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THE LATTICE OF TOPOLOGIES: A SURVEY

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In 1936, Garrett Birkhoff noted that inherent in the study of topology is the notion of the comparison of two different topologies on the same basic set. In his paper, *On the combination of topologies,* Birkhoff first explicitly described this comparison by ordering the family of all topologies on a given set, and looking at the resulting lattice. Birkhoff's ordering was the natural one of set inclusion; that is, if $\mathcal{T}$ and $\mathcal{T}'$ are both topologies on a given set, $\mathcal{T}$ is less than or equal to $\mathcal{T}'$ iff $\mathcal{T}$ is a subset of $\mathcal{T}'$.

The topological definitions of this paper coíncide mainly with those of W. J. Thron as found in *Topological Structures,* and the lattice definitions follow Garrett Birkhoff's *Lattice Theory.* However, for the reader's convenience we will begin with some lattice-theory preliminaries, as well as listing several of the less familiar topological definitions in the glossary. In order to make reading easier, a consistent notation will be used throughout the paper; however, it should be noted that in some of the entries in the bibliography, the lattice considered is the dual of the lattice of topologies to which we refer.

This survey is divided into five sections. The first two sections deal with the lattice of topologies and the lattice of T_1-topologies. The third section contains short summaries of lattices of different subsets of the set of topologies, while the fourth section summarizes lattices of structures which contain, or partially contain, the set of topologies. The final section discusses minimal and maximal topologies.

Before proceeding, we should remark that we have consciously omitted some of the more detailed results. Furthermore, in a survey of this nature, we have surely overlooked work which should have been included. We gladly welcome any additions or corrections.

Lattice Terminology and Notation. A *partially ordered set,* $(L, \leqq)$ is a set L on which a binary relation $\leqq$ has been defined. The relation is reflexive, anti-symmetric and transitive.

A partially ordered set $(L, \leqq)$ is called a *lattice* if any two elements in the set have a greatest lower bound or meet denoted by $a \wedge b$ and a least upper bound or join denoted by $a \vee b$.

A lattice is called *complete* if any of its subsets have a meet and join in the set.

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$(L, \geqq)$ is called the *dual* of the lattice $(L, \leqq)$.

$(A, \leqq)$ is called a *sublattice* of $(L, \leqq)$ if $A \subseteq L$ and finite meets and joins are preserved. $(A, \leqq)$ is called a *complete sublattice* of $(L, \leqq)$ if arbitrary meets and joins are preserved.

By "*a covers b*" in a lattice $(L, \leqq)$ we mean $b \leqq a$ and $b \leqq c \leqq a$ implies $b = c$ or $c = a$.

The *least element* of a lattice is designated O and the *greatest element* is designated I.

An *atom* is an element which covers the least element. A lattice is *atomic* if every element other than O can be written as the join of atoms.

An *anti-atom* is an element which is covered by I. A lattice is *anti-atomic* if every element other than I can be written as the meet of anti-atoms.

An element a is called the *complement* of b in a lattice if $a \wedge b = O$ and $a \vee b = I$. A lattice is called *complemented* if every element has at least one complement, *uniquely complemented* if every element has exactly one complement.

A lattice is called *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all a, b, c in the lattice.

A lattice is called *modular* if $a \leqq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$.

A lattice L is called *upper semi-modular* iff for distinct a and b in L such that a and b both cover c, then $a \vee b$ covers both a and b. *Lower semi-modular* is defined dually.

If L is a complete atomic lattice with the set of atoms A, then L is called *tall* iff for every $P \subseteq A$, where $p = \bigvee \{a \mid a \in P\}$, $\{a \mid a \in A,\ a \leqq p\} = \bigcap \{B \mid P \subseteq B \subseteq A,\ a,\ b \in B$ and $c \leqq a \vee b$ implies $c \in B\}$.

A map from a lattice L to a lattice K is called a *lattice homomorphism* if it preserves finite meets and joins. The map is called a *complete homomorphism* if it preserves arbitrary meets and joins. A *lattice isomorphism* is a lattice homomorphism which is one to one and onto.

A lattice $(L, \leqq)$ is called *self dual* if it is lattice isomorphic to $(L, \geqq)$.

I. The Lattice of Topologies on a Set. If X is a set, let $\Sigma(X) = \{\mathcal{T} \mid \mathcal{T}$ is a topology on $X\}$.

THEOREM 1.1. *$\Sigma(X)$ forms a complete lattice under set inclusion. The least element is the indiscrete topology and the greatest element is the discrete topology. The least upper bound of two topologies $\mathcal{T}$*

and $\mathcal{T}'$ *is the topology generated by* $\{G \cap G' \mid G \in \mathcal{T}, G' \in \mathcal{T}'\}$, *and the greatest lower bound is given by* $\mathcal{T} \cap \mathcal{T}'$. [19]

Orders other than set inclusion have been defined on $\Sigma(X)$, [85], [121] but these are not considered in this survey and accordingly we will henceforth use $\Sigma(X)$ to represent both the set of topologies on X and the lattice of topologies on X.

Atomic and Anti-Atomic Properties of $\Sigma(X)$.

THEOREM 1.2. *$\Sigma(X)$ is an atomic lattice. The atoms are topologies of the form* $\{\emptyset, G, X\}$, *where* $\emptyset \subsetneq G \subsetneq X$. *If* $|X| = n$, $\Sigma(X)$ *contains* $2^n - 2$ *atoms. If X is infinite,* $\Sigma(X)$ *contains* $2^{|X|}$ *atoms.* [117]

If $\mathcal{U}$ is an ultrafilter on X and $x \in X$, such that $x \notin \cap\, \mathcal{U}$, then the topology $\mathcal{T}(x, \mathcal{U}) = \{G \mid x \notin G \text{ or } G \in \mathcal{U}\}$ is called an *ultratopology* on X. An ultratopology $\mathcal{T}(x, \mathcal{U})$ is called principal or non-principal depending on whether the ultrafilter $\mathcal{U}$ is principal or non-principal.

THEOREM 1.3. *$\Sigma(x)$ is an anti-atomic lattice. The anti-atoms are the ultratopologies. If* $|X| = n$, $\Sigma(X)$ *contains* $n(n-1)$ *anti-atoms. If X is infinite,* $\Sigma(X)$ *contains* $2^{2^{|X|}}$ *anti-atoms.* [37]

Vaidyanathaswamy described the principal ultratopologies on a set and recognized that they were anti-atoms, but the proof of Theorem 1.3 is due to Fröhlich.

Cardinality of $\Sigma(X)$. When Fröhlich realized that the ultratopologies on a set can be described in terms of ultrafilters on the same basic set, he was able to conclude that the number of topologies definable on an infinite set, X, is the set-theoretic maximum; that is, $2^{2^{|X|}}$. The number of topologies definable on a finite set involves much more computation.

THEOREM 1.4. $|\Sigma(X)| = 2^{2^{|X|}}$ *if X is infinite.* [37]

There is no known formula for the number of topologies on a finite set, although there has been considerable investigation into this problem. These investigations have led to the following partial results.

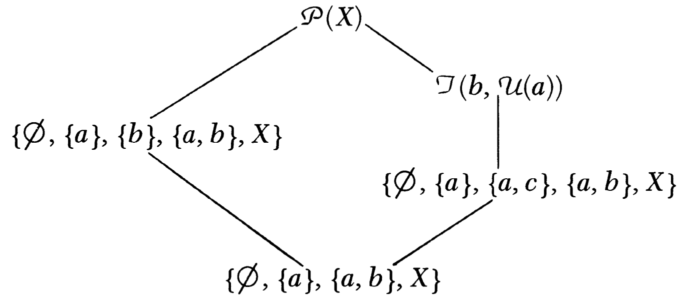
THEOREM 1.5. *If* $|X| = 1, 2, 3, 4, 5, 6,$ *or* 7, $|\Sigma(X)| = 1, 4, 29, 355, 6{,}942, 209{,}527,$ *or* $9{,}535{,}241$. *If* $|X| = n \neq 1$, $2^n \leqq |\Sigma(X)| \leqq 2^{n(n-1)}$. [29], [35], [62]

The cardinality of $\Sigma(X)$, when $X = 5$ or 6, is listed incorrectly in [97].

The Irregular Lattice Structure of $\Sigma(X)$. The following three theorems help to point out the complex structure of $\Sigma(X)$.

THEOREM 1.6. *If $|X| > 2$, $\Sigma(X)$ is non-distributive, non-modular, and neither upper nor lower semi-modular.* [**117**], [**100**], [**67**]

Let $X = \{a, b, c\}$ and let $\mathcal{U}$ be the principal ultrafilter of a. The following diagram from [**67**] illustrates Theorem 1.6.



It is interesting to note that if X is finite and contains more than three elements, that there are more atoms in $\Sigma(X)$ than anti-atoms. However, if X is infinite, the reverse is true. This points out the following theorem.

THEOREM 1.7. *If $|X| > 3$, $\Sigma(X)$ is not self-dual.* [**100**]

THEOREM 1.8. *$\Sigma(X)$ is tall iff X is finite.* [**41**]

Lattice Embeddings in $\Sigma(X)$.

THEOREM 1.9. *For any lattice L, there exists a set X, such that L may be embedded in $\Sigma(X)$.*

This is an extension of Whitman's well-known result that any lattice may be embedded in the lattice of partitions on some set. [**123**] Since the lattice of all partition topologies on X forms a complete sublattice of $\Sigma(X)$, [**117**] and since the lattice of partition topologies on X is isomorphic to the dual of the lattice of all partitions on X, [**89**] the result follows.

Morphisms of $\Sigma(X)$.

THEOREM 1.10. *If $|X| \neq 2$, $\Sigma(X)$ has only trivial lattice homomorphisms. That is, any lattice homomorphism of $\Sigma(X)$ onto a lattice L, is either a lattice isomorphism or L consists of a single element.* [**41**]

THEOREM 1.11. *If X contains one or two elements, or X is infinite, the group of lattice automorphisms of $\Sigma(X)$ is isomorphic to the sym-*

metric group on X. If X is finite and contains more than two elements, the group of lattice automorphisms of $\Sigma(X)$ *is isomorphic to the direct product of the symmetric group on X with the two element group.* [41], [37]

In the proof of Theorem 1.11, Hartmanis used the atomic structure of $\Sigma(X)$. Later, Fröhlich proved the same theorem using the anti-atomic structure of $\Sigma(X)$. Theorem 1.11 has the following consequence.

If X is an infinite set and P is any topological property, then the set of topologies in $\Sigma(X)$ possessing property P may be identified simply from the lattice structure of $\Sigma(X)$. This follows from Theorem 1.11 since the only lattice automorphisms of $\Sigma(X)$ for infinite X are those which simply permute the elements of X. Therefore, any automorphism of $\Sigma(X)$ must map all the topologies in $\Sigma(X)$ onto homeomorphic images. Thus the topological properties of elements of $\Sigma(X)$ must be determined by the position of the topologies in $\Sigma(X)$. An example is shown in the following theorem.

THEOREM 1.12. *If* $\mathcal{T}$ *is an anti-atom in* $\Sigma(X)$, *then* $\mathcal{T}$ *is* T_1 *iff* $\mathcal{T}$ *possesses no maximum complement in* $\Sigma(X)$. [96]

Complementation of $\Sigma(X)$. Of all the questions concerning the lattice structure of $\Sigma(X)$, the question of complementation seems to have aroused the most interest. For more details on the origins of this problem, some of the partial solutions, and other aspects of its interesting history, see Schnare. [94]

THEOREM 1.13. $\Sigma(X)$ *is complemented.* [100], [118]

THEOREM 1.14. *If X is infinite, every topology in* $\Sigma(X)$, *other than the discrete or indiscrete topologies, has at least* $|X|$ *complements in* $\Sigma(X)$. [95]

THEOREM 1.15. *If X is infinite, there exists a subset of* $\Sigma(X)$, *of cardinality* $|X|$ *such that any two elements in the subset are complements of each other.* [3]

II. **The Lattice of T_1-Topologies.** Investigations into the lattice of T_1-topologies have shown that this lattice possesses a substantially different lattice structure than the lattice of topologies. The following theorems point out some of these differences.

Let $\Lambda(X)$ be the set of all T_1-topologies on X, and let $\mathcal{G} = \{G \mid G \subseteq X, X \sim G \text{ is finite}\} \cup \{\emptyset, X\}$.

THEOREM 2.1. $\Lambda(X)$ *is a complete sublattice of* $\Sigma(X)$. *The least element in* $\Lambda(X)$ *is the cofinite topology,* $\mathcal{G}$, *and the greatest element is the discrete topology.* [19]

Atomic and Anti-atomic Properties of $\Lambda(X)$. One of the first distinctions between the two lattices is seen in the atomic structure. While the lattice of topologies is atomic, the lattice of T_1-topologies is not.

THEOREM 2.2. *$\Lambda(X)$ possesses atoms, but is not atomic. The atoms are precisely the topologies of the form $\mathcal{G} \cup \{x\}$.* [117], [8]

To see that $\Lambda(X)$ is not atomic, we need look no further than the usual topology on the reals. In fact, if X is the set of real numbers, the usual topology on the reals is greater than $\mathcal{G}$, but not greater than any of the atoms in $\Lambda(X)$.

Vaidyanathaswamy incorrectly claimed that $\Lambda(X)$ possessed no anti-atoms. This was corrected by Liu in 1959. [74]

THEOREM 2.3. *$\mathcal{T}$ is an anti-atom in $\Lambda(X)$ iff $\mathcal{T} = \{G \mid x \notin G$ or $G \in \mathcal{U}\}$, where $x \in X$ and $\mathcal{U}$ is a non-principal ultrafilter on X.* [74]

THEOREM 2.4. *$\Lambda(X)$ is anti-atomic.* [37]

Cardinality of $\Lambda(X)$. A second difference can easily be seen in the cardinality of $\Lambda(X)$ and $\Sigma(X)$ when X is finite.

THEOREM 2.5. *$|\Lambda(X)| = 1$ if X is finite and $|\Lambda(X)| = 2^{2^{|X|}}$ if X is infinite.* [37]

Complementation of $\Lambda(X)$. Even though several types of T_1-topologies do have complements in $\Lambda(X)$, [2], [101], [102], [103] there are T_1-topologies which do not possess complements in $\Lambda(X)$, which leads to the following distinction between $\Sigma(X)$ and $\Lambda(X)$.

THEOREM 2.6. *$\Lambda(X)$ is not a complemented lattice.* [7], [56], [101]

There is an interesting anecdote connected with the history of Theorem 2.6. The result of Theorem 2.6, as well as an appropriate counterexample was given as a concluding remark, although not set off as a theorem, in Bagley's paper in 1955. [7] In 1958, and using Bagley's paper as a reference, Hartmanis [41] gave a partial solution to the general complementation problem, but left the complementation of $\Lambda(X)$ as an open question. Then, in 1966, again using Bagley's and Hartmanis' papers as references, A. Steiner answered the general complementation problem, but left complementation of $\Lambda(X)$ as still unanswered. [100] Finally, in the latter part of 1966 A. Steiner published the result of Theorem 2.6. [101]

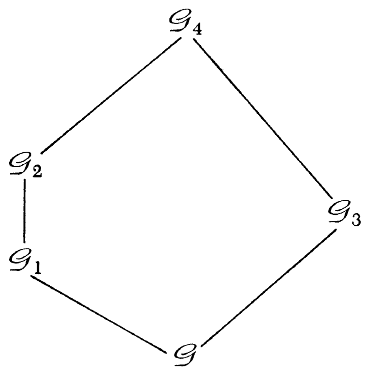
Modularity of $\Lambda(X)$. A fourth difference between the two lattices can be seen in the local behavior of elements of $\Lambda(X)$.

THEOREM 2.7. *$\Lambda(X)$ is not modular, and hence not distributive.* [8]

The following example to illustrate Theorem 2.7 is taken from [67]. Let X be an infinite set such that A and $X \sim A$ are both infinite. Let $\mathcal{G}$ be the cofinite topology on X and choosing $x \notin A$, let

$$\mathcal{G}_1 = \mathcal{G} \vee \{\emptyset, A, X\}$$
$$\mathcal{G}_2 = \mathcal{G} \vee \{\emptyset, A \cup \{x\}, X\}$$
$$\mathcal{G}_3 = \mathcal{G} \vee \{\emptyset, \{x\}, X\}$$
$$\mathcal{G}_4 = \mathcal{G} \vee \{\emptyset, \{x\}, A, A \cup \{x\}, X\}.$$

The following diagram is valid.



THEOREM 2.8. *$\Lambda(X)$ is both upper and lower semi-modular.* [67]

THEOREM 2.9. *Any nontrivial interval in $\Lambda(X)$ contains a covering relation.* [68]

Lattice Embeddings in $\Lambda(X)$.

THEOREM 2.10. *If L is any finite distributive lattice, there exists a set X and topologies $\mathcal{T}$ and $\mathcal{T}'$ in $\Lambda(X)$ such that L is isomorphic to the interval between $\mathcal{T}$ and $\mathcal{T}'$.* [68]

Morphisms of $\Lambda(X)$.

THEOREM 2.11. *If X is infinite, the group of automorphisms of $\Lambda(X)$ is isomorphic to the symmetric group on X.* [34]

THEOREM 2.12. *If X is infinite, the lattice of complete homomorphisms of $\Lambda(X)$ is isomorphic to the lattice of finite subsets of X and the set X ordered under set inclusion.* [41]

III. **Lattices of Subfamilies of $\Sigma(X)$.** Before proceeding with this section, we present the following chart which shows how several topological properties are preserved under lattice operations in $\Sigma(X)$. For proofs or references, see [39].

In the chart, we will use the following notation:

- $\leqq$ preservation under weakening of topologies.
- $\bigwedge$ preservation under arbitrary meets.
- $\wedge$ preservation under finite meets.
- $\geqq$ preservation under strengthening of topologies.
- $\bigvee$ preservation under arbitrary joins.
- $\vee$ preservation under finite joins.
- $+$ indicates that the topological property is preserved.

Property	Preservation under lattice operations					
	$\leqq$	$\bigwedge$	$\wedge$	$\geqq$	$\bigvee$	$\vee$
T_1	$-$	$+$	$+$	$+$	$+$	$+$
T_0, T_D, T_2, and totally disconnected	$-$	$-$	$-$	$+$	$+$	$+$
T_3, T_{3a}, regular, completely regular, and zero-dimensional	$-$	$-$	$-$	$-$	$+$	$+$
1st and 2nd countable	$-$	$-$	$-$	$-$	$-$	$+$
principal	$-$	$+$	$+$	$-$	$-$	$+$
bicompact, Lindelöf, connected, and separable	$+$	$+$	$+$	$-$	$-$	$-$
locally connected	$-$	$+$	$+$	$-$	$-$	$-$
T_4, T_5, normal, completely normal, paracompact, and locally bicompact	$-$	$-$	$-$	$-$	$-$	$-$

This listing points out that of the above common properties, the T_1 separation axiom holds a special place, in that it is preserved under arbitrary meets and joins in $\Sigma(X)$. Of course, subfamilies of $\Sigma(X)$ may form lattices under set inclusion even if they do not preserve arbitrary meets and joins in $\Sigma(X)$. We present five such examples.

The Lattice of Principal Topologies. A topological space is called principal if it is discrete or if it can be written as the meet of principal ultratopologies. A. Steiner proved that this is equivalent to requiring that the arbitrary intersection of open sets is open.

While $\Lambda(X)$ possesses a lattice structure which is quite different from $\Sigma(X)$, the lattice of principal topologies compares closely as a lattice with $\Sigma(X)$. Most of the results concerning this lattice are due to A. Steiner. [**100**]

The lattice of principal topologies is a complete lattice whose least element is the indiscrete topology and greatest element is the discrete topology. This lattice is both atomic and anti-atomic; its atoms coincide with those of $\Sigma(X)$. Although the lattice of principal topologies is a sublattice of $\Sigma(X)$, it is not a sub-complete lattice of $\Sigma(X)$.

This lattice is complemented, non-modular if $|X| \geqq 3$, and non-self dual if $|X| \geqq 4$. If X is finite, the lattice of principal topologies is simply $\Sigma(X)$. If X is infinite, the lattice of principal topologies is of cardinality $2^{|X|}$.

The lattice of principal topologies on X is isomorphic to the dual of the lattice of pre-orders on X. [**100**] Alexandroff pointed out this correspondence between orders and topologies when he proved that there is a one-to-one correspondence between the principal T_0-topologies on X and the partial orders on X. [**1**] For more information on this subject, see [**124**].

The Lattice of Partition Topologies. A topology on X is called a partition topology if it possesses a base which is a partition of X. The lattice of all partition topologies on a set has been studied in several different forms. O. Ore seems to have been the first to extensively investigate this lattice in the form of the lattice of equivalence relations on a set. [**78**] Vaidyanathaswamy pointed out that the lattice of partition topologies is a sublattice of $\Sigma(X)$. [**117**] More recently, M. Rayburn studied this lattice as the lattice of closed-open topologies and as the lattice of complete Boolean Algebras on X. [**89**] Finally, M. Huebener has proved that the lattice of partition topologies on X is precisely the lattice of principal-regular topologies on X. [**46**]

The least and greatest elements of this lattice are the indiscrete and discrete topologies. Ore has shown that this lattice is completely complemented; that is, every interval in the lattice is a complemented lattice. He also proved that this lattice is atomic and anti-atomic, where the atoms are the partition topologies containing precisely two minimal open sets, and the anti-atoms are the partition topologies which have one minimal open set consisting of a pair of points, and the other minimal open sets each consisting of a single point.

The lattice of partition topologies on X is lower semi-modular, but non-modular if $|X| \geqq 4$. If X is infinite, the cardinality of this lattice is $2^{|X|}$. If $|X| = n$, the cardinality of the lattice of partition topologies on X, denoted p_n, is given by the recursion formula

$$p_{n+1} = \sum_{i=0}^{n} \binom{n}{i} p_i$$

Ore also proved that the group of automorphisms of the lattice of partition topologies on X is isomorphic to the symmetric group on X. [78]

As a final comment, we point out, once again, Whitman's result that any lattice may be embedded in the lattice of partition topologies on an appropriate set. [**123**]

The Lattice of Regular Topologies and the Lattice of Completely Regular Topologies. The join of regular (completely regular) topologies in $\Sigma(X)$ is itself a regular (completely regular) topology. However, this is not necessarily true of the meet of regular (completely regular) topologies. [43] [**75**] Hence, neither of these lattices is a sublattice of the lattice of topologies. The least and greatest elements of each lattice are the indiscrete and discrete topologies. If X is a finite set, the lattice of regular topologies on X is precisely the lattice of partition topologies on X. [**46**] M. Huebener has described certain classes of regular topologies which have complements in the lattice of regular topologies, [**46**] but it is apparently unknown if the entire lattice is complemented. Since the non-principal ultratopologies in $\Sigma(X)$ are both regular and completely regular, [**100**] the cardinality of both lattices on an infinite set is $2^{2^{|X|}}$.

The Lattice of Countably Accessible Topologies. This lattice was introduced by R. Larson [**66**] in a construction which paralleled A. Steiner's construction of the lattice of principal topologies.

A topology is called a countably accessible topology if it can be written as the meet of ultratopologies whose associated ultrafilters contain countable sets, but no finite sets. A topology is countably accessible if and only if every non-closed set, G, contains a countable subset with a limit point lying outside of G.

The set of countably accessible topologies on a set is a subset of the collection of T_1-topologies and contains all first countable T_1-topologies on the set. Arbitrary meets in this lattice and $\Sigma(X)$ coincide, although arbitrary joins may differ. The lattice is non-atomic, anti-atomic, non-complemented, and non-modular. If X is countable, the lattice of countably accessible topologies is precisely $\Lambda(X)$. If X is uncountable, the cardinality of the lattice of countably accessible topologies is $2^{|X|}$.

The Lattice of "Joins of Hyperplanes". R. Bagley defined this lattice in [7] to be the lattice consisting of the cofinite topology, the atoms in $\Lambda(X)$, and any topology which can be written as the join of atoms in $\Lambda(X)$. This lattice is a sub-complete lattice on $\Lambda(X)$, and is isomorphic to the complete Boolean algebra of subsets of X.

IV. $\Sigma(X)$ is contained, in a natural way, in several more general lattices, which arise from considering some of the alternative approaches to topological structure through closure functions or convergence functions.

Lattices of Closure Functions. A closure function in the sense of Čech is an increasing, order-preserving, function f from $P(X)$ to $P(X)$ such that $f(\emptyset) = \emptyset$. Ore's definition adds to these the property that f is idempotent. If, in addition, the function preserves unions, as is assumed in the Kuratowski axioms, the structure is equivalent to the usual topological structure. When ordered by $f \leqq g$ iff $g(A) \subseteq f(A)$ for all $A \subseteq X$, the set of Čech-closures and the set of Ore-closures on X both form complete lattices containing the lattice of topologies on X. [61], [77]

The lattice of Čech-closures has the virtue of simple joins and meets defined by $(f \vee g)(A) = f(A) \cap g(A)$, and $(f \wedge g)(A) = f(A) \cup g(A)$ for all $A \subseteq X$. It is a proper sublattice of the lattice of all functions from $\mathcal{P}(X)$ to $\mathcal{P}(X)$. Unfortunately, neither the lattice of Ore-closures nor the lattice of topologies is a proper sublattice of the lattice of Čech-closures. The Ore-closures have the same join as the Čech-closures, but different meet. The lattice of topologies has the same meet as the lattice of Ore-closures, but a different join. [61], [77]

The lattice of Ore-closures can be represented as the lattice of complete intersection rings over X. It is a lower semi-modular atomic lattice, but is not anti-atomic and not complemented. [77]

Lattices of Sequential Topologies. When Garrett Birkhoff introduced the lattice of topologies, he also discussed the lattice of sequential topologies. In its most general form, a sequential topology on a set X is just a relation between the set of sequences on X and X itself.

If f is a sequence topology on X, then a sequence $\{x_n\}$ converges to a point x if $x \in f(\{x_n\})$. We denote this as $x_n \rightarrow x$. A Frechet

$\mathcal{L}$-space is a sequential topology that satisfies:

(1) $f(\{x_n\})$ is empty or singleton.
(2) If $x = x_n$ for every n, then $x_n \to x$.
(3) $x_n \to x$ implies that every subsequence of $\{x_n\}$ converges to x.

The sequential topologies, when ordered by $f \leqq g$ iff $g(\{x_n\}) \subseteq f(\{x_n\})$ form a complete, distributive lattice, of which the lattice of Frechet $\mathcal{L}$-spaces is a sub-complete lattice. [19] If the Frechet $\mathcal{L}$-spaces are further assumed to satisfy the condition that the addition of a finite number of terms to a sequence affects neither its convergence nor its limits, then the corresponding lattice is complete and completely distributive. [117]

These lattices do not, however, contain $\Sigma(X)$ in any natural way. The lattice of topologies and the lattice of sequential topologies can be mapped into each other by the functions ζ and ψ, defined by $\zeta(\mathcal{T})(\{x_n\}) = \{x \mid$ every open neighborhood of x in $\mathcal{T}$ contains all but a finite number of the $x_n\}$ and $\psi(f) = \{A \mid A \subseteq X,\ f(\{x_n\}) \subseteq A$ implies that A contains all but a finite number of the $x_n\}$. It is clear that, for any topology $\mathcal{T}$, $\psi(\zeta(\mathcal{T}))$ is always T_1. Furthermore, even when $\mathcal{T}$ is T_1, $\psi(\zeta(\mathcal{T}))$ does not always equal $\mathcal{T}$. [117] Thus ζ fails to include, not only the lattice of topologies, but also the lattice of T_1-topologies, in the lattice of sequential topologies. A satisfactory topological characterization of the topologies for which $\psi(\zeta(\mathcal{T})) = \mathcal{T}$, and thus of the class of topologies which can be included in the lattice of sequential topologies on X, has apparently not been determined.

Lattices of Convergence Structures. Convergence structures generalizing topological structure can be found by using convergences, not of sequences, but of filters, or alternatively, of nets. The definitions and notation used here are those of Kent. [50], [51] Let $F(X)$ be the set of all filters on X, and let $\mathcal{F}_x$ be the filter generated by x, for each $x \in X$. A convergence function f is a function from $F(X)$ into $\mathcal{P}(X)$, such that

(1) $\mathcal{F} \subseteq \mathcal{G} \Rightarrow f(\mathcal{F}) \subseteq f(\mathcal{G})$, for all $\mathcal{F}, \mathcal{G} \in F(X)$, and
(2) $x \in f(\mathcal{F}_x)$, for all $x \in X$.

For each $x \in X$, let $\mathcal{V}_f(x) = \bigcap \{\mathcal{F} \mid \mathcal{F}$ is a filter and $x \in f(\mathcal{F})\}$. A series of progressively stronger structures, culminating in one equivalent to topological structure, follows. A convergence function f is a convergence structure iff

(3) $x \in f(\mathcal{F}) \Rightarrow x \in f(\mathcal{F} \cap \mathcal{F}_x)$.

A convergence structure f is a limitierung [36] iff

(4) $f(\mathcal{G}_1) \cap f(\mathcal{G}_2) \subseteq f(\mathcal{G}_1 \cap \mathcal{G}_2)$, for all $\mathcal{G}_1, \mathcal{G}_2 \in F(X)$.

A limitierung f is a pseudo-topology iff

(5) $x \in f(\mathcal{G}')$ for all ultrafilters $\mathcal{G} \subseteq \mathcal{G}' \Rightarrow x \in f(\mathcal{G})$.

A pseudotopology f is a pretopology iff

(6) $x \in f(\mathcal{V}_f(x))$, for all $x \in X$.

A pretopology is topological iff

(7) for every $x \in X$, $\mathcal{V}_f(x)$ has a filter base $\mathcal{B}_f(x) \subseteq \mathcal{V}_f(x)$ such that $y \in G(x) \in \mathcal{V}_f(x) \Rightarrow G(x) \in \mathcal{V}_f(y)$.

For each of these properties, the set of all such functions on a fixed set X forms a complete lattice when ordered in the natural way by $f_1 \leqq f_2$ iff $f_2(\mathcal{G}) \subseteq f_1(\mathcal{G})$, for all $\mathcal{G} \in F(X)$.

The lattice of pretopologies on X has a representation as a sublattice of filters on X^X, and, as a consequence, is atomic, anti-atomic, modular, distributive, and compactly generated, but not completely distributive, and not complemented unless X is finite, in which case it is uniquely complemented. [26] Carstens has stated that the lattice of pseudotopologies on X can be represented as the lattice of subsets of a set and is therefore a complete Boolean lattice. [27]

The lattice of convergence structures, $C(X)$, is a sub-complete lattice of the lattice of convergence functions, $C'(X)$. Both $C(X)$ and $C'(X)$ have join and meet that can be defined very simply for any family Q of functions as $(\vee Q)(\mathcal{G}) = \cap \{f(\mathcal{G}) \mid f \in Q\}$, and $(\wedge Q)(\mathcal{G}) = \cup \{f(\mathcal{G}) \mid f \in Q\}$, for all $\mathcal{G} \in F(X)$. Each of the other lattices is an additive subsystem of $C'(X)$, in that it has the same join, both finite and infinite. But none is a sub-complete lattice of $C'(X)$ because meets are not preserved. In fact, Kent has shown that every convergence structure is the infimum of a set of topologies. [51] Therefore, for any two lattices intermediate between $\Sigma(X)$ and $C'(X)$, one can never be a sub-complete lattice of the other. However, Carstens has shown that $\Sigma(X)$ is a sublattice of the lattice of pretopologies on X.

For each convergence function f, there is a finest limitierung, a finest pseudo-topology, a finest pretopology, and a finest topology coarser than, or equal to f. [36], [50]

V. **Strengthening of Topologies.** If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$, $\mathcal{T}'$ has been called an expansion, [43] an extension, [73], [21] and an enlargement of $\mathcal{T}$. [30] Most of the well known separation axioms are preserved under arbitrary expansions. Some topological properties which are only preserved under special types of expansions have been studied in [73], [21], [30]. Examples of such properties are regularity, complete regularity, normality, separability, first and second countability, connectedness, and various types of compactness.

Minimal and Maximal Topologies. While the set of all topologies in $\Sigma(X)$ sharing a given topological property may not have a least or greatest element, it may have minimal or maximal elements. A topology on X is said to be maximal P (minimal P) if the topology possesses property P, but no stronger (weaker) topology on X possesses property P. Topologies which qualify as being minimal or maximal with respect to appropriate topological properties are characterized in the following theorem.

THEOREM 5.1. *A topology in $\Sigma(X)$ is minimal P or maximal P with respect to some topological property P iff every one-to-one continuous map of the space onto itself is a homeomorphism.* [85]

A similar theorem may be given for minimum and maximum topologies.

THEOREM 5.2. *A topology in $\Sigma(X)$ is minimum P or maximum P with respect to some topological property P iff every one-to-one map of the space onto itself is a homeomorphism.* [65]

Another example of a general type of theorem which characterizes an entire class of maximal P spaces is given by D. Cameron in [24, Theorem 2.4].

The following two charts indicate several topological properties whose minimal or maximal topologies have been characterized. In many cases, the topological behavior of these minimal and maximal topologies has been investigated. For further information on these results, see the references listed.

For some topological properties, there are no maximal topologies. For example, if X is an uncountable set, then there are no maximal separable or maximal second countable topologies in $\Sigma(X)$. Of course, if X is finite or countable, the discrete topology is maximum separable and maximum second countable. Note that the discrete topology is maximum for many topological properties: T_0, T_1, T_2, regular, normal, disconnected, etc. It appears that maximal connected spaces have not been characterized in such a way as to answer many questions about them. For example, it is unknown whether there is a maximal connected topology which is finer than the usual topology on the reals.

Additional information on most of the following minimal topologies may be found in *A survey of minimal topological structures*, by Berri, Porter, and Stephenson. [**16**]

Property	Characterization of maximal elements	References
Bicompact	maximal iff the closed subsets of the topology are precisely the bicompact subsets.	[**88**], [**98**], [**71**], [**30**], [**25**]
Countably compact	maximal iff the closed subsets of the topology are precisely the countably compact subsets.	[**25**], [**67**]
Perfect	maximal iff $\mathcal{T}$ is perfect and every subset which has no isolated points is open.	[**43**]
Perfect regular	maximal iff $\mathcal{T}$ is perfect regular and whenever A and B are complementary subsets of X which have no isolated points, then A and B are open.	[**43**]
Sequentially compact	maximal iff the closed subsets of the topology are precisely the sequentially compact subsets.	[**25**], [**67**]
Lindelöf	maximal iff the closed subsets of the topology are precisely the Lindelöf subsets.	[**25**]
Regular non-discrete	maximal iff it is a nonprincipal ultratopology or it is of the form $\mathcal{T}(x, \mathcal{U}(y)) \cap \mathcal{T}(y, \mathcal{U}(x))$.	[**100**]
Nested	maximal iff it is nested and T_D.	[**124**]
Connected Principal	maximal iff $X = \bigcup \{B_x \mid \{x\} \notin \mathcal{T}\}$, and $x \neq y$ implies $B_x \cap B_y$ contains at most one point, and every two points in X can be connected by exactly one simple chain consisting of minimal open sets. $(B_x = \bigcap \{G \mid x \in G, G \in \mathcal{T}\})$	[**113**]
Non-T_0	maximal iff it is of the form $\mathcal{T}(x, \mathcal{U}(y)) \cap \mathcal{T}(y, \mathcal{U}(x))$.	[**111**]
Non-T_1	maximal iff it is a principal ultratopology.	[**111**]

Property	Characterization of minimal elements	References
T_0	minimal iff it is T_0, nested, and the complements of the point closures generate the topology.	[81], [64], [39]
T_δ and T_ζ	minimal iff minimal T_0 and principal.	[124]
T_D	minimal iff it is T_D and nested.	[64], [39]
T_1	minimum iff the closed proper subsets of the topology are precisely the finite subsets.	[19], [13]
T_2	minimal iff it is T_2, H-closed, and semi-regular.	[48], [86]
T_2	minimal iff it is T_2, and every open filter with a unique cluster point converges.	[22], [13]
T_{2a}	minimal iff it is T_{2a} and every Urysohn filter with a unique cluster point converges.	[45], [91]
T_3	minimal iff it is T_3, and every regular filter with a unique cluster point converges.	[17]
T_{3a}, T_4, T_5, T_2-para-compact, Metrizable, T_2-locally compact, T_2-zero dimensional	minimal iff it possesses the property under consideration and is bicompact.	[13], [45] [98], [105]
T_2-perfectly normal	minimal iff it is T_2-perfectly normal and countably compact.	[105]
T_2-first countable, E_1	minimal iff T_2-first countable and every open filter with a countable base and a unique cluster point converges.	[105], [83]
E_0	minimum iff X is countable and $\mathcal{J}$ is the minimum T_1 topology.	[83]

Glossary

Topological Definitions:

A space $(X, \mathcal{T})$ is called a

T_D-space iff $\{x\}'$ (the derived set of $\{x\}$) is a closed set for every $x \in X$.

T_δ-space iff whenever $\{x\}' \neq \emptyset$, $\{x\}'$ is a point closure.

T_ζ-space iff for each $x \in X$, $\{x\}'$ is the union of a family of point-closures, $\{\overline{\{y\}} \mid y \in Y\}$ such that for all distinct $r, s \in Y$, r and s are separated.

T_{2a}-space iff it is a Urysohn space.

T_3-space iff it is T_1 and regular.

T_{3a}-space iff it is T_1 and completely regular.

T_4-space iff it is T_1 and normal.

T_5-space iff it is T_1 and completely normal.

E_0-space iff every point in X can be written as the countable intersection of neighborhoods of x.

E_1-space iff every point in X can be written as the countable intersection of closed neighborhoods of x.

A filter in a space $(X, \mathcal{T})$ is called an *open filter* iff it has a filter base consisting of open sets.

A filter in a space $(X, \mathcal{T})$ is called a *closed filter* iff it has a filter base consisting of closed sets.

A filter in a space $(X, \mathcal{T})$ is called a *regular filter* iff it is both an open and closed filter.

A filter, $\mathcal{F}$ in a space $(X, \mathcal{T})$ is called a *Urysohn filter* iff it is an open filter and for each $x \in X$, such that x is not a cluster point of $\mathcal{F}$, there is an open neighborhood U of x and $V \in \mathcal{F}$ such that $\bar{U} \cap \bar{V} = \emptyset$.

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