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LOWER BOUNDS FOR POLYNOMIAL APPROXIMATIONS TO RATIONAL FUNCTIONS

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1. **Introduction and preliminary definitions.** For a complex valued function f defined on a compact set E in the plane, let $\|f\|_E = \sup_{z \in E} |f(z)|$.

If Γ is a closed Jordan curve and $R(z)$ is a rational function having at least one pole inside Γ, then one can easily show that there exists a $\delta > 0$ such that $\|R - p\|_\Gamma \geqq \delta$ for all polynomials p. Obviously the same δ will not work for all Γ and all R since $\|R\|_\Gamma$ can be arbitrarily small. However, if we normalize the problem by requiring that R be of the form $R(z) = q_{n-1}(z)/\prod_{i=1}^{n} (z - a_i)$, where q_{n-1} is a polynomial of degree $n - 1$ (or less), all the a_i's are inside Γ and $\|R\|_\Gamma = 1$, then one might inquire as to the existence of a $\delta_n > 0$, independent of Γ and R, with the property that $\|R - p\|_\Gamma \geqq \delta_n$ for all polynomials p. The authors plan to give a more detailed treatment of this problem and its implications including the proofs of the following theorems, elsewhere.

2. **Some partial answers.** A weaker question than the one just stated pertains to the existence of a $\delta_n(\Gamma) > 0$, independent of R but not of Γ, such that $\|R - p\|_\Gamma \geqq \delta_n(\Gamma)$ for all polynomials p. The following theorem establishes the existence of a $\delta_n(\Gamma) > 0$ in the case where Γ is the unit circle $U = \{|z| = 1\}$.

THEOREM 1. *For $n = 1, 2, \cdots$ there exists $\delta_n(U) > 0$ such that if $R_n(z)$ is a rational function of the form $R_n(z) = q_{n-1}(z)/\prod_{k=1}^{n} (z - a_k)$ where q_{n-1} is a polynomial of degree $n - 1$, $|a_k| < 1$ for $k = 1, 2, \cdots, n$ and $\|R_n\|_U = 1$ then, $\|R_n - p\|_U \geqq \delta_n(U)$ for all polynomials p.*

PROOF. If we define δ_n by the recursive formula $\delta_n = \delta_{n-1}/(3 + 2\delta_{n-1})$ with $\delta_1 = 1/2$ then our proof proceeds by way of induction. We now weaken our original problem by considering only those rational functions whose poles have a common locus.

THEOREM 2. *For $n = 1, 2, \cdots$ there exists $\delta_{n^*} > 0$ such that if Γ is any closed Jordan curve and if $R_n^*(z) = q_{n-1}(z)/(z - a)^n$, where q_{n-1} is a polynomial of degree $n - 1$, the point $z = a$* lies in the *interior of Γ and $\|R_n^*\|_\Gamma = 1$ then $\|R_n^* - p\|_\Gamma > \delta_{n^*}$ for all polynomials p. Furthermore, we may choose δ_{n^*} to be given by*

$$\delta_{n^*} = \{4^n + 4^{n-1}(1 + 4^n) + 4^{n-2}(1 + 4^n + 4^{n-1}(1 + 4^n)) + \cdots + 4(1 + 4^n + 4^{n-1}(1 + 4^n) + \cdots + 4^2(1 + 4^n + 4^{n-1}(1 + 4^n) + \cdots))\}^{-1}.$$

3. **A related question.** Let Γ and the point $z = a$ be as in Theorem 2, and let $\delta_{n^*}(\Gamma, a)$ be the largest δ_{n^*} that satisfies the conditions in that theorem. Since the lower bounds we mention for these constants tend to zero as n increases, we are naturally led to the question of whether the $\lim_{n\to\infty}\delta_{n^*}(\Gamma, a) = 0$ for each Γ and each a. Our final theorem answers this question affirmatively in the case where $\Gamma = U$ is the unit circle.

THEOREM 3. *Let $\delta_{n^*}(\Gamma, a)$ be as defined above. Then* $\lim_{n\to\infty}\delta_{n^*}(U, a) = 0$, (for all $|a| < 1$).

PROOF. The theorem is first proved in the case where $a = 0$ by considering the sequence of rational functions defined by

$$\delta_n(z) = \sum_{k=2}^{n} \frac{z^{-k}}{k \log k} - \sum_{k=2}^{n} \frac{z^{k}}{k \log k}$$

and its convergence on U [1, p. 253]. The theorem is then easily extended to any point $z = a$ inside U.

4. **An application to rational approximation.** If f is defined and continuous on Γ, a closed Jordan curve, and $\epsilon > 0$ there exists [2, p. 100] a rational function $Q_{n,k}$ of the form

$$Q_{n,k}(z) = q_{n-1}(z) / \prod_{k=1}^{n} (z - a_k) + p_k(z)$$

where q_{n-1} and p_k are polynomials of respective degrees $n - 1$ and k (for some n and some k). The a_k's are inside Γ and such that, $\|f - Q_{n,k}\|_\Gamma < \epsilon$. Since $\|Q_{n,k}\|_\Gamma < 2\|f\|_\Gamma$ if ϵ is sufficiently small, a natural question to ask is whether $\|q_{n-1}(z)/\prod_{k=1}^{n}(z - a_k)\|_\Gamma$ is bounded in any way. If as in § 1, there exists a $\delta_n > 0$ (possibly independent of Γ) we would then have:

$$\left\| q_{n-1}(z) / \prod_{1}^{n} (z - a_k) \right\|_\Gamma < 2\|f\|_\Gamma / \delta_n.$$

In this way, one can immediately state corollaries to Theorems 1 and 2.

REFERENCES

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