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UNIQUENESS OF SOLUTIONS OF AN INFINITE SYSTEM OF EQUATIONS

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1. **Introduction and Results.** Let $A = (a_{i,j})$, $i, j = 1, 2, \cdots$, be an infinite non-zero matrix of complex numbers such that for each i, the sequence $\{a_{i,j}\}$ where $j = 1, 2, \cdots$ is in ℓ^2, the space of all square summable sequences. In this note, we will discuss some uniqueness theorems on the ℓ^2 solutions of the following system of linear equations:

$$\sum_{j=1}^{\infty} a_{i,j}x_j = y_i, \quad i = 1, 2, \cdots. \tag{1}$$

Let $\{e_i\}$ be an orthonormal basis of a Hilbert space H. Then the uniqueness of the solutions of the system (1) is equivalent to the completeness of the system $\{Ae_i\}$,

$$Ae_i = \sum_{j=1}^{\infty} a_{i,j}e_j, \quad i = 1, 2, \cdots,$$

in H. It is a rule of thumb that a perturbed basis is still a basis provided that the perturbation is sufficiently small. Thus, it is also a purpose of this note to give some limit on the size of a perturbation A so that $\{Ae_i\}$ is again a basis of H. We obtain the following results.

THEOREM 1. *Let $A_n = (a_{i,j})$, $1 \leqq i, j \leqq n$, be the $n \times n$ matrices obtained from A. Either one of the following conditions is sufficient for the uniqueness of the solutions of the system* (1):

$$\text{(i)} \qquad \liminf_{n\to\infty} \frac{\prod_{i=1}^{n}\left(\sum_{j=1}^{\infty} |a_{i,j}|^2\right)}{|\det A_n|^2} < \infty.$$

$$\text{(ii)} \qquad \liminf_{n\to\infty} \frac{\left(\sum_{i=1}^{n}\sum_{j=n+1}^{\infty} |a_{i,j}|^2\right)\left(\prod_{i=1}^{n}\left[\sum_{j=1}^{n} |a_{i,j}|^2\right]\right)}{|\det A_n|^2} < \infty.$$

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There is a vast literature on estimating lower bounds of determinants. We mention only Brenner [1, 2], Ostrowski [6], and Price [7]. Using some of these bounds, we may get some other more workable sufficient conditions from (i) or (ii) for the uniqueness of ℓ^2 solutions of the system (1). We also wish to mention that some similar, but somewhat different, results can be found in Hilding [3] and Kato [4]. In the following section, we will compare our above result with theirs.

For upper triangular matrices, we have a better result:

THEOREM 2. *Let $a_{i,j} = 0$ whenever $j < i$. Then the condition*

$$\sum_{j>i} |a_{i,j}| \leqq (1+\delta)|a_{i,i}| \neq 0 \tag{2}$$

with $\delta = 0$ for all $i = 1, 2, \cdots$ is sufficient for the uniqueness of the solutions of the system (1). *But for each $\delta > 0$, there exists an upper triangular matrix satisfying* (2) *such that the solutions for the system* (1) *are not unique.* (We remark that Theorem 2 is well-known for $-1 < \delta < 0$).

2. **Proof of Theorem 1.** Let $\{b_i\}$ be an ℓ^2 solution of the system (1) with all $y_i = 0$. We have to prove that $b_i = 0$ for all i. We write

$$\epsilon_{N,i} = \sum_{j=1}^{N} a_{i,j}b_j = -\sum_{j=N+1}^{\infty} a_{i,j}b_j. \tag{3}$$

Hence, if $\det A_N \neq 0$, we have, from Cramer's rule, that

$$b_1 = \frac{1}{\det A_N}\begin{vmatrix} \epsilon_{N,1} & a_{1,2} & \cdots & a_{1,N} \\ \epsilon_{N,2} & a_{2,2} & \cdots & a_{2,N} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot \\ \epsilon_{N,N} & a_{N,2} & \cdots & a_{N,N} \end{vmatrix} .$$

If all but a finite number of the b_i are zero, then it is clear that all the b_i are zero. Otherwise, we set

$$\delta_{N,i} = \epsilon_{N,i} \Big/ \left(\sum_{k=N+1}^{\infty} |b_k|^2 \right)^{1/2},$$

which gives

$$b_1 = \frac{\left(\sum_{k=N+1}^{\infty} |b_k|^2\right)^{1/2}}{\det A_N} \begin{vmatrix} \delta_{N,1} & a_{1,2} & \cdots & a_{1,N} \\ \delta_{N,2} & a_{2,2} & \cdots & a_{2,N} \\ \cdot & \cdot & \cdots & \cdot \\ \delta_{N,N} & a_{N,2} & \cdots & a_{N,N} \end{vmatrix}$$

By the Hadamard determinant theorem (cf. [5]) we have

$$(4) \qquad |b_1|^2 \leqq \frac{\sum_{k=N+1}^{\infty} |b_k|^2}{|\det A_N|^2} \prod_{i=1}^{N} (|\delta_{N,i}|^2 + |a_{i,2}|^2 + \cdots + |a_{i,N}|^2)$$

and

$$(5) \qquad |b_1|^2 \leqq \frac{\sum_{k=N+1}^{\infty} |b_k|^2}{|\det A_N|^2} \left[\prod_{j=2}^{N} \left(\sum_{i=1}^{N} |a_{i,j}|^2\right)\right] \left[\sum_{i=1}^{N} |\delta_{N,i}|^2\right].$$

Also, from (3) using the Schwarz inequality, we have

$$(6) \qquad |\delta_{N,i}|^2 = \frac{|\epsilon_{N,i}|^2}{\sum_{k=N+1}^{\infty} |b_k|^2} \leqq \sum_{j=N+1}^{\infty} |a_{i,j}|^2.$$

Then (4) and (5) yield

$$(7) \qquad |b_1|^2 \leqq \frac{\sum_{k=N+1}^{\infty} |b_k|^2}{|\det A_N|^2} \prod_{i=1}^{N} \left(\sum_{j=1}^{\infty} |a_{i,j}|^2\right)$$

and

$$(8) \, |b_1|^2 \leqq \frac{\sum_{k=N+1}^{\infty} |b_k|^2}{|\det A_N|^2} \left[\prod_{j=2}^{N} \left(\sum_{i=1}^{N} |a_{i,j}|^2\right)\right] \left[\sum_{i=1}^{N} \sum_{j=N+1}^{\infty} |a_{i,j}|^2\right]$$

$$\leqq \frac{\sum_{k=N+1}^{\infty} |b_k|^2}{|\det A_N|^2 \left[\sum_{i=1}^{N} |a_{i,1}|^2\right]} \left[\prod_{j=1}^{N} \left(\sum_{i=1}^{N} |a_{i,j}|^2\right)\right] \left[\sum_{i=1}^{N} \sum_{j=N+1}^{\infty} |a_{i,j}|^2\right],$$

respectively.

Now since

$$\lim_{N\to\infty} \sum_{k=N+1}^{\infty} |b_k|^2 = 0\ ,$$

and since $\det A_N \neq 0$ for infinitely many N, we have for all j,

$$\sum_{i=1}^{\infty} |a_{i,j}|^2 > 0.$$

Thus, by the hypothesis (i) or (ii), we have proved that $b_1 = 0$, and by a similar proof, we can conclude that all the b_i are zero. This completes the proof of Theorem 1.

As a consequence of this theorem, we have the following

COROLLARY 1. *Let $\{e_i\}$, $i = 1, 2, \cdots$, be an orthonormal basis of a Hilbert space H, and let A be a linear operator in H. Then $\{Ae_i\}$, $i = 1, 2, \cdots$, is complete in H if*

$$\lim_{n\to\infty} \frac{1}{|\det A_n|} \prod_{i=1}^{n} \|Ae_i\| < \infty\,. \tag{9}$$

The following result can be found in [4], page 266:

THEOREM A. *Let $\{e_j\}$ be a complete orthonormal family in a Hilbert space H. Then a sequence $\{f_j\}$ of non-zero vectors of H is a basis of H if*

$$\sum_{j=1}^{\infty} \left(\|f_j - e_j\|^2 - \frac{|(f_j - e_j, f_j)|^2}{\|f_j\|^2} \right) < 1. \tag{10}$$

We now compare (10) and our Corollary 1 for the sequence $\{f_j\}$ such that

$$f_j = \sum_{k=j}^{\infty} a_{j,k} e_k\ ,$$

where $a_{j,j}$ are real and $\|f_j\| = 1$ for all j. By (10), we know that $\{f_j\}$ is complete if

$$\sum_{j=1}^{\infty} (1 - |a_{j,j}|) < 1,$$

but by (9), we can conclude that $\{f_j\}$ is complete if

$$\prod_{j=1}^{\infty} |a_{j,j}| > 0,$$

which is equivalent to

$$\sum_{j=1}^{\infty} (1 - |a_{j,j}|) < \infty .$$

As another consequence of Theorem 1, we have the following results:

COROLLARY 2. *Let* $A_n = (a_{i,j})$, $1 \leqq i, j \leqq n$, *be* $n \times n$ *matrices of complex numbers. Then the following conditions are sufficient for the uniqueness of the solutions of the system* (1):

(i) $$\sum_{j=1}^{\infty} |a_{i,j}|^2 \leqq 1 \ ,$$

for all $i = 1, 2, \cdots$ *and*

(ii) $$\limsup_{n \to \infty} |\det A_n| > 0.$$

COROLLARY 3. *If* $\{e_j\}$ *is a complete orthonormal sequence and the* $f_i = \sum_j a_{ij} e_j$ *are orthonormal, then* $\{f_i\}$ *is complete if*

$$L = \limsup_{n \to \infty} |\det A_n| > 0,$$

or equivalently,

$$\limsup_{n \to \infty} |\det(\langle f_i, e_j \rangle), 1 \leqq i, j \leqq n| > 0.$$

We remark that Theorem 1 shows that certain perturbed bases are still bases even when the perturbation is large. For example, the rows of

$$\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ & & & & \vdots & & & & \end{array}$$

are complete. Also, Theorem 1 (i) and Corollary 3 are equivalent, since the rows of A are complete if and only if their Gram-Schmidt orthogonalizations are complete. But orthonormalization only improves the hypothesis of Theorem 1 (i).

3. **Proof of Theorem 2.** For $\delta = 0$, let $\{b_i\}$ be an ℓ^2 solution of the system (1) with all $y_i = 0$. We have to show that all $b_i = 0$. Since $b_i \to 0$ as $i \to \infty$, we can find a k such that

$$|b_k| = \max(|b_i| : i = 1, 2, \cdots).$$

We assume, on the contrary, that $b_k \neq 0$. Then by the hypothesis, we have

$$a_{k,k}b_k = -\sum_{s=k+1}^{\infty} a_{k,s}b_s.$$

Since $b_s \to 0$, $|b_s| < |b_k|$ for large s, and hence, from (2) we have

$$\begin{aligned}|a_{k,k}|\,|b_k| &\leqq \sum_{s=k+1}^{\infty} |a_{k,s}|\,|b_s| \\ &< |a_{k,k}|\,|b_k|,\end{aligned}$$

which is a contradiction. As for $\delta > 0$, we let $a_{i,i} = 1$, $a_{i,i+1} = 1 + \delta$ for all $i = 1, 2, \cdots$, and let $a_{i,j} = 0$ otherwise. Then

$$\sum_{j=i+1}^{\infty} |a_{i,j}| \leqq (1 + \delta)a_{i,i} \ ,$$

for each $i = 1, 2, \cdots$. However, the sequence

$$\left(1, \frac{-1}{(1+\delta)}, \frac{1}{(1+\delta)^2}, \frac{-1}{(1+\delta)^3}, \cdots\right)$$

is clearly an ℓ^2 solution of the system

$$\sum_{j=1}^{\infty} a_{i,j}x_j = 0, i = 1, 2, \cdots.$$

The above example is an "analytic Toeplitz matrix", that is,

$$a_{i,j} = 0 \text{ if } i > j, a_{ij} = b_{i-j} \text{ if } i \leqq j,$$

where

$$\sum_{n=0}^{\infty} |b_n|^2 < \infty.$$

It is well-known that a necessary and sufficient condition for the completeness of the rows of A is that

$$f(z) = \sum_{n=0}^{\infty} b_n z^n$$

is an outer function in H^2. Although the proof of Theorem 2 is quite simple, this theorem has some interesting consequences.

COROLLARY 4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a function of the Hardy class* H^2 *on* $|z| < 1$, *such that*

$$\sum_{n=2}^{\infty} |a_n| \leqq |a_1| \neq 0. \tag{11}$$

Then the space generated by the functions $1, f(z), f(z^2), \cdots$ *is dense in* H^2.

By a similar proof, we can also conclude that Corollary 4 holds for any Hardy space H^p with $1 \leqq p < \infty$. However, we remark that this corollary does not hold for the Banach space A of functions continuous on $|z| \leqq 1$ and holomorphic in $|z| < 1$ with the supremum norm. This can be seen from the following:

EXAMPLE. Let $f(z) = z - z^3$. Then $f \in A$ and (11) is satisfied. But $f(1) = f(-1) = 0$, so that any function g that can be approximated uniformly on $|z| \leqq 1$ by linear combinations of $1, f(z), f(z^2), \cdots$ must satisfy $g(1) = g(-1)$.

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