{3}(\lambda-\bar{\lambda})r^2, \tag{4.22}$$

where q is defined by (4.8). Suppose, at first, that $V = 0$ and $\rho = 0$. Then we have from (4.8), (4.20), (4.22) that

$$t(u) \geqq -\tfrac{3}{4}q(u). \tag{4.23}$$

Hence, using the homogeneity of q and t, (4.22) and (4.23) imply that

$$(\phi'(u), u) \geqq -q(u) \geqq \tfrac{1}{3}(\lambda-\bar{\lambda})r^2 > 0. \tag{4.24}$$

Thus (4.21) is satisfied in this case for any $h \in (0, 1)$. If we do not assume that $V=0$ and $\rho=0$, then in view of the fact that V and ρ are smaller order than T and t, the same conclusion holds if r_2 is sufficiently small.

As the second case, writing u in the form $u_1 + u_2 + u_3$, where $u_1 \in N$, $u_2 \in F$, $u_3 \in G$, we assume that

$$\|u_1\| \leqq \frac{\sqrt{3}}{2}r\,. \tag{4.25}$$

Condition (4.21) is implied if there exists $g \in (0, 1)$ and $v \neq 0$ such that

$$(v, u) = 0, \quad (\phi'(u), v) > g\|\phi'(u)\|\,\|v\|. \tag{4.26}$$

If r_2 is sufficiently small, then by hypotheses (B) and (C) and by (4.19) we have that $|t(u) + \rho(u)| \leqq \text{const}(\lambda - \bar{\lambda})r^2$ and hence, by (4.20), that

$$|q(u)| \leqq \text{const}(\lambda - \bar{\lambda})r^2. \tag{4.27}$$

Using (A), we have that

$$q(u_1) = -\tfrac{1}{2}(\lambda - \bar{\lambda})\|u_1\|^2. \tag{4.28}$$

Hence, by (4.8) and (4.27),

$$|q(u_2) + q(u_3)| \leqq \text{const}(\lambda - \bar{\lambda})r^2. \tag{4.29}$$

(4.25) implies that

$$\|u_2\|^2 + \|u_3\|^2 \geqq \tfrac{1}{4}r^2. \tag{4.30}$$

From (3.1) and (3.2),

$$q(u_2) \leqq -\tfrac{1}{2}(\lambda - \bar{\lambda} + \delta_1)\|u_2\|^2, \tag{4.31}$$

$$q(u_3) \geqq -\tfrac{1}{2}(\lambda - \bar{\lambda} - \delta_2)\|u_3\|^2. \tag{4.32}$$

It follows from (4.8), (4.28), (4.30), (4.31), (4.32) that if $\delta_6 > 0$ is chosen sufficiently small and $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_6)$, then

$$q(u_2) \leqq -\tfrac{1}{8}\min(\delta_1, \delta_2)r^2, \qquad q(u_3) \geqq \tfrac{1}{8}\min(\delta_1, \delta_2)r^2.$$

Hence, using the continuity of q,

$$\|u_2\| \geqq \text{const } r, \ \|u_3\| \geqq \text{const } r. \tag{4.33}$$

Now we let $v = u_2 - (\|u_2\|^2/\|u_3\|^2)u_3$. Then $(v, u) = 0$ and, by (A),

$$(\phi'(u), v) = 2q(u_2) - 2(\|u_2\|^2/\|u_3\|^2)q(u_3) + (T(u) + V(u), v). \tag{4.34}$$

It follows from (4.30–4.33) that if δ_6 is chosen sufficiently small, then $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_6)$ implies that

$$2q(u_2) - 2(\|u_2\|^2/\|u_3\|^2)q(u_3) \leqq -\text{const } r^2. \tag{4.35}$$

From (4.33),

$$\|v\| \leqq \text{const } r, \tag{4.36}$$

and, using the continuity and homogeneity properties of L and T, and hypotheses (C) for V, we have that, for r_2 sufficiently small,

$$\|\phi'(u)\| \leqq \text{const } r. \tag{4.37}$$

Also, by (B), (C), and (4.34), (4.35), we have that for r_2 sufficiently small, $|(\phi'(u), v)| \geqq \text{const } r^2$, hence, by (4.36) and (4.37), $|(\phi'(u), v)| \geqq \text{const } \|\phi'(u)\| \, \|v\|$. Thus we have established conditions of the form (4.26), from which an inequality of the form (4.21) follows.

The third and last remaining case is that where

$$\|u_1\| \geqq \frac{\sqrt{3}}{2} r, \tag{4.38}$$

$$q(u) \geqq -\tfrac{1}{3}(\lambda - \bar{\lambda}) r^2. \tag{4.39}$$

By (4.9), (4.28), (4.38), and (4.39),

$$q(u_2) + q(u_3) \geqq \tfrac{1}{24}(\lambda - \bar{\lambda}) r^2. \tag{4.40}$$

By (4.40) and the continuity of q,

$$\|u_2\| + \|u_3\| \geqq \operatorname{const}(\lambda - \bar{\lambda})^{1/2} r. \tag{4.41}$$

From (4.41) it follows that either

$$\|u_2\| \geqq \tfrac{1}{2} \operatorname{const} (\lambda - \bar{\lambda})^{1/2} r \tag{4.42}$$

holds or the corresponding inequality with u_3 replacing u_2 holds. Let us assume that (4.42) holds. Letting $w = u_1 - (\|u_1\|^2/\|u_2\|^2) u_2$, we have $(w, u) = 0$ and

$$(\phi'(u), w) = \bar{\lambda}\|u_1\|^2 - (\|u_1\|^2/\|u_2\|^2) 2\, q(u_2) + (T(u) + V(u), w). \tag{4.43}$$

Using (3.1) and (4.38), if δ_6 is sufficiently small, then

$$\bar{\lambda}\|u_1\|^2 - (\|u_1\|^2/\|u_2\|^2) 2\, q(u_2) \geqq \tfrac{1}{2} \delta_1 r^2. \tag{4.44}$$

Using (B) and (C), if r_2 is sufficiently small, then

$$\|T(u) + V(u)\| \leqq \operatorname{const} r^k. \tag{4.45}$$

By (4.38) and (4.42),

$$\|w\| \leqq \operatorname{const}(\lambda - \bar{\lambda})^{-1/2} r^2. \tag{4.46}$$

From (4.19), (4.45), and (4.46) follows

$$|(T(u) + V(u), w)| \leqq \operatorname{const} \epsilon^{k-1}(\lambda - \bar{\lambda})^{1/2} r^2. \tag{4.47}$$

From (4.43), (4.44), and (4.47) it follows that if δ_6 is sufficiently small, then

$$(\phi'(u), w) \geqq \operatorname{const} r^2. \tag{4.48}$$

As before, we may estimate $\|\phi'(u)\|$ and we find that

$$\|\phi'(u)\| \leqq \operatorname{const} r \tag{4.49}$$

for r_2 sufficiently small. Combining (4.46), (4.48), and (4.49),

$$|(\phi'(u), w)| \geqq \operatorname{const} (\lambda - \bar{\lambda})^{1/2} \|\phi'(u)\| \, \|w\|.$$

Hence we have established conditions of the form (4.26) and consequently also an inequality of the form (4.21).

We have demonstrated that an inequality of the form (4.21) holds in an exhaustive set of cases, and this establishes the lemma.

LEMMA 4.4. *Assume that* (A), (B), (C) *hold. Then there exist constants* $\delta_7, e_0 > 0$ *such that if* $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_7)$ *and* $r = e_0(\lambda - \bar{\lambda})^{1/(k-1)}$ *then there exist* $L \leqq U < 0$ *such that*

$$L \leqq b_{i,j,r} < U \tag{4.50}$$

for $i = 1, 2, \cdots, n - \alpha$, $j = 1, 2, 3, \cdots$; *and if* p, q *are integers such that* $1 \leqq p \leqq p + q - 1 \leqq n - \alpha$ *and* $b_{p,j,r} - b_{p+q-1,j,r} = \epsilon$, *then there exist* q *distinct non-zero pairs* $\pm u_i$, $i = 1, 2, \cdots, q$, *and there exists* $h \in (0, 1)$ *such that*

$$b_{p+q-1,j,r} < \phi_j(u_i) < b_{p,j,r}, \tag{4.51}$$

$$\|\phi_j{}'(u_i)\| \leqq \max(\sqrt{\epsilon/(1 - h^2)}, 2\epsilon/((1 - h^2)\bar{r})), \tag{4.52}$$

$$\bar{r} \leqq \|u_i\| \leqq r, \tag{4.53}$$

$$\|u_i - u_\ell\| \geqq r, \ \ell \neq i, \tag{4.54}$$

for any $\bar{r}$ *such that* $\phi(u) > U$ *for all* $u \in B_{\bar{r}}$.

PROOF. Let us take K_1, K_2 to be as in Lemma 4.2, take $e_0 < (12K_1)^{1/2}$, and take $\delta_7 < \min(\delta_5, (r_1/e_0)^{k-1})$, where δ_5, r_1 are as in Lemma 4.2. Then the hypotheses of Lemma 4.2 are satisfied and hence (4.50) holds if we take

$$L = -K_1(\lambda - \bar{\lambda})^{(k+1)/(k-1)}, \ U = -K_2(\lambda - \bar{\lambda})^{(k+1)/(k-1)}.$$

Let us further restrict δ_7 to be less than $\min(\delta_6, (r_2/e_0)^{k-1})$, where δ_6, r_2 are as in Lemma 4.3. Then the hypotheses of Lemma 4.3 are satisfied, hence there exists $h \in (0, 1)$, independent of j, such that every ϕ_j satisfies condition (I) with constants L, U, r, h where we are using the hypothesis that $e_0 < (12K_1)^{1/2}$.

Hypotheses (A), (B), (C) imply ϕ' is Lipschitz continuous, and it follows that each $\phi_j{}'$ is Lipschitz continuous. The conclusion of Lemma 4.4 now follows as an application of Lemma 2.6.

We reduce (1.1) to a pair of equations, as in the method of Lyapunov-Schmidt (See [**6**, **10**]). Let P be the orthogonal projection onto N, as before, and let Q be the orthogonal projection onto $N^\perp = F \oplus G$. Then (1.1) is equivalent to the system

$$(4.55)\qquad (\lambda-\bar{\lambda})v = P(T(v+w) + V(v+w)),$$

$$(4.56)\qquad \lambda w = Lw + Q(T(v+w) + V(v+ w)),$$

where $v = Pu$, $w = Qu$.

We consider the following equation in H_j, which is similar to (1.1):

$$(4.57)\qquad \lambda u = \Pi_j(Lu + T(u) + V(u)) + s,$$

where Π_j is the orthogonal projection onto H_j and $u, s \in H_j$. We note that (4.57) is equivalent to $\phi_j{}'(u) = s$. Again introducing the Lyapunov-Schmidt reduction, we let Q_j be the orthogonal projection onto $F_j \oplus G_j$, and then (4.57) is equivalent to the system

$$(4.58)\qquad (\lambda-\bar{\lambda})v = P(T(v+w_j) + V(v+w_j) + s),$$

$$(4.59)\qquad \lambda w_j = Q_j(Lw_j + T(v + w_j) + V(v + w_j) + s),$$

where $v = Pu$, $w_j = Q_j u$.

We may consider v and s fixed in (4.56) and (4.59) and seek to solve for w and w_j.

In the following lemma we use the notation $B(r)$ in place of B_r.

Lemma 4.5. *Assume that* (A), (B), (C) *hold, and let* $\delta \in (0, \delta_1)$. *Then there exist* $r_3, r_4 > 0$ *such that for* $(\lambda, v) \in \mathfrak{D} \equiv [\bar{\lambda}, \bar{\lambda} + \delta] \times (B(r_3) \cap N)$, (4.56) *has a unique solution* $w = \psi(\lambda, v)$; *and for*

$$(\lambda, v, s) \in \mathfrak{D}_j \equiv [\bar{\lambda}, \bar{\lambda} + \delta] \times (B(r_3) \cap N) \times (B(r_4) \cap H_j),$$

(4.59) *has a unique solution* $w_j = \psi_j(\lambda, v, s)$. ψ *is uniformly continuous, and* $\{\psi_j\}$ *is a uniformly equicontinuous family. If* $s_j \in B(r_4) \cap H_j$ *is such that* $s_j \to 0$, *then* $\psi_j(\lambda, v, s) \to \psi(\lambda, v)$ *uniformly.*

Proof. For $\lambda \in [\bar{\lambda}, \bar{\lambda} + \delta_1]$ the restriction of $L-\lambda I$ to $N^{\perp}$ has a bounded inverse $K = K(\lambda)$, so that $K(L-\lambda I)w = w, w \in N^{\perp}$, and $\|K\| \leqq \max(|\lambda - \bar{\lambda} + \delta_1|^{-1}, |\lambda - \bar{\lambda} - \delta_2|^{-1})$, where $\|\cdot\|$ denotes the norm on operators from $N^{\perp}$ to $N^{\perp}$. In terms of K, (4.56) may be written as

$$(4.60)\qquad w = -KQ(T(v+w) + V(v+w)).$$

Defining $J \equiv -KQ(T + V)$, (4.60) may be written as

$$(4.61)\qquad w = J(v+w)$$

For each $\delta \in (0, \delta_1)$, $\|K\|$ is bounded uniformly for $\lambda \in [\bar{\lambda}, \bar{\lambda} + \delta]$ Using (3.3) and (3.4) we may choose r_3 so that for $\lambda \in [\bar{\lambda}, \bar{\lambda} + \delta]$ and $u_1, u_2 \in B(2r_3)$

$$(4.62)\qquad \|J(u_1) - J(u_2)\| \leqq \tfrac{1}{2}\|u_1 - u_2\|.$$

In an analogous manner we may construct a linear operator K_j, defined on $F_j \oplus G_j$, so that $K_j(L - \lambda I)w = w$, $w \in F_j \oplus G_j$, and $\|K_j\| \leqq \max(|\lambda - \bar{\lambda} + \delta_1|^{-1}, |\lambda - \bar{\lambda} - \delta_2|^{-1})$, and (4.59) may be written in the form

$$w = J_j(v + w, s), \tag{4.63}$$

where $J_j(u, s) = -K_j Q_j(T(u) + V(u) + s)$. By choosing r_3 smaller, if necessary, we may arrange that for all j and s, and for all $\lambda \in [\bar{\lambda}, \bar{\lambda} + \delta]$. and $u_1, u_2 \in B(2r_3)$

$$\|J_j(u_1, s) - J_j(u_2, s)\| \leqq \tfrac{1}{2}\|u_1 - u_2\|. \tag{4.64}$$

Equations (4.61) and (4.63) may be solved by the well known method of iterations. Let

$$w^0 = J(v),\ w^{i+1} = J(v + w^i),\ i = 0, 1, 2, \cdots, \tag{4.65}$$

$$w_j^0 = J_j(v, s),\ w_j^{i+1} = J_j(v + w_j^i, s),\ i = 0, 1, 2, \cdots. \tag{4.66}$$

Inequality (4.62) implies that for $u \in B(2r_3)$, $J(u) \in B(r_3)$. Hence, if $v \in B(r_3) \cap N$, then by induction $w^i \in B(r_3)$. The uniform convergence of w^i to a solution w of (4.61) for $(\lambda, v) \in \mathfrak{D} \equiv [\bar{\lambda}, \bar{\lambda} + \delta] \times (B(r_3) \cap N)$ follows easily, and it is easily shown that w is a uniformly continuous function $\psi(\lambda, v)$ on $\mathfrak{D}$. Uniqueness of a solution of (4.61) for $(\lambda, v) \in \mathfrak{D}$ follows from (4.62). Similarly, for r_4 sufficiently small we have that $s \in B(r_4) \cap H_j$ implies that $K_j Q_j s \in B(r_4/4)$, hence $(v, s) \in (B(r_3) \cap N) \times (B(r_4) \cap H_j)$ implies, using induction, that $w_j^i \in B(r_3)$ for all i, j. We find easily that w_j^i converges uniformly, as $i \to \infty$, to a solution w_j of (4.63) for $(\lambda, v, s) \in \mathfrak{D}_j \equiv [\lambda, \lambda + \delta] \times (B(r_3) \cap N) \times (B(r_4) \cap H_j)$, which is unique, and that the family of functions $(\lambda, v, s) \to \psi_j(\lambda, v, s) = w_j$ is uniformly equicontinuous.

To prove the remaining assertion of the theorem, it clearly suffices to consider only the special case where all $s_j = 0$ in view of the uniform equicontinuity of the ψ_j. Thus we consider (λ, v) as fixed and let w be the solution of (4.61), w_j the solution of (4.63) with $s = 0$, and let w_j^i be defined by (4.66) with $s = 0$. We must show that $w_j \to w$ uniformly for $(\lambda, v) \in \mathfrak{D}$. We have seen that $w^i \to w$ uniformly for $(\lambda, v) \in \mathfrak{D}$ and that $w_j^i \to w_j$ uniformly for $(\lambda, v, j) \in \mathfrak{D} \times \{1, 2, 3, \cdots\}$, hence it will suffice to show that $w_j^i \to w^i$ for each i uniformly for $(\lambda, v) \in \mathfrak{D}$.

The latter assertion will be shown by induction on i. For this purpose, we introduce the sets

$$S_i = \{w_j^i | (\lambda, v) \in \mathfrak{D}, j = 1, 2, 3, \cdots\},\ T_i = \{w^i \mid (\lambda, v) \in \mathfrak{D}\}, \tag{4.67}$$

where it is understood that w^i and w_j^i are the functions of (λ, v) defined by (4.65) and (4.66) with $s = 0$. We have seen that J and J_j are continuous on $\mathfrak{D}$, which is compact, and it follows from their definitions that $J_j(u) \to J(u)$, as $j \to \infty$, for $u \in \mathfrak{D}$, hence $w_j^0 \to w^0$ uniformly for $(\lambda, v) \in \mathfrak{D}$. Also S_0 and T_0 are compact. Suppose now that for $i = k$ it is true that $w_j^i \to w^i$ uniformly for $(\lambda, v) \in \mathfrak{D}$ and that S_i and T_i are compact. Using (4.65) and (4.66) we may write

(4.68)
$$w_j^{i+1} - w^{i+1} = [J_j(v+w_j^i, 0) - J(v+w_j^i)] + [J(v+w_j^i) - J(v + w^i)] .$$

The argument $v+w_j^i$ lies in the compact set $(B(r_3) \cap N) + S_i$, hence by the continuity of J_j and the convergence of J_j to J, we have that the first term on the right hand side of (4.68) converges to zero uniformly for $(\lambda, v) \in \mathfrak{D}$. Similarly for the second term on the right hand side of (4.68), hence $w_j^{i+1} \to w^{i+1}$ uniformly for $(\lambda, v) \in \mathfrak{D}$. The compactness of S_{i+1} and T_{i+1} follows easily from the compactness of S_i and T_i. This completes the induction step. Hence $w_j^i \to w^i$ uniformly for $(\lambda, v) \in \mathfrak{D}$, for all i, and with this assertion we have completed the proof of the lemma.

LEMMA 4.6. *Under hypotheses* (A), (B), (C) *there exist* $\delta_3, \delta_4 > 0$ *such that for* $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_3)$ (*resp.* $\lambda \in (\bar{\lambda} - \delta_4, \bar{\lambda})$) *equation* (1.1) *has at least* $n - \alpha$ (*resp.* $n - \beta$) *distinct non-zero pairs* $\pm u$ *of solutions in* B_r, *where* $r = e_0(\lambda - \bar{\lambda})^{1/(k-1)}$ *with* e_0 *as Lemma* 4.4.

PROOF. Let us consider the case where $\lambda > \bar{\lambda}$. We choose δ_3 positive and less than $\min(\delta, \delta_7, (r_3/e_0)^{k-1})$, where δ_7 and e_0 are as in Lemma 4.4 and δ and r_3 are as in Lemma 4.5. Letting $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_3)$ and $r = e_0(\lambda - \bar{\lambda})^{1/(k-1)}$, by Lemma 4.4 there exist $L \leqq U < 0$ such that $L \leqq b_{i,j,r} \leqq U$ for $i = 1, 2, \cdots, n - \alpha$, $j = 1, 2, 3, \cdots$. Hence, for each $i = 1, 2, \cdots, n - \alpha$, a limit point ξ_i of $b_{i,j,r}$ as a sequence in j exists, and since $b_{i,j,r}$ is decreasing in i, it follows that $L \leqq \xi_{n-\alpha} \leqq \cdots \leqq \xi_2 \leqq \xi_1 \leqq U$.

Suppose that some of the values ξ_i coincide. For example, suppose that $\xi_1 = \cdots = \xi_p = \xi$. By passing to subsequences, if necessary, we may assume that $b_{i,j,r} \to \xi$ as $j \to \infty$, for $i = 1, 2, \cdots, p$. By Lemma 4.4, for each j there exist p distinct pairs $\pm u_{i,j}$, $i = 1, 2, \cdots, p$ and there exists $\bar{r} \in (0, r)$ independent of j, such that $\phi_j(u_{i,j}) \to \xi$ as $j \to \infty$, $\phi_j'(u_{i,j}) \to 0$ as $j \to \infty$, $\bar{r} \leqq \|u_{i,j}\| \leqq r$, and $\|u_{i,j} - u_{\ell,j}\| \geqq \bar{r}, \ell \neq i$. We let $u_{i,j} = v_{i,j} + w_{i,j}$, where $v_{ij} \in N$, $w_{i,j} \in F_j \oplus G_j$, and let $\phi_j'(u_{i,j}) = s_{i,j}$.

By the discussion preceding (4.58) and (4.59) the equation $\phi_j'(u_{i,j}) = s_{i,j}$ is equivalent to (4.58) and (4.59) with v, w_j, s replaced by $v_{i,j}$, $w_{i,j}$, $s_{i,j}$ respectively. We have $\|v_{i,j}\| \leqq r$, and since N is finite dimensional, each sequence $v_{i,j}$ in j has a convergent subsequence. By passing to a subsequence, we may assume that $v_{i,j}$ converges to a point v_i as $j \to \infty$. Since $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta_3)$, and by the definition of r and the restriction that $\delta_3 < \delta$, we have that $v_{i,j} \in B(r_3) \cap N$, where r_3 is as in Lemma 4.5, and since $s_{i,j} \to 0$ as $j \to \infty$, we have $\|s_{i,j}\| \leqq r_4$ for j sufficiently large, where r_4 is as in Lemma 4.5. Hence by Lemma 4.5 it must be the case that $w_{i,j} = \psi_j(\lambda, v_{i,j}, s_{i,j})$. By the uniform equicontinuity property of $\{\psi_j\}$ and by the last statement of Lemma 4.5 we have $\psi_j(\lambda, v_{i,j}, s_{i,j}) \to \psi(\lambda, v_i)$ as $j \to \infty$. If we set $w_i = \psi(\lambda, v_i)$, then $w_{i,j} \to w_i$ and by the definition of ψ, $v = v_i, w = w_i$ is a solution of (4.56). Also, since $v = v_{i,j}, w_j = w_{i,j}, s = s_{i,j}$ is a solution of (4.58), it follows by passing to the limit as $j \to \infty$ that $v = v_i, w = w_i$ is a solution of (4.55). Hence $u_i = v_i + w_i$ is a solution of (1.1).

Since $\|u_{i,j} - u_{\ell,j}\| \geqq \bar{r}, i \neq \ell$, and $u_{i,j} \to u_i$, the $u_i, i = 1, 2, \cdots, p$, are distinct. It is clear that $\phi(u_i) = \xi_i$, $i = 1, 2, \cdots, n - \alpha$. Hence, if a value ξ occurs p times in the sequence $\xi_1, \xi_2, \cdots, \xi_{n-\alpha}$, then there are at least p pairs $\pm u$ such that $\phi(u) = \xi$ and $\phi'(u) = 0$. Therefore, there are at least $n - \alpha$ pairs $\pm u$ such that $\phi'(u) = 0$. Moreoever, these pairs lie in B_r, where r has the value previously stated.

It is easy to see that the result stated in the lemma for the case $\lambda < \bar{\lambda}$ can be obtained by dealing with the functional $-\phi$ in place of ϕ.

This completes the proof of the lemma.

Lemma 4.7. *Let hypotheses* (A), (B), (C) *hold, let a be a positive constant, and let u be a solution of* (1.1) *such that*

$$\|u\| \leqq a|\lambda - \bar{\lambda}|^{1/(k-1)}. \tag{4.69}$$

Then u is of the form

$$|\lambda - \bar{\lambda}|^{1/(k-1)}u_1 + o(|\lambda - \bar{\lambda}|^{1/(k-1)}) \text{ as } \lambda \to \bar{\lambda}, \tag{4.70}$$

where u_1 satisfies

$$u_1 = \operatorname{sgn}(\lambda - \bar{\lambda})PTu_1. \tag{4.71}$$

Proof. We let $v = Pu$, $w = Qu$, and use the decomposition of (4.55), (4.56). Solving (4.56) for w in terms of u and using (A), (B), and (C), we have that for a sufficiently small positive constant δ_0, if $|\lambda - \bar{\lambda}| < \delta_0$, then

$$\|w\| \leqq \text{const } \|u\|^k. \tag{4.72}$$

We let

$$v = |\lambda - \bar{\lambda}|^{1/(k-1)}\bar{v}, w = |\lambda - \bar{\lambda}|^{1/(k-1)}\bar{w}, \bar{u} = \bar{v} + \bar{w}. \tag{4.73}$$

Then (4.55) is equivalent to

$$\bar{v} = \operatorname{sgn}(\lambda - \bar{\lambda})PT\bar{u} + r, \tag{4.74}$$

where

$$r = \operatorname{sgn}(\lambda-\bar{\lambda})|\lambda-\bar{\lambda}|^{-k/(k-1)}PV(|\lambda-\lambda|^{1/(k-1)}\bar{u}).$$

Hence, from (4.69), (4.73), and (C) it follows that

$$\|r\| = o(|\lambda - \bar{\lambda}|) \text{ as } \lambda \to \bar{\lambda}. \tag{4.75}$$

From (4.72), (4.73), the Lipschitz continuity of T, and (4.75), it follows that $\bar{v}$ satisfies

$$\bar{v} = \operatorname{sgn}(\lambda-\bar{\lambda})PT\bar{v} + o(|\lambda-\bar{\lambda}| \text{ as } \lambda \to \bar{\lambda}. \tag{4.76}$$

An easy consequence of hypotheses (B) is that the set of solutions of (4.71) is compact. Denoting this set by $\mathcal{S}$, it is easily seen that if $\bar{v}$ satisfies (4.76) then $\operatorname{dist}(\bar{v}, \mathcal{S}) \leqq \sigma(|\lambda-\bar{\lambda}|)$, where $\sigma(s) \to 0$ as $s \to 0$. Hence, using (4.69), (4.72), and (4.73), it follows that u is of the form (4.70).

Theorem 3.1 now follows immediately from Lemmas 4.6 and 4.7.

Corollary 3.3 is an immediate consequence of Lemma 4.1, since the proof only requires that we show that $\alpha + \beta \leqq n$, and this is obvious by Lemma 4.1.

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