ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 2, Spring 1974

RATIONAL APPROXIMATIONS WITH APPLICATIONS TO THE SOLUTION OF FUNCTIONAL EQUATIONS*

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1. **Introduction.** The subject of Padé approximations was begun about a century ago. Although considerable progress has been made in recent times on various aspects of the subject, analytical expressions for the pertinent polynomials of the Padé matrix table, recurrence formulas for the polynomials and general pointwise convergence theorems are known for only a small class of functions.

In this exposition, we take the approach of developing rational approximations for a large class of functions which can be defined by series expansions, differential equations, integral transforms, etc., without insisting on the definition which characterizes the Padé approximation. In this fashion we show how closed form analytical expressions for the polynomials emerge naturally. Also, a general theorem concerning the existence of recurrence relations is proved. Further, by seizing upon the functional properties noted above, the error in the approximation process can be characterized in an analytic manner, from which important data on convergence and assessment of the error can be deduced. In some instances, these rational approximations do reduce to familiar Padé approximations for a certain class of well known functions.

2. **Rational Approximations for Functions Defined by a Series.** Suppose

$$F(z) = \sum_{r=0}^{n-1} b_r z^r + R_n(z), \tag{1}$$

where b_r is independent of n and z and $R_n(z)$ is the error in the above polynomial approximation to $F(z)$. Expressions of the type (1) can be obtained in a variety of ways — viz. solutions of differential equations and from integral transforms such as those of Laplace and Mellin-Barnes integrals. In (1) replace n by $k+1-a$, $a=0$ or $a=1$, multiply both sides by $A_{n,k}\gamma^{-k}$ and sum from $k=0$ to $k=n$. Here γ is a free parameter, $A_{n,k}$ is independent of z and γ, and $A_{n,k}=0$ if $k>n$. Then

Received by the editors February 8, 1973.
*This paper was prepared under Air Force Office of Scientific Research grant number 71-2127.

$$F(z)h_n(\gamma) = \psi_n(z,\gamma) + S_n(z,\gamma), \tag{2}$$

$$h_n(\gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^{-k}, \tag{3}$$

$$\psi_n(z,\gamma) = \sum_{k=a}^{n} \gamma^{-k} \sum_{r=o}^{n-k} \tau_{n,r+k} b_r \delta^r,\ \delta = z/\gamma, \tag{4}$$

$$S_n(z,\gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^{-k} R_{k+1-a}(z). \tag{5}$$

Thus $\psi_n(z,\gamma)/h_n(\gamma)$ is a rational approximation to $F(z)$ with remainder $S_n(z)/h_n(\gamma)$. So long as γ is distinct from z, the approximation is a polynomial in z of degree n. But since γ is free, we can put $\gamma = z$ and so achieve an approximation in the form of a ratio of two polynomials in z — the numerator and denominator polynomials being of degree $n - a$ and n respectively. Thus the approximation process has great flexibility. Indeed, the importance of the above summability scheme is that there exists a large class of functions for which the process is convergent not only for the case when the series in (1) when extended to infinity is convergent, but also in cases where this infinite series is divergent but asymptotic.

An important aspect of the above developments is the possibility of recurrence relationships for the polynomials in the approximation. Suppose there exist constants K_m, L_m and M_n such that

$$\sum_{m=0}^{t} (K_m + L_m/\gamma)h_{n-m}(\gamma) = M_n,\ \ K_0 = 1,\ L_0 = 0,\ n \geqq t. \tag{6}$$

Then

$$\begin{aligned} &\sum_{m=0}^{t} (K_m + L_m/\gamma)\psi_{n-m}(z,\gamma) = Q_n(z,\gamma), \\ &Q_n(z,\gamma) = \gamma^{-a} \sum_{r=0}^{n-a} b_r\delta^r \sum_{m=0}^{t} K_m A_{n-m,r+a}. \end{aligned} \tag{7}$$

It follows that

$$\sum_{m=0}^{t} (K_m + L_m/\gamma)S_{n-m}(z,\gamma) = F(z)M_N - Q_n(z,\gamma). \tag{8}$$

Thus $S_n(z,\gamma)$ can be characterized and studied once we establish a basis of solutions of the difference equation (6) with $M_n = 0$.

3. **Rational Approximations for Functions Defined by a Differential Equation.** The τ-method, first introduced by Lanczos and exploited by Luke [**9**] to achieve rational approximations mostly for hypergeometric functions, is an excellent example of how to get such approximations for functions defined by a differential equation.

Let

$$L(D)y(z) = g(z), D = d/dz, \tag{9}$$

where $L(D)$ is a differential operator whose coefficients are polynomials in z. If $L(D)$ is nonlinear, we suppose that the power to which each derivative is raised is a positive integer or zero. Let $g(z)$ also be a polynomial in z. Suppose a particular solution of (9) is

$$y(z) = \sum_{k=0}^{\infty} a_k z^k, |z| < c, \tag{10}$$

where it is assumed that the above series is nonterminating. Clearly the a_k's are found in virtue of given boundary conditions and recurrence relations which ensue when one seeks a power series solution to (10). Suppose we ask for a polynomial approximation to (9) of the form

$$y_n(z) = \sum_{k=0}^{n} b_k z^k. \tag{11}$$

Now (11) cannot satisfy (9). Our idea is to perturb the differential equation (9) so that (11) is a solution of this perturbed equation. Let us write

$$L(D)y_n(z) = g(z) + \sum_{u=0}^{p} \tau_{m-u} h_{m-u}(z/\gamma) \tag{12}$$

where the τ_i's are constants, and the $h_u(z)$'s are preassigned polynomials in z of degree u. Also γ is a parameter whose role will emerge in our later discussion. The τ_i's enter the recursion relation involving the b_j's and both the τ_i's and b_j's are found as the solution of this recursion system. The values of m and p depend on the nature of $L(D)$ and $g(z)$ and we make no attempt to spell out how these are determined. Suffice it to say that p and m are selected so that (11) is a unique solution of (12). This much is certain. If $L(D)$ is linear, $b_0 = a_0$, and the recursion relation which generates the a_j's is composed of $(s + 2)$ terms, then $p = s$. Only the case $s = 0$ has been studied in detail, and in this event the rational approximations are of the form in (2)–(4), and the role of γ is the same as in (2)–(4).

To gain further insight, suppose $L(D)$ is linear. If

$$\epsilon_n(z) = y(z) - y_n(z), \tag{13}$$

is the error in the approximation process, then

$$L(D)\epsilon_n(z) = -\sum_{u=0}^{p} \tau_{m-u} h_{m-u}(z/\gamma). \tag{14}$$

Thus the error can be analyzed once a basis of solutions of the homogeneous equation $L(D)y(z) = 0$ is known.

Notice that if $F(z)$ as given by (1) also satisfies a linear differential equation, it is possible to achieve rational approximations by the methods of this section which are identical with those given in Section 1. If the discussion surrounding (6)–(8) holds, then the error can be studied from two different points of view.

4. **Rational Approximation for the $_pF_q$ and a Certain G-Function.** In the case of the generalized hypergeometric function $_pF_q(z)$, $p \leqq q+1$, rational approximations have been studied by Luke [**9**] from the views enunciated in Sections 2 and 3. If $p > q+1$, the $_pF_q(z)$ series diverges. Nonetheless, it is the asymptotic expansion for a certain generalization of $_pF_q(z)$ known as the G-function. This G-function can be represented as a Mellin-Barnes integral which can be used to deduce a representation like (1) valid for all p and q, from which the forms (2)–(5) can be derived. In the cited reference, the $A_{n,k}$'s are chosen to be closely related to the coefficients in the classical orthogonal Jacobi polynomial, commonly notated as $P_n^{(\alpha,\beta)}(x)$. In this event, the recursion formula for $h_n(\gamma)$ is homogeneous, that is in (6), $M_n = 0$ and t is a fixed quantity independent of n. Let $\beta = 0$. Then α can always be chosen in a certain manner (actually we almost always take $\alpha = 0$) so that in (7), $Q_n(z, z) = 0$, whence both $h_n(z)$ and $\psi_n(z, z)$ satisfy the same recurrence formula. Thus the rational approximations are easy to generate. In the cited reference, the error is studied mostly from the view of constructing the solution of the differential equation satisfied by the error. Fields and Wimp [**8**] have used the difference equation (8) with $M_n = Q_n(z, z) \equiv 0$ to analyze the related error term $S_n(z, z)$ for Tricomi's ψ-function, and this has been generalized by Fields [**7**] for the G-function noted above.

Space does not permit us to give a detailed description of the recurrence formulas, the convergence properties of the rational approximations, etc. For details the reader is referred to the noted references [**7, 8, 9**]. The following is a minimal statement of our findings. We can always construct approximations for ${}_pF_q(z)$ with $\gamma = z$ which converge for all z if $p \leqq q$, and which converge for all z, with $|\arg(1 - z)| < \pi$ if $p = q + 1$. In the case where $p > q + 1$ and ${}_pF_q(z)$ is the asymptotic representation of a certain G-function, then with $\gamma = z$, we can always construct rational approximations for this G-function which converge for all z, $|\arg(-z)| < \pi$.

5. **Padé Approximations.** When the G-function noted in Section 4 reduces to ${}_2F_1(1, \sigma; \rho + 1; -z)$ or to two confluent forms of the latter which correspond to two forms of the incomplete gamma function, respectively we can choose the free parameters α and β so as to recover the main diagonal ($a = 0$) and first sub-diagonal ($a = 1$) Padé approximations for these functions. Our analysis is particularly enlightening since a complete analytical description of the error is deduced from which asymptotic data are derived which are remarkably accurate even for low order approximations. For details, see Luke [**9**]. Another aspect of the analysis is that under rather liberal conditions, the Padé approximations give upper and lower bounds for the functions being approximated. In [**10**] Luke has extended these results to get upper and lower bounds for ${}_pF_q(z)$, $p = q$ and $p = q + 1$, and for a certain G-function which includes Tricomi's or Whittaker's function, which in turn includes Bessel functions and the parabolic wave cylinder functions as special cases.

6. **Applications.** The rational approximations have been found useful not only to compute hypergeometric functions, but to locate zeros and poles and invert Laplace transforms. A number of examples are given in [**9**]. Now hypergeometric functions can be viewed as building blocks for functions not of the hypergeometric family and also for functions not commonly conceived as members of this family. Thus our approximations for hypergeometric functions are valuable to develop approximations for larger classes of functions. Indeed in [**11, 12**] we employed this idea to get approximations for the gamma function and its logarithmic derivative.

Recently, we proved that the backward recursion technique for the evaluation of Bessel functions generates rational approximations which are of the family given by (2)–(5) when $\gamma = z$. Further, the analysis leads to closed form representations of the error from which effective a priori estimates of the error can be deduced. For details, see Luke [**13**].

The τ-method has been applied mostly to linear differential equations and then nearly always to the one satisfied by ${}_pF_q(z)$. Fair [**1**] has used the τ-method to get the main diagonal Padé approximations for the nonlinear first order Riccati equation, and Fair and Luke [5] have used a linear fractional transformation technique to get like approximations for a generalized second order Riccati equation. See also Fair [2]. A summary of [1, 5] is also available in [9].

Recently Luke, Fair and Wimp [14] have developed predictor-corrector schema for the numerical integration of differential equations based on rational interpolants. The technique is highly advantageous since it automatically forecasts and computes the zeros and poles on or near the path of integration. Further, the process is 'continuous' in the sense that one passes through a pole just as though it were an ordinary point.

Many of the notions expounded here are being extended to the case of functions with matrix argument. In this connection, see a paper by Fair and Luke [6]. Papers by Fair [3, 4] are also pertinent.

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