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DOMINANCE OF N-TH ORDER LINEAR EQUATIONS

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ABSTRACT. Consider the nth order linear equation

$$(1) \qquad y^{(n)} + \sum_{k=1}^{n} p_k(x) y^{(n-k)} = 0, \text{ where } p_k(x) \in c[a, \infty) \quad \text{for } 1 \leqq k \leqq n.$$

Introducing a new concept called dominance, the authors compare the asymptotic properties of the set of oscillatory solutions with the set of nonoscillatory solutions for the equation (1) when dominance occurs. These results also give information about the number of linearly independent oscillatory or nonoscillatory solutions of (1). The third order equation is given concentrated attention.

Consider the nth order linear equation

$$(1)_n \; y^{(n)} + \sum_{k=1}^{n} p_k(x) y^{(n-k)} = 0 \text{ where } p_k(x) \in c[a, \infty) \quad \text{for } 1 \leqq k \leqq n.$$

A nontrivial solution of equation $(1)_n$ is said to be oscillatory on $[a, \infty)$ if it has infinitely many zeros on $[a, \infty)$; otherwise, it is said to be nonoscillatory.

Many of the known results for nth order oscillation theory are catalogued in Swanson [1]. Also a good discussion for 3rd order equations can be found in Barrett [2].

It is the intent of this paper to compare the asymptotic properties of oscillatory and nonoscillatory solutions of equation $(1)_n$ by means of the concept of dominance. Thus if information about the asymptotic behavior of all nonoscillatory solutions is known for certain equations then asymptotic behavior of all oscillatory solutions can be determined and visa versa. This approach also yields information about the number of linearly independent nonoscillatory and linearly independent oscillatory solutions of equation $(1)_n$.

We conclude with an examination of third order equations, $(1)_3$, as examples of types of dominance.

1. **Dominance.** Let $\mathcal{S}$ denote the linear space of all solutions of $(1)_n$, $\mathcal{N}$ the subset of $\mathcal{S}$ of nonoscillatory solutions and $\mathcal{O}$ the subset of $\mathcal{S}$ of oscillatory solution. Let $\mathcal{N}^+$ be the subset of $\mathcal{N}$ of solutions

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which are eventually positive and $\mathcal{N}^- = \mathcal{N} - \mathcal{N}^+$.

DEFINITION 1. If $z \in \mathcal{N}$ and $y \in \mathcal{O}$ then we say that z dominates y at $c \in R$, $z > y$ at c, if there is a neighborhood, U, of c such that $z + \lambda y \in \mathcal{N}$ for all $\lambda \in U$. Similarly, we say that y dominates z at $c \in R$, $y > z$ at c, if there is a neighborhood, U, of c such that $y + \lambda z \in \mathcal{O}$ for all $\lambda \in U$. c may be $\pm\infty$ in which case U is an appropriate half ray.

DEFINITION 2. If $\mathcal{P} \subseteq \mathcal{N}$ and $\mathcal{T} \subseteq \mathcal{O}$ then we say that $\mathcal{P}$ dominates $\mathcal{T}$ at $c \in R$, $\mathcal{P} > \mathcal{T}$ at c, if $z > y$ at c for all $z \in \mathcal{P}$ and all $y \in \mathcal{T}$. Similarly we say that $\mathcal{T}$ dominates $\mathcal{P}$ at $c \in R$, $\mathcal{T} > \mathcal{P}$ at c, if $y > z$ at c for all $y \in \mathcal{T}$ and $z \in \mathcal{P}$. For notational purposes we agree that $\mathcal{P} > \emptyset$ and $\mathcal{T} > \emptyset$ at ∞.

Of special interest is the case when $\mathcal{P} = \mathcal{N}$ and $\mathcal{T} = \mathcal{O}$. The following theorems indicate that $c = 0$ or ∞ are the important cases.

THEOREM 1. *If $c \neq 0$ is an extended real number and if $\mathcal{O} > \mathcal{N}\{\mathcal{N} > \mathcal{O}\}$ at c, then $\mathcal{O} > \mathcal{N}\{\mathcal{N} > \mathcal{O}\}$ at k for all $-\infty \leqq k \leqq \infty$.*

PROOF. We show the theorem only for the case that $\mathcal{O} > \mathcal{N}$ at c.

Suppose $c \neq 0$ is finite and $k \neq 0$, $k \in (-\infty, \infty)$. Let $y \in \mathcal{O}$ and $z \in \mathcal{N}$, then $y + \lambda z = (k/c)[(cy/k) + (c\lambda/k)z]$; $cy/k \in \mathcal{O}$, $z \in \mathcal{N}$ and $\mathcal{O} > \mathcal{N}$ at c implies there exists an $\epsilon > 0$ such that $c - \epsilon < c\lambda/k < c + \epsilon$ implies that $y + \lambda z \in \mathcal{O}$. But this describes a λ neighborhood of k. Since we can do this for any $k \neq 0$. We have $\mathcal{O} > \mathcal{N}$ at $\pm\infty$ as well. Also for $k = 0$, $y + kz = y \in \mathcal{O}$ hence we have $\mathcal{O} > \mathcal{N}$ at k for all $-\infty \leqq k \leqq \infty$.

Suppose $c = +\infty$ then for $y \in \mathcal{O}$ and $z \in \mathcal{N}$ there is an $M > 0$ such that $y + \lambda z \in \mathcal{O}$ for $\lambda > M$. But this implies that $\mathcal{O} > \mathcal{N}$ at $2M$ and consequently $\mathcal{O} > \mathcal{N}$ at k for $-\infty \leqq k \leqq \infty$ by the previous argument. The case, $c = -\infty$, can be handled in a similar manner.

As a consequence of this theorem, if $\mathcal{O} > \mathcal{N}$ $\{\mathcal{N} > \mathcal{O}\}$ at $c \neq 0$ we say $\mathcal{O} > \mathcal{N}\{\mathcal{N} > \mathcal{O}\}$ at ∞. The following is a list of all possibilities for dominance between sets $\mathcal{O}$ and $\mathcal{N}$.

1. $\mathcal{O} > \mathcal{N}$ at 0 but not at ∞
2. $\mathcal{O} > \mathcal{N}$ at ∞
3. $\mathcal{O} \not> \mathcal{N}$ at 0
4. $\mathcal{N} > \mathcal{O}$ at 0 but not at ∞
5. $\mathcal{N} > \mathcal{O}$ at ∞
6. $\mathcal{N} \not> \mathcal{O}$ at 0.

Cases 1, 2, 3, 4, 5 are mutually exclusive. This follows from Theorems 4 and 5. Examples, later will be given for each of these possibilities.

If $T \subseteq \mathcal{S}$ then $Sp\ T$ is the vector space span of T in $\mathcal{S}$.

THEOREM 2. *If* $\mathcal{O} > \mathcal{N}(\mathcal{N} > \mathcal{O})$ *at* 0 *then* $Sp\ \mathcal{O} = \mathcal{S}(Sp\ \mathcal{N} = \mathcal{S})$.

PROOF. We will argue the case that $\mathcal{O} > \mathcal{N}$ at 0 only. If $\mathcal{N} = \emptyset$ then the result is obvious. Suppose $\mathcal{N} \neq \emptyset$ and suppose $\{y_1, \cdots, y_q, z_{q+1}, \cdots, z_n\}$ is a basis for $\mathcal{S}$ such that $\{y_1, \cdots, y_q\} \subset \mathcal{O} \subset Sp\{y_1, \cdots, y_q\}$. Since $\mathcal{O} > \mathcal{N}$ at 0 there exists a $\lambda \neq 0$ sufficiently small such that $y_1 + \lambda z_{q+1} \in \mathcal{O}$. This contradicts the linear independence of the set $\{y_1, \cdots, y_q, z_{q+1}\}$ hence $q = n$.

Next we have a characterization of $\mathcal{O} > \mathcal{N}$ at ∞ and $\mathcal{N} > \mathcal{O}$ at ∞.

THEOREM 3. $\mathcal{O} > \mathcal{N}$ *at* ∞ *if and only if* $\mathcal{O} \neq \emptyset$ *and* $Sp\ \mathcal{N} = \mathcal{N} \cup \{0\}$. $\mathcal{N} > \mathcal{O}$ *at* ∞ *if and only if* $\mathcal{N} \neq \emptyset$ *and* $Sp\ \mathcal{O} = \mathcal{O} \cup \{0\}$.

PROOF. We will argue the first statement only. Of course $\mathcal{N} \subset Sp\ \mathcal{N}$. Suppose $y \in Sp\ \mathcal{N} - \mathcal{N} \cup \{0\}$ then $y \in \mathcal{O}$. Hence y is a linear combination of elements in $\mathcal{N}$. Choose a y with this property such that y possesses the smallest linear combination; that is,

$$y = \sum_{1}^{n} \alpha_i z_i \text{ where } n \text{ is a minimum, and } z_i \in \mathcal{N}.$$

Consequently $\alpha_1 \neq 0$. Let $y_1 = -\alpha_1^{-1} y \in \mathcal{O}$. Then $\mathcal{O} > \mathcal{N}$ at ∞ implies $z_1 + \lambda y_1 \in \mathcal{O}$ for all $\lambda \neq 0$, in particular $z_1 + y_1 \in \mathcal{O}$. But $z_1 + y_1 = z_1 - \alpha_1^{-1} y = \alpha_1^{-1}(\alpha_1 z_1 - y) = \alpha_1^{-1}(\sum_2^n \alpha_i z_i) = \sum_2^n \alpha_1^{-1}\alpha_i z_i \in \mathcal{O}$ and this contradicts the minimality of n. Hence no such y exists and $Sp\ \mathcal{N} = \mathcal{N} \cup \{0\}$.

Conversely, suppose $\mathcal{O} \neq \emptyset$, $Sp\ \mathcal{N} = \mathcal{N} \cup \{0\}$. Let $y \in \mathcal{O}$ and $z \in \mathcal{N}$. If there is a $\lambda \neq 0$ such that $z + \lambda y \in \mathcal{N}$ then $\lambda y = (z + \lambda y) - z$ is in $\mathcal{O}$ and is a linear combination of elements of $\mathcal{N}$. From this impossibility we conclude that $\mathcal{O} > \mathcal{N}$ at ∞.

The next theorems relate dominance to asymptotic properties of solutions of $(1)_n$.

THEOREM 4. (i) $\mathcal{N} > \mathcal{O}$ *at* ∞ *for* $(1)_n$ *if and only if* $\lim_{x\to\infty} y(x)/z(x) = 0$ *for all* $y \in \mathcal{O}$ *and* $z \in \mathcal{N}$.

(ii) $\mathcal{O} > \mathcal{N}$ *at* ∞ *for* $(1)_n$ *if and only if* $\liminf_{x\to\infty} y(x)/z(x) = -\infty$ *and* $\limsup_{x\to\infty} y(x)/z(x) = \infty$ *for all* $y \in \mathcal{O}$ *and* $z \in \mathcal{N}$.

PROOF. (i) If $y_1 \in \mathcal{N}$, $y_2 \in \mathcal{O}$, $\mathcal{N} > \mathcal{O}$ at ∞, then there is an open set U containing ∞ such that $y_1 + \lambda y_2 \in \mathcal{N}$ for each real number

$\lambda \in U$. Assume that $\alpha = \underline{\lim}_{x\to\infty}(y_2/y_1)(x) < 0$.

Let $\mu \in (\alpha, 0)$ and such that $-1/\mu \in U$ then there exists a divergent sequence $\{t_n\}_{n=1}^{\infty} \to \infty$ such that $(y_2/y_1)(t_n) < \mu, n = 1, 2, 3, \cdots$. Hence, $-\mu + (y_2/y_1)(t_n) < 0$. Since $\beta = \overline{\lim}_{x\to\infty}(y_2/y_1)(x) \geqq 0$, there is a divergent sequence $\{\tau_n\} \to \infty$ such that $(y_2/y_1)(\tau_n) > \mu$.

Consequently, $-\mu + (y_2/y_1)(\tau_n) > 0$. Hence, $y_1 - (1/\mu)y_2 \in \mathcal{O}$ which is a contradiction. Consequently $\alpha = 0 = \beta$.

Conversely, suppose for all $y_1 \in \mathcal{N}$, $y_2 \in \mathcal{O}$ that $\lim_{x\to\infty}(y_2/y_1)(x) = 0$. Since, for each real number λ, $y_1 + \lambda y_2 = y_1[1 + \lambda(y_2/y_1)]$ holds eventually, $\lim_{x\to\infty}[1 + \lambda(y_2/y_1)](x) = 1$.

It follows that $y_1 + \lambda y_2 \in \mathcal{N}$, λ real.

Consequently $\mathcal{N} > \mathcal{O}$ at ∞.

(ii) Suppose

$$\mathcal{O} > \mathcal{N} \text{ at } \infty$$

then $y_1 \in \mathcal{O}$ and $y_2 \in \mathcal{N}$ implies that

$y_1 + \lambda y_2 \in \mathcal{O}$ for each real number λ in some neighborhood U of ∞.

Assume y_2 is eventually positive, then for each $\lambda < 0$ for which $\lambda \in U$, there is a divergent sequence of numbers $\{t_n\}_{n=1}^{\infty}$ such that $(y_1 + \lambda y_2)(t_n) \geqq 0$, hence, $(y_1/y_2)(t_n) \geqq -\lambda$, $n = 1, 2, \cdots$. Consequently, $\overline{\lim}_{x\to\infty}(y_1/y_2)(x) = \infty$, since U is a neighborhood of ∞. Similarly, $\underline{\lim}_{x\to\infty}(y_1/y_2)(x) = -\infty$.

The argument for the converse is quite apparent.

THEOREM 5. (i) $\mathcal{N} > \mathcal{O}$ *at* 0 *for* $(1)_n$ *if and only if* $\limsup_{x\to\infty}|y(x)/z(x)| < \infty$ *for all* $z \in \mathcal{N}$ *and* $y \in \mathcal{O}$.

(ii) $\mathcal{O} > \mathcal{N}$ *at* 0 *for* $(1)_n$ *if and only if* $\liminf_{x\to\infty} y(x)/z(x) < 0 < \limsup_{x\to\infty} y(x)/z(x)$ *for all* $y \in \mathcal{O}$ *and* $z \in \mathcal{N}$.

PROOF. Let $y \in \mathcal{O}$ and $z \in \mathcal{N}$ where $\mathcal{N} > \mathcal{O}$ at 0. Then there is a $\lambda_0 > 0$ such that $|\lambda| \leqq \lambda_0$ implies $z + \lambda y \in \mathcal{N}$. In particular $z \pm \lambda_0 y \in \mathcal{N}$ or equivalently $\lambda_0 z[(1/\lambda_0) \pm (y/z)] \in \mathcal{N}$ and hence $(1/\lambda_0) \pm (y/z)$ is eventually positive or negative. Hence $|y(x)/z(x)| \leqq 1/\lambda_0$ for x sufficiently large and $\limsup_{x\to\infty}|y(x)/z(x)| < \infty$. If, conversely, $\limsup_{x\to\infty}|y(x)/z(x)| < \lambda_0 < \infty$ then for $|\lambda| > \lambda_0$, $|\lambda \pm (y(x)/z(x))| \geqq |\lambda| - |y(x)/z(x)| > \lambda_0 - |y(x)/z(x)| > 0$ for x sufficiently large. Hence $z(x) \pm (1/\lambda)y(x) \in \mathcal{N}$ for $|\lambda| > \lambda_0$ which means that $\mathcal{N} > \mathcal{O}$ at 0.

Secondly, suppose $\mathcal{O} > \mathcal{N}$ at 0 and $y \in \mathcal{O}$, $z \in \mathcal{N}$. Then there is a $\lambda_0 > 0$ such that $y \pm \lambda_0 z \in \mathcal{O}$ and consequently $(y/z) \pm \lambda_0$ is oscillatory; that is, $\limsup_{x\to\infty}(y(x)/z(x)) \geqq \lambda_0$ and $\liminf_{x\to\infty}(y(x)/z(x)) \leqq -\lambda_0$ and consequently $\limsup_{x\to\infty} y(x)/z(x) \geqq \lambda_0 > 0 > -\lambda_0 \geqq$

$\liminf_{x\to\infty} y(x)/z(x)$. Conversely, if $\liminf_{x\to\infty} y(x)/z(x) < 0 < \limsup_{x\to\infty} y(x)/z(x)$ then a $\lambda_0 > 0$ can be found such that

$$\liminf_{x\to\infty} \frac{y(x)}{z(x)} < -\lambda_0 < 0 < \lambda_0 < \limsup_{x\to\infty} \frac{y(x)}{z(x)} \quad \text{and}$$

the remainder of the argument is obvious by reversing the above steps with λ_0 replaced by any λ such that $|\lambda| \leqq \lambda_0$.

One observes as a consequence of theorems 4 and 5 that when one form of dominance occurs then if specific information about the asymptotic properties of $\mathcal{N}\{\mathcal{O}\}$ is known then information about $\mathcal{O}\{\mathcal{N}\}$ follows from these theorems.

In the case that p_i are constants in $(1)_n$ one can determine dominance in terms of the roots of the characteristic equation

$$r^n + \sum_{k=1}^{n} p_k r^{n-k} = 0.$$

Let R be the set of real roots and C be the set of complex roots of this equation then the following results can be obtained as a consequence of theorems 4 and 5.

(i) $\mathcal{O} > \mathcal{N}$ *at* ∞ *if and only if* $\max\{r \mid r \in R\} < \min\{\operatorname{Re} c \mid c \in C\}$
(ii) $\mathcal{N} > \mathcal{O}$ *at* ∞ *if and only if* $\min\{r \mid r \in R\} > \max\{\operatorname{Re} c \mid c \in C\}$
(iii) $\mathcal{N} > \mathcal{O}$ *only at* 0 *if and only if* $\min\{r \mid r \in R\} = \max\{\operatorname{Re} c \mid c \in C\} \equiv k$

and the multiplicity of complex roots with real part equal k is 1.

In all other cases there is no dominance.

If the p_k are all constants for $k = 1, 2, \cdots, n$ the adjoint of $(1)_n$ has a characteristic equation of the form $(-r)^n + \sum_{k=1}^{n} p_k(-r)^{n-k} = 0$. Hence if $\mathcal{O}^*$ and $\mathcal{N}^*$ are respectively the oscillatory and nonoscillatory solutions of the adjoint equation then $\mathcal{O} > \mathcal{N}$ at ∞ if and only if $\mathcal{N}^* > \mathcal{O}^*$ at ∞ and $\mathcal{N} > \mathcal{O}$ at ∞ if and only if $\mathcal{O}^* > \mathcal{N}^*$ at ∞.

An open question is whether this result is true for arbitrary coefficients p_k.

Example. The fourth order equation $y^{IV} - y = 0$ satisfies no dominance property for it has a fundamental set of solutions $\{e^x, e^{-x}, \sin x, \cos x\}$ and since

$$\lim_{x\to\infty} \frac{\sin x}{e^x} = 0 \quad \text{and} \liminf_{x\to\infty} \frac{\sin x}{e^{-x}} = -\infty$$

no type of dominance can occur as a result of theorems 4 and 5.

2. **Third Order Linear Equation.** It will be shown that for C_I and C_{II} equations, as defined by Hanan [3], that dominance results can be obtained. To this end we review the definitions of C_I and C_{II}.

Definition. The linear third-order equation, $(1)_3$, is said to be in $C_I[b, \infty)$ $\{C_{II}[b, \infty)\}$ if for each number $c \in (b, \infty)$ and nontrivial solution y of $(1)_3$ satisfying the conditions $y(c) = 0 = y'(c)$ it is true that $y(x) \neq 0$ for $x \in [b, c)$ $\{x \in (c, \infty)\}$.

THEOREM 6. *If* $(1)_3 \in C_1[a, v)$ *and* $\mathcal{O} \neq \emptyset$ *then* $\mathcal{O} > \mathcal{N}$ *at* 0.

PROOF. It is known that $\mathcal{N} \neq \emptyset$ if $(1)_3 \in C_1[a, \infty)$. Let $y \in \mathcal{O}$ and $z \in \mathcal{N}$. If $b > a$ is such that $z(x) > 0$ on $[b, \infty)$ then $\{y(x)/z(x) \mid b \leqq x < \infty\}$ must contain an open neighborhood of 0, say $(-\lambda_0, \lambda_0)$, otherwise $y(x) \geqq 0$ or $y(x) \leqq 0$ which implies $y(x)$ has infinitely many double zeros contrary to $C_I[a, \infty)$. But then for each $\lambda \in (-\lambda_0, \lambda_0)$ there is an $x_0 \in [b, \infty)$ such that $y(x_0)/z(x_0) = \lambda$ or $y(x) - \lambda z(x)$ has a zero for each $\lambda \in (-\lambda_0, \lambda_0)$. By a theorem of Hanan [2, Theorem 3.4] since $\mathcal{O} \neq \emptyset, y - \lambda z \in \mathcal{O}$ for all $\lambda \in (-\lambda_0, \lambda_0)$.

THEOREM 7. *If* $(1)_3 \in C_{II}[a, \infty)$, *then* $\mathcal{N} > \mathcal{O}$ *at* 0.

PROOF. It is known that $\mathcal{N} \neq \emptyset$ if $(1)_3 \in C_{II}[a, \infty)$. If $\mathcal{O} = \emptyset$ then $\mathcal{N} > \mathcal{O}$ at 0 vacuously. Hence let $z \in \mathcal{N}$ and $y \in \mathcal{O}$. Suppose z is eventually positive. Then there are numbers x_1 and x_2, $a < x_1 < x_2$, such that $z(x) > 0$ on $[x_1, x_2]$, $y(x_1) = y(x_2) = 0$, and $y(x) < 0$ on (x_1, x_2). Hence there is a $\lambda_0 > 0$ such that $z + \lambda_0 y$ has a double zero on (x_1, x_2). The $C_{II}[a, \infty)$ condition implies that $z + \lambda_0 y \in \mathcal{N}^+$. Let $0 \leqq \lambda < \lambda_0$ then $z + \lambda y = (\lambda/\lambda_0)(z + \lambda_0 y) + (1 - (\lambda/\lambda_0))z \in \mathcal{N}^+$ because both terms are in $\mathcal{N}^+$. By a similar argument one can show there is a $\mu_0 < 0$ such that if $\mu_0 < \mu < 0$ then $z + \mu y \in \mathcal{N}$ and hence $\mathcal{N} > \mathcal{O}$ at 0. Barrett [2], Hanan [3], Jones [4], Lazer [5], and Utz [**6**] have studied the behavior of solutions of

$$(3) \qquad y''' + p(x)y' + q(x)y = 0, \quad \text{where } p, q, \in C[a, \infty)$$

for various sign conditions on p and q. Using these same sign conditions, dominance results can be obtained for this equation.

THEOREM 8. *If* $p \leqq 0$ *and* $q > 0$ *for* $x \in [a, \infty)$ *and* $\mathcal{O} \neq \emptyset$ *then* $\mathcal{O} > \mathcal{N}$ *at* 0.

This theorem is deduced by observing that Lemma 1.1 of Lazer [5, p. 436] implies that $(3) \in C_1[a, \infty)$ and hence $\mathcal{O} > \mathcal{N}$ at 0 as a consequence of theorem 6.

LEMMA. *Suppose* $p \leqq 0$ *and* $q > 0$ *on* $[a, \infty)$. *If* $\mathcal{O} \neq \emptyset$ *then either* $\lim_{x\to\infty} z(x) = 0$ *for all* $z \in \mathcal{N}$ *or* $\lim_{x\to\infty} z(x) \neq 0$ *for* $z \in \mathcal{N}$.

PROOF. By Theorem 1.2 of Lazer [5; p. 439] every nonoscillatory solution has a limit at ∞. Suppose $\lim_{x\to\infty} z_1(x) = 0 \neq \lim_{x\to\infty} z_2(x)$ where $z_1, z_2 \in \mathcal{N}^+$. Since z_1 and z_2 are both positive, for $x_0 \in [a, \infty)$ there is a $\lambda_0 > 0$ such that $z_2(x_0) - \lambda_0 z_1(x_0) < 0$. Then the solution with a zero in $[x_0, \infty)$. This contradicts Theorem 1.2 of Lazer [5], and we have proved the Lemma.

THEOREM 9. *Suppose* $p \leqq 0$ *and* $q > 0$ *on* $[a, \infty)$. If $\mathcal{O} \neq \emptyset$ *and there is a solution* $z_0 \in \mathcal{N}^+$ *such that* $\lim_{x\to\infty} z_0(x) \neq 0$ *then* $\mathcal{O} > \mathcal{N}$ *at* ∞.

PROOF. Let $z \in \mathcal{N}^+$ and $y \in \mathcal{O}$. Then $\lim_{x\to\infty} z(x) > 0$ by the previous Lemma. Since $\mathcal{O} > \mathcal{N}$ at 0, $\liminf_{x\to\infty} y(x)/z(x) < 0 < \limsup_{x\to\infty} y(x)/z(x)$. If $\limsup_{x\to\infty} y(x)/z(x) = k < \infty$, then

$$\limsup_{x\to\infty} [y(x) - kz(x)]/z(x) = 0$$

and $y - kz \in \mathcal{N}$. But then $y - kz$ has a finite nonzero limit. This is impossible and hence $\limsup_{x\to\infty} y(x)/z(x) = \infty$. Similarly $\liminf_{x\to\infty} y(x)/z(x) = -\infty$ and $\mathcal{O} > \mathcal{N}$ at ∞.

THEOREM 10. *If* $p \leqq 0$ *and* $q \leqq 0$ *then* $\mathcal{N} > \mathcal{O}$ *at* 0 *for* (3).

The sign conditions easily imply that (3) $\in C_{II}[a, \infty)$ and hence Theorem 7 implies Theorem 10.

Recent results of Dolan and Klaasen [7] on the third order equation

$$y''' + p(x)y = 0, \quad \text{where } p(x) \in C[a, \infty) \tag{4}$$

yield information about dominance for this equation by comparing it to equations of Euler.

Dolan and Klaasen [7] have shown that equation (4) is in $C_I[b, \infty)$, in $C_{II}[b, \infty)$, or disconjugate on $[b, \infty)$ for some $b \geqq a$ as $p(x) \geqq -2\sqrt{3}/9x^3$, $p(x) \leqq 2\sqrt{3}/9x^3$ or $|p(x)| \leqq 2\sqrt{3}/9x^3$ for sufficiently large x. Hence, as a consequence of Theorem 6 and Theorem 7 we have:

THEOREM 11. (i) *If* $\mathcal{O} \neq \emptyset$ *and* $p(x) \geqq -2\sqrt{3}/9x^3$ *for large* x *then* $\mathcal{O} > \mathcal{N}$ *at* 0.

(ii) *If* $p(x) \leqq 2\sqrt{3}/9x^3$ *for large* x *then* $\mathcal{N} > \mathcal{O}$ *at* 0.

(iii) *If* $|p(x)| \leqq 2\sqrt{3}/9x^3$ *for large* x *then* $\mathcal{N} > \mathcal{O}$ *at* ∞.

If we consider Euler equation

$$y''' + (K/x^3)y = 0 \tag{5}$$

then if $K > 2\sqrt{3}/9$ we have that $\mathcal{O} > \mathcal{N}$ at ∞ and if $K < -2\sqrt{3}/9$

then $\mathscr{N} > \mathscr{O}$ at ∞. This suggests attempting similar results for (4) where $p(x) > 2\sqrt{3}/9x^3$ or $p(x) < -2\sqrt{3}/9x^3$. No results in this direction have been proved. Kondratév [8] gives some information about behavior of solutions in this case.

Bibliography

1. C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.
2. J. H. Barrett, *Oscillation Theory of Ordinary Linear Differential Equations*, Advances in Mathematics 3 (1969), 415–509.
3. M. Hanan, *Oscillation criteria for third-order linear differential equations*, Pacific J. Math. 11 (1961), 919–944.
4. G. D. Jones, *A Property of* $y''' + p(x)y' + 1/2\, p'(x)y = 0$, Proc. Am. Math. Soc. 33 (1972), 420–422.
5. A. C. Lazer, *The behavior of solutions of the differential equation* $y''' + p(x)y' + q(x)y = 0$, Pacific J. Math. 17 (1966), 435–466.
6. W. R. Utz, *The nonoscillation of a solution of a third order equation*, SIAM Math. Anal. 1 (1970), 535–537.
7. Dolan & Klausen, *On the nonoscillatory behavior of* $y''' + py = 0$, (submitted to SIAM).
8. V. A. Kondratév, *On the oscillation of solutions of linear differential equations of third and fourth order* (in Russian), Trudy Mosk. Mat. Obse. 8 (1959), 259–282; Dohl. Akad. Nauk. U.S.S.R. 118 (1958), 22–29.

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