ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 2, Spring 1973

SOME ASPECTS OF NONLINEAR EIGENVALUE PROBLEMS[1]

PAUL H. RABINOWITZ

Introduction. Let E be a real Banach space and $G : R \times E \to E$ where G is continuous. Consider the equation

$$u = G(\lambda, u) \tag{0.1}$$

where $\lambda \in R$ and $u \in E$. A *solution* of (0.1) is a pair $(\lambda, u) \in R \times E$. Equations of the form (0.1) are generally called *nonlinear eigenvalue problems*. As has been amply demonstrated at this symposium, they occur in many parts of mathematical physics. Our main interest here is in studying the structure of the set of solutions of (0.1). We restrict ourselves to real λ and to real Banach spaces since this is the situation usually encountered in applications.

Our survey will focus on two major approaches to the study of nonlinear eigenvalue problems, namely the theory of topological degree of Leray and Schauder and the theory of critical points of Ljusternik and Schnirelmann. The applications of degree theory will be mainly to bifurcation problems although some results will be given for the case where bifurcation need not occur. A very general treatment of the Ljusternik-Schnirelmann theory has been presented in his lectures by Professor Browder. Here a simpler version will be given for a very special case — namely the manifolds dealt with will be spheres in a Hilbert space — our goal being to bring out the underlying ideas in essentially as simple a context as possible.

In §1 applications of degree theory to bifurcation theory will be developed. In particular a generalization of a theorem of Krasnoselski is obtained showing that in a certain context bifurcation is a global phenomenon. A useful constructive local theorem for bifurcation from simple eigenvalues is also given. Several applications to ordinary and partial differential equations of the results of §1 are carried out in §2.

The notion of genus is introduced in §3 and its properties studied.

Received by the editors February 11, 1972.

AMS (MOS) subject classifications (1970). Primary 34-XX, 35-XX, 47-XX; Secondary 45-XX, 46-XX.

[1]The contents of this paper are the series of lectures given by the author at the Rocky Mountain Mathematics Consortium Symposium on Nonlinear Eigenvalue Problems at Santa Fe, June–July, 1971. Some of the topics only sketched there are expanded upon here and a few items have been added. Preparation for this paper was sponsored in part by the United States Army under Contract No. DA-31-124-ARO-D-462.

Using this notion a finite-dimensional version of the critical point theorem of Ljusternik and Schnirelmann is given in §4; namely, it is proved that if f is a continuous even real valued function on $\boldsymbol{R}^n$, it possesses at least n distinct pairs of critical points on each sphere centered about the origin. A Galerkin argument is used in §5 to extend this result to infinite dimensions.

Lastly in §6 some applications of degree theory are made to (0.1) in situations where bifurcation need not occur and continua of solutions are obtained.

1. **Bifurcation theory and degree theory.** Of course (0.1) is too general an equation to study without imposing more conditions on G. One type of situation that arises often in practice is $G(\lambda, u) = \lambda Lu + H(\lambda, u)$ where $L : E \to E$ is a bounded linear operator and $H(\lambda, u)$ is continuous on $\boldsymbol{R} \times E$ with $H = o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. For this case, (0.1) becomes

$$u = \lambda Lu + H(\lambda, u) \tag{1.1}$$

which possesses the line of solutions $\{(\lambda, 0) \mid \lambda \in \boldsymbol{R}\}$ henceforth referred to as the *trivial solutions*. This leads us to the notion of a *bifurcation point*: $(\mu, 0)$ is a bifurcation point for (1.1) with respect to the line of trivial solutions or more briefly a bifurcation point if every neighborhood of $(\mu, 0)$ contains nontrivial solutions. Physical examples of this phenomenon occur e.g. in fluid dynamics and elasticity theory. See [1] for many examples.

A necessary condition for $(\mu, 0)$ to be a bifurcation point is that μ^{-1} belongs to the spectrum of L. To see this, note that if μ^{-1} is not in the spectrum of L, $I - \lambda L$ will be invertible for all λ near μ with uniformly bounded inverse. Writing (1.1) for $u \neq 0$ in the equivalent form

$$u/\|u\| = (I - \lambda L)^{-1}H(\lambda, u)/\|u\| \tag{1.2}$$

and applying the $o(\|u\|)$ condition on H shows $(\mu, 0)$ cannot be the limit of nontrivial solutions of (1.1). Therefore $(\mu, 0)$ is not a bifurcation point.

The above necessary condition is not sufficient as simple examples show. E.g. let $E = \boldsymbol{R}^2$, $u = (x, y)$ and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix}. \tag{1.3}$$

Multiplying the first equation of (1.3) by y, the second by x, and subtracting shows $(1, (0, 0))$ is not a bifurcation point.

Next a sufficient condition for $(\mu, 0)$ to be a bifurcation point will be derived. Henceforth we assume that L and H are *compact* on E, $\boldsymbol{R} \times E$ respectively, i.e., are continuous and map bounded sets into relatively compact sets. Following Krasnoselski [2], we will say μ is a *characteristic value* of L if there exists $v \in E$, $v \neq 0$, such that $v = \mu L v$, i.e., μ^{-1} is a nonzero eigenvalue of L. As is well known [3], since L is compact the characteristic values of L are discrete and of finite multiplicity. (The *multiplicity* of a characteristic value μ is the dimension of $\bigcup_{n=1}^{\infty} \ker (I - \mu L)^n$ where $\ker A$ denotes the kernel of A.) For what follows let $r(L)$ denote the set of real characteristic values of L.

THEOREM 1.4. *If $\mu \in r(L)$ is of odd multiplicity, $(\mu, 0)$ is a bifurcation point.*

Theorem 1.4 is due to Krasnoselski [2] and follows from the more general Theorem 1.10 below but, as an exercise in the methods used, we will prove it. Some preliminary remarks are needed. Let $\Omega \subset E$ be a bounded open set and $\partial\Omega$ denote the boundary of Ω. Suppose $\Psi : \overline{\Omega} \to E$ where $\Psi = I - T$ with I the identity map and T compact. Let $b \in E$, $b \notin \Psi(\partial\Omega)$. Then the *Leray-Schauder degree* of Ψ with respect to the set Ω and the point b is well defined and will be denoted by $d(\Psi, \Omega, b)$. The usual properties of Leray-Schauder degree are assumed for what follows (see [2], [4], [5] or [6]). When $b = 0$ as is generally the case below, we write $d(\Psi, \Omega)$ for $d(\Psi, \Omega, 0)$. A closed ball in E of radius r and centered at u_0 will be denoted by $B_r(u_0)$. If $u_0 = 0$, we write $B_r(0) \equiv B_r$. If u_0 is an isolated solution of $\Psi(u) = b$, the *index* of u_0, $i(\Psi, u_0, b) = \lim_{r \to 0} d(\Psi, B_r(u_0), b)$. Let $i(\Psi, u_0, 0) = i(\Psi, u_0)$.

The following change of index lemma [2], [5] is crucial for the proof of Theorem 1.4.

LEMMA 1.5. *Suppose $T : E \to E$ is compact, $T(0) = 0$, T is Fréchet differentiable at $u = 0$, and $1 \notin r(T'(0))$. If $\Psi(u) \equiv u - T(u)$, then $i(\Psi, 0) = (-1)^{\beta}$ where β is the sum of the multiplicities of the characteristic values of $T'(0)$ in $(0, 1)$.*

PROOF OF THEOREM 1.4. Suppose $(\mu, 0)$ is not a bifurcation point for (1.1). For convenience assume $\mu > 0$; $\mu < 0$ is treated similarly. Then there is a neighborhood of $(\mu, 0)$ containing no nontrivial solutions of (1.1). In particular there exists an $\epsilon > 0$ such that there are no solutions of (1.1) on $[\mu - \epsilon, \mu + \epsilon] \times \partial B_\epsilon$ and μ is the only characteristic value of L in $|\lambda - \mu| \leqq \epsilon$. Let $\Phi(\lambda, u) = u - G(\lambda, u)$. Then $d(\Phi(\lambda, \cdot\,), B_\epsilon)$ is well defined for $|\lambda - \mu| \leqq \epsilon$ and, by the homotopy invariance property of degree,

$$d(\Phi(\lambda, \cdot), B_\epsilon) \equiv \text{constant} = c, \qquad |\lambda - \mu| \leqq \epsilon. \tag{1.6}$$

But by Lemma 1.5,

$$\begin{aligned} d(\Phi(\mu - \epsilon, \cdot), B_\epsilon) &= i(\Phi(\mu - \epsilon, \cdot), 0) \equiv (-1)^{\beta^-} \equiv i^-, \\ d(\Phi(\mu + \epsilon, \cdot), B_\epsilon) &= i(\Phi(\mu + \epsilon, \cdot), 0) \equiv (-1)^{\beta^+} \equiv i^+. \end{aligned} \tag{1.7}$$

The β^+ sum differs from the β^- sum by a term equal to the multiplicity of the characteristic value $\mu/(\mu + \epsilon)$ of $(\mu + \epsilon)L$. Since this is just the multiplicity of μ which by assumption is odd, $i^+ = -i^- \neq 0$ contrary to (1.6)–(1.7). The theorem is proved.

Let $\mathcal{S}$ denote the closure of the set of nontrivial solutions of (1.1). Theorem 1.4 implies the intersection of $\mathcal{S}$ with any neighborhood of $(\mu, 0)$ is nonempty when $\mu \in r(0)$ is of odd multiplicity. Actually under the hypotheses of Theorem 1.4 a global statement concerning $\mathcal{S}$ and $(\mu, 0)$ can be made by exploiting more strongly the ingredients that went into the proof of the theorem. In order to show this, more preliminaries are needed. In particular, a stronger version of the homotopy invariance property of degree is required. This is also due to Leray and Schauder [5].

Let $\Lambda = [\lambda_1, \lambda_2]$ and $\mathcal{A} \subset \Lambda \times E$. For $\lambda \in \Lambda$, let $\mathcal{A}_\lambda = \{u \in E \mid (\lambda, u) \in \mathcal{A}\}$ and $(\partial\mathcal{A})_\lambda = \{u \in E \mid (\lambda, u) \in \partial\mathcal{A}\}$.

LEMMA 1.8. *Suppose $\mathcal{A} \subset \Lambda \times E$ is bounded and open and $\Phi(\lambda, u) = u - G(\lambda, u)$ where $G : \bar{\mathcal{A}} \to E$ is compact. If $0 \notin \Phi(\lambda, (\partial\mathcal{A})_\lambda)$ for all $\lambda \in \Lambda$, then $d(\Phi(\lambda, \cdot), \mathcal{A}_\lambda) \equiv$ constant for $\lambda \in \Lambda$.*

By a component of a topological space we mean a closed connected subset maximal with respect to inclusion.

LEMMA 1.9. *Let $\mu \in r(L)$ and let $\mathcal{C}$ denote the component of $\mathcal{S} \cup \{(\mu, 0)\}$ to which $(\mu, 0)$ belongs. If $\mathcal{C}$ is bounded and contains no points of the form $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in r(L)$, then there exists a bounded open set $\mathcal{O} \subset \boldsymbol{R} \times E$ such that $\mathcal{C} \subset \mathcal{O}$, $\partial\mathcal{O} \cap \mathcal{S} = \emptyset$, and the only trivial solutions in $\mathcal{O}$ form a segment $\{(\lambda, 0) \mid |\lambda - \mu| < \epsilon\}$ where $\epsilon < \epsilon_0$, the distance from μ to $r(L) - \{\mu\}$.*

PROOF. The compactness of G and boundedness of $\mathcal{C}$ imply $\mathcal{C}$ is compact. Let U be a δ neighborhood of $\mathcal{C}$ in $\boldsymbol{R} \times E$ where $\delta < \epsilon_0$. Let $K = \bar{U} \cap \mathcal{S}$. Then K is a compact metric space under the induced topology from $\boldsymbol{R} \times E$. By construction $\partial U \cap \mathcal{C} = \emptyset$. Hence by a lemma from point set topology [7, Chapter 1], there exist disjoint compact subsets $A, B \subset K$ such that $\mathcal{C} \subset A$, $\partial U \cap \mathcal{S} \subset B$ and $K = A \cup B$. Let $\mathcal{A}$ be a ρ neighborhood in $\boldsymbol{R} \times E$ of A where ρ is less than the distance from A to B. By possibly removing some trivial solutions from $\mathcal{A}$, we obtain $\mathcal{O}$ as in the statement of the lemma.

The global analogue of Theorem 1.4 can now be given.

THEOREM 1.10. *If $\mu \in r(L)$ is of odd multiplicity, $\mathcal{S}$ contains a component $\mathcal{C}$ which either* (i) *is unbounded or* (ii) *contains $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in r(L)$.*

PROOF. If not, there exists a bounded open set $\mathcal{O}$ as in Lemma 1.9. Let $\Phi(\lambda, u) = u - G(\lambda, u)$. For $s \in (\epsilon, \epsilon_0)$ and $0 < |\lambda - \mu| \leqq s$ define $\rho(\lambda) = \frac{1}{2} \inf\{\xi > 0 |$ there exists $0 \neq u \in B_\xi$ such that $\Phi(\lambda, u) = 0\}$. Since by earlier remarks for $0 < |\lambda - \mu| \leqq s$, $(\lambda, 0)$ is an isolated zero of $\Phi(\lambda, u)$ in $\{\lambda\} \times E$, $\rho(\lambda) > 0$. Moreover it is easy to see that $\rho(\lambda)$ is bounded away from zero on compact subsets of $0 < |\lambda - \mu| \leqq s$. Let $\mathcal{O}_\lambda = \{u \in E \mid (\lambda, u) \in \mathcal{O}\}$ and $(\partial \mathcal{O})_\lambda = \{u \in E \mid (\lambda, u) \in \partial \mathcal{O}\}$. Choose numbers ρ^+, ρ^- so small that $B_{\rho^\pm} \cap (\partial \mathcal{O})_\lambda = \emptyset$ for $|\mu - \lambda| > s$ and define $\rho(\lambda) = \rho^+$ for $\lambda > \mu + s$, $\rho(\lambda) = \rho^-$ for $\lambda < \mu - s$.

Note that, for $\lambda \neq \mu$, $d(\Phi(\lambda, \cdot\,), \mathcal{O}_\lambda - B_{\rho(\lambda)})$ is well defined. We will show that

$$d(\Phi(\lambda, \cdot\,), \mathcal{O}_\lambda - B_{\rho(\lambda)}) = 0, \qquad \lambda \neq \mu, \tag{1.11}$$

and then show that (1.11) is not possible for all λ near μ.

Suppose that $\lambda > \mu$ and $\sigma - \lambda$ is greater than the diameter of $\mathcal{O}$. If $\rho = \inf_{x \in [\lambda, \sigma]} \rho(x)$, then $\rho > 0$. Finally let $U = \mathcal{O} - [\lambda, \sigma] \times B_\rho$. Then U is a bounded open set in $[\lambda, \sigma] \times E$ and by construction $\Phi(\nu, u) \neq 0$ for $(\nu, u) \in \partial U$ (where by ∂U we mean relative to $[\lambda, \sigma] \times E$).

By Lemma 1.8,

$$d(\Phi(\nu, \cdot\,), \mathcal{O}_\nu - B_\rho) \equiv \text{constant} = c, \qquad \nu \in [\lambda, \sigma]. \tag{1.12}$$

Since $\mathcal{O}_\sigma = \emptyset$, $c = 0$ and therefore

$$d(\Phi(\lambda, \cdot\,), \mathcal{O}_\lambda - B_\rho) = 0. \tag{1.13}$$

Since $d(\Phi, (\lambda, \cdot\,), \operatorname{int} B_{\rho(\lambda)} - B_\rho) = 0$ (where int A denotes the interior of A), (1.13) and the additivity of degree yield (1.11). The argument for $\lambda < \mu$ is the same.

Next choose $\underline{\lambda}, \bar{\lambda}$ such that $\mu - \epsilon < \underline{\lambda} < \mu < \bar{\lambda} < \mu + \epsilon$. By Lemma 1.8 again,

$$d(\Phi(\lambda, \cdot\,), \mathcal{O}_\lambda) \equiv \text{constant}, \qquad \lambda \in [\underline{\lambda}, \bar{\lambda}]. \tag{1.14}$$

The additivity of degree implies

$$(1.15)\quad d(\Phi(\underline{\lambda},\,\cdot\,),\mathcal{O}_{\underline{\lambda}}) = i(\Phi(\underline{\lambda},\,\cdot\,),0) + d(\Phi(\underline{\lambda},\,\cdot\,),\mathcal{O}_{\underline{\lambda}} - B_{\rho(\underline{\lambda})}).$$

By (1.11), the last term on the right equals zero. Similarly

$$(1.16)\qquad d(\Phi(\bar{\lambda},\,\cdot\,),\mathcal{O}_{\bar{\lambda}}) = i(\Phi(\bar{\lambda},\,\cdot\,),0).$$

Combining (1.14)–(1.16) yields

$$(1.17)\qquad i(\Phi(\underline{\lambda},\,\cdot\,),0) = i(\Phi(\bar{\lambda},\,\cdot\,),0).$$

But since $\mu \in r(L)$ is of odd multiplicity, by (1.7) and the remarks following it, (1.17) is not possible. Hence we have a contradiction and the theorem is proved.

Remarks. Both alternatives of Theorem 1.10 are possible. The simplest example of (i) is the linear case $H \equiv 0$. Examples of (ii) are more difficult to construct but this can already be done for $E = \mathbf{R}^2$. See e.g. [8]–[10]. In a recent numerical study of a pair of nonlinear ordinary differential equations arising in a buckling problem, Bauer, Keller, and Riess [11] found (ii) occurring at every eigenvalue. When it is a priori known that (i) does not occur, a small modification of the proof of Theorem 1.10 shows that if $\Gamma = \{\gamma \in r(L) \mid \gamma \neq \mu \text{ and } (\gamma, 0) \in \mathcal{C}\}$, then Γ contains an odd number of characteristic values of odd multiplicity [8].

The most common situation in which Theorem 1.10 is applicable is when $\mu \in r(L)$ is of multiplicity one, i.e., is *simple.* For this case an improved version of the theorem obtains [8]. Namely there exists a neighborhood $\mathcal{N}$ of $(\mu, 0)$ such that $\mathcal{C} \cap \mathcal{N} \equiv \mathcal{C}^+ \cup \mathcal{C}^-$ where $\mathcal{C}^+, \mathcal{C}^-$ are subcontinua of $\mathcal{C}$ which meet only at $(\mu, 0)$. Moreover each of $\mathcal{C}^+, \mathcal{C}^-$ have extensions $\hat{\mathcal{C}}^+, \hat{\mathcal{C}}^-$ to the complement of $\mathcal{N}$ although possibly $\hat{\mathcal{C}}^+$ and $\hat{\mathcal{C}}^-$ meet outside of $\mathcal{N}$.

We conclude this section with a version of a constructive result that tells us more about the structure of $\mathcal{S}$ near $(\mu, 0)$ if $\mu \in r(L)$ is simple and slightly more is assumed for H. Suppose there exists a neighborhood U of $(\mu, 0)$ and a continuous monotonic function $K: \mathbf{R}^+ \to \mathbf{R}^+$ such that $K(0) = 0$ and, for all $(\lambda, u), (\nu, w) \in U$,

$$(1.18)\quad \begin{aligned} &\|H(\lambda, u) - H(\nu, w)\| \\ &\qquad \leqq K(\|u\| + \|w\|)[\,\|u - w\| + (\|u\| + \|w\|)|\lambda - \nu|\,]. \end{aligned}$$

Then $\mathcal{S}$ near $(\mu, 0)$ will consist of a continuous curve of solutions passing through $(\mu, 0)$. Condition (1.18) is satisfied e.g. if $H(\lambda, u) = \lambda N(u)$ where N is continuously Fréchet differentiable near $u = 0$. For what follows if A and B are topological spaces, let $C(A, B)$ denote the set of continuous maps from A into B. A precise statement of our result is then

THEOREM 1.19. *Suppose H satisfies* (1.18) *and $\mu \in r(L)$ is simple. If v is an eigenvector of L corresponding to μ and $\hat{E}$ is any complement in E of span $\{v\}$, there exists an interval $[-\bar{\alpha}, \bar{\alpha}]$, a pair of functions $\rho \in C([-\bar{\alpha}, \bar{\alpha}], R)$, $w \in C([-\bar{\alpha}, \bar{\alpha}], \hat{E})$ such that $\rho(0) = 0$, $w(0) = 0$, and $(\lambda(\alpha), u(\alpha)) \equiv (\mu + \rho(\alpha), \alpha(v + w(\alpha)))$ satisfies* (1.1) *for $|\alpha| \leqq \bar{\alpha}$. Moreover if $(\hat{\lambda}, \hat{u})$ satisfies* (1.1) *and lies near $(\mu, 0)$, either $(\hat{\lambda}, \hat{u}) = (\lambda(\alpha), u(\alpha))$ for some $|\alpha| \leqq \bar{\alpha}$ or $\hat{u} = 0$.*

PROOF. First the existence of a solution of the above form is established and then the uniqueness assertion is proved. The theory of compact linear operators implies $I - \mu L$ is an isomorphism from $\hat{E}$ to $(I - \mu L)E \equiv F$ and F is closed and of codimension one. Since F is a closed hyperplane in E, $F = \{e \in E \mid \langle l, e\rangle = 0\}$ for some $l \in E'$ where $\langle \cdot, \cdot \rangle$ denotes the duality between E' and E. Observe that $v \notin F$ since $(I - \mu L)e = v$ implies $(I - \mu L)^2 e = 0$ violating the simplicity assumption for $\mu \in r(L)$. Thus it can be assumed that $\langle l, v\rangle = 1$ and $\|v\| = 1$.

We try for a solution of (1.1) of the form $\lambda = \mu + \rho(\alpha)$, $u = \alpha(v + w(\alpha))$ where ρ, w are as in the statement of Theorem 1.19. Substituting into (1.1) leads to

$$w = \rho v/\mu + (\mu + \rho)Lw + \alpha^{-1}H(\mu + \rho, \alpha(v + w)). \tag{1.20}$$

Equation (1.20) will be solved for $\bar{\alpha}$ small enough via the contracting mapping theorem. Letting $\bar{\alpha}$ be free for the moment, a mapping is defined as follows: Let $\rho \in C([-\bar{\alpha}, \bar{\alpha}], R)$, $w \in C([-\bar{\alpha}, \bar{\alpha}], \hat{E})$ with $\rho(0) = 0$, $w(0) = 0$ and set

$$\hat{\rho}(\alpha) = -\mu\rho(\alpha)\langle l, Lw\rangle - \alpha^{-1}\mu\langle l, H(\mu + \rho(\alpha), \alpha(v + w(\alpha)))\rangle, \tag{1.21}$$

$$\begin{aligned} \hat{w}(\alpha) = (I - \mu L)^{-1}[\hat{\rho}(\alpha)\mu^{-1}v + \rho(\alpha)Lw \\ + \alpha^{-1} H(\mu + \rho(\alpha), \alpha(v + w(\alpha)))]. \end{aligned} \tag{1.22}$$

Note that (1.21) implies the term in brackets on the right-hand side of (1.22) lies in F and therefore the right-hand side of (1.22) is well defined since $(I - \mu L)^{-1}$ is an isomorphism from F to $\hat{E}$. Equations (1.21)–(1.22) imply $\hat{\rho}(0) = 0$, $\hat{w}(0) = 0$. Finally observe that a fixed point of the mapping $(\rho, w) \to (\hat{\rho}, \hat{w})$ is a solution of (1.20).

To find such a fixed point via the contracting mapping theorem it suffices to show there exist constants $a, b > 0$ such that for $\bar{\alpha}$ sufficiently small: (i) $\max_{[-\bar{\alpha},\bar{\alpha}]}|\rho(\alpha)| \leqq a$ implies $\max_{[-\bar{\alpha},\bar{\alpha}]}|\hat{\rho}(\alpha)| \leqq a$, (ii) $\max_{[-\bar{\alpha},\bar{\alpha}]}\|w(\alpha)\| \leqq b$ implies $\max_{[-\bar{\alpha},\bar{\alpha}]}\|\hat{w}(\alpha)\| \leqq b$ and

$$\max_{[-\bar{\alpha},\bar{\alpha}]} \|\hat{w}(\alpha) - \hat{z}(\alpha)\| + \max_{[-\bar{\alpha},\bar{\alpha}]} \|\hat{\rho}(\alpha) - \hat{\zeta}(\alpha)\|$$
$$\leqq \frac{3}{4} \max_{[-\bar{\alpha},\bar{\alpha}]} \|w(\alpha) - z(\alpha)\| + \frac{3}{4} \max_{[-\bar{\alpha},\bar{\alpha}]} \|\rho(\alpha) - \zeta(\alpha)\|.$$

Choose a, b, and $\bar{\alpha}$ such that

$$\begin{aligned} b &\leqq \min \{1, [(1 + |\mu|^{-1}\|(I - \mu L)^{-1}\|)(8|\mu|\,\|l\|\,\|L\|)]^{-1}, \\ &\qquad (8\|L\|\,\|(I - \mu L)^{-1}\|)^{-1}\}, \\ a &\leqq \min \{1, (2|\mu|\,\|l\|\,\|L\|)^{-1}, \\ &\qquad (3\|(I - \mu L)^{-1}\|(|\mu|^{-1} + \|L\|))^{-1}b, b\}, \\ K(4\bar{\alpha}) &\leqq \min \{[8(|\mu|\,\|l\| + \|(I - \mu L)^{-1}\|)(1 + |\mu|)]^{-1}, \\ &\qquad [32(1 + |\mu|^{-1}\|(I - \mu L)^{-1}\|)|\mu|\,\|l\|]^{-1}, \\ &\qquad (32\|(I - \mu L)^{-1}\|)^{-1}\}. \end{aligned} \tag{1.23}$$

Then (1.21) implies

$$|\hat{\rho}(\alpha)| \leqq |\mu|\,\|l\|\,\|L\|\,|\rho|\,\|w\| + |\mu|\,\|l\|(4 + 2|\mu|)K(2\bar{\alpha}) \leqq a. \tag{1.24}$$

Hence (i) is satisfied. Next from (1.22), (1.23),

$$\begin{aligned} &\|\hat{w}(\alpha)\| \\ &\leqq \|(I - \mu L)^{-1}\|\,[a|\mu|^{-1} + \|L\|ab + (4 + 2|\mu|)K(2\bar{\alpha})] \leqq b. \end{aligned} \tag{1.25}$$

Lastly (1.21) and (1.23) lead to

$$\begin{aligned} &\|\hat{\rho}(\alpha) - \hat{\zeta}(\alpha)\| \leqq \min (4^{-1}, 4^{-1}|\mu|\,\|(I - \mu L)^{-1}\|^{-1}) \\ &\qquad \cdot \left[\max_{[-\alpha,\alpha]} |\rho(\alpha) - \zeta(\alpha)| + \max_{[-\bar{\alpha},\bar{\alpha}]} \|w(\alpha) - z(\alpha)\| \right] \end{aligned} \tag{1.26}$$

which when combined with (1.22)–(1.23) yield (iii). Thus the contracting mapping theorem gives a solution $(\lambda(\alpha), u(\alpha))$ of the form claimed in Theorem 1.19.

It remains only to prove the uniqueness assertion. Actually the contracting mapping theorem implies the uniqueness of the solutions we have obtained within the class of functions satisfying our Ansatz for the solution. Moreover it implies a pointwise uniqueness assertion, i.e., for each $\alpha \in [-\bar{\alpha}, \bar{\alpha}]$ there exists a unique solution of the form $\lambda = \mu + \rho(\alpha)$, $u = \alpha(v + w(\alpha))$ with $|\rho(\alpha)| \leqq a$, $\|w(\alpha)\| \leqq b$. Thus to complete the uniqueness proof it suffices to show there exists a neighborhood V of $(\mu, 0)$ and a continuous nondecreasing function g defined on $\boldsymbol{R}^+$ near $x = 0$ with $g(0) = 0$ such that if $(\hat{\lambda}, \hat{u}) \in V$ and satisfy (1.1), then

$$\|\hat{w}\| + |\alpha|\,|\hat{\rho}| \leqq |\alpha| g(|\alpha|) \tag{1.27}$$

where $\hat{\rho} = \hat{\lambda} - \mu$, $\hat{u} = \alpha v + \hat{w}$, $\hat{w} \in \hat{E}$, and $\alpha = \langle l, \hat{u} - \hat{w} \rangle$.

Since $(\hat{\lambda}, \hat{u})$ satisfy (1.1) as in (1.20),

$$(I - \mu L)\hat{w} = \hat{\rho}\mu^{-1}\alpha v + \hat{\rho} L \hat{w} + H(\mu + \hat{\rho}, \alpha v + \hat{w}). \tag{1.28}$$

Note that the right-hand side of (1.28) belongs to F. Therefore

$$\begin{aligned} \|\hat{w}\| \leqq \|(I - \mu L)^{-1}\| [\, |\hat{\rho}|\, |\mu|^{-1} |\alpha| + \|L\|\, |\hat{\rho}|\, \|\hat{w}\| \\ + K(|\alpha| + \|\hat{w}\|) \cdot (|\alpha| + \|\hat{w}\|) \cdot (1 + 2|\mu|)]. \end{aligned} \tag{1.29}$$

It can be assumed that $\|(I - \mu L)^{-1}\|\, \|L\|\, |\hat{\rho}| < \frac{1}{3}$ and since $|\alpha| + \|\hat{w}\|$ is small when $\|\hat{u}\|$ is small, that $K(|\alpha| + \|\hat{w}\|)\|(I - \mu L)^{-1}\| < 1/3(1 + 2|\mu|)$. Then (1.29) implies

$$\|\hat{w}\| \leqq (c_1 |\hat{\rho}| + c_2 K(|\alpha| + \|\hat{w}\|))|\alpha| \tag{1.30}$$

where c_1 and c_2 are constants. Hence if $\alpha = 0$, $\hat{w} = 0 = \hat{u}$. Thus near $(\mu, 0)$ only the trivial solutions have $\alpha = 0$ so for what follows we can assume $\alpha \neq 0$. Moreover from (1.30) we can write $\hat{w} = \alpha \tilde{w}(\alpha)$ where $\tilde{w}(\alpha)$ is a bounded function of α for $0 < |\alpha|$ small. Suppose $\|\tilde{w}(\alpha)\| \leqq M$. Then (1.30) becomes

$$\|\hat{w}\| \leqq c_1 |\hat{\rho}|\, |\alpha| + c_2 K((M + 1)|\alpha|)|\alpha| \tag{1.31}$$

where the second term on the right of (1.31) is $O(|\alpha|)$. Operating on (1.28) with l yields

$$0 = \hat{\rho}\alpha\mu^{-1} + \hat{\rho}\langle l, L\hat{w}\rangle + \langle l, H(\mu + \hat{\rho}, \alpha v + \hat{w})\rangle. \tag{1.32}$$

Using the form obtained for $\hat{w}$ in (1.32),

$$-\hat{\rho} = \mu\hat{\rho}\langle l, L\tilde{w}\rangle + \alpha^{-1}\mu\langle l, H(\mu + \hat{\rho}, \alpha(v + \tilde{w}))\rangle. \tag{1.33}$$

Hence if $|\hat{\rho}| < \min\, [1, (2|\mu|\, \|l\|\, \|L\| c_1)^{-1}]$, $c_2 |\mu|\, \|l\|\, \|L\| K((M + 1)\, |\alpha|)$ is small enough from (1.31), (1.33),

$$|\hat{\rho}| \leqq c_3 K((M + 1)|\alpha|). \tag{1.34}$$

Finally (1.34) combined with (1.31) gives (1.27) and uniqueness is proved.

REMARKS. If E is a Hilbert space, a natural choice for $\hat{E}$ is the orthogonal complement of span v. Note that the compactness of H played no role in the proof of Theorem 1.19. The proof given above is based on a more general result in [12] where neither L nor H

need be compact and one begins with a curve rather than a line of known solutions. Since the proof in [12] uses the implicit function theorem slightly more smoothness for H was required there. If H is analytic, the analytic version of the implicit function theorem shows ρ and w are analytic functions of α. One may then determine ρ and w by expanding in a power series in α and obtain the coefficients by solving linear problems with appropriate "orthogonality" conditions as is often done in practice.

A natural question to ask now having obtained $\mathcal{C}$ near $(\mu, 0)$ is whether one can repeatedly use the above argument or the implicit function theorem to constructively obtain all of $\mathcal{C}$. In general this approach will not work but it has been carried out successfully in some special cases involving positive solutions [13], [14], [15].

2. **Applications of Theorem 1.10.** Perhaps the nicest applications of Theorem 1.10 are to nonlinear eigenvalue problems for second order ordinary differential equations. Consider

$$\mathcal{L}u \equiv -(p(x)u')' + q(x)u = \lambda F(x, u, u'), \qquad 0 < x < \pi, \tag{2.1}$$

together with the separated boundary conditions (B.C.)

$$a_0 u(0) + b_0 u'(0) = 0, \qquad a_1 u(\pi) + b_1 u'(\pi) = 0 \tag{2.2}$$

and $(a_0{}^2 + b_0{}^2)(a_1{}^2 + b_1{}^2) \neq 0$. The function $F(x, \xi, \eta)$ is assumed to be continuous in its arguments and $F(x, \xi, \eta) = a(x)\xi + o((\xi^2 + \eta^2)^{1/2})$ near $(\xi, \eta) = (0, 0)$. As is usual we assume p is continuously differentiable and positive, q is continuous, and a is continuous and positive on $[0, \pi]$. Associated with (2.1)–(2.2) is the linear eigenvalue problem

$$\mathcal{L}v = \mu a v, \qquad 0 < x < \pi, \tag{2.3}$$

together with the B.C. (2.2). This problem possesses an increasing sequence of simple eigenvalues $\mu_1 < \mu_2 < \cdots$ with $\mu_n \to \infty$ as $n \to \infty$ [16]. Any eigenfunction v_n corresponding to μ_n has precisely $n - 1$ simple zeros in $(0, \pi)$. For technical convenience it will be assumed that 0 is not an eigenvalue of (2.3)–(2.2). (The general case can be handled by an approximation argument.)

By using the Green's function $g(x, y)$ for $\mathcal{L}$ with B.C., (2.1)–(2.2) can be converted to the equivalent integral equation

$$\begin{aligned} u(x) &= \lambda \int_0^\pi g(x, y) F(y, u(y), u'(y))\, dy \\ &\equiv G(\lambda, u) \equiv \lambda L u + H(\lambda, u) \end{aligned} \tag{2.4}$$

where $Lu = \int_0^\pi g(x, y)a(y)u(y)\, dy$.

Let $E = C^1[0, \pi] \cap$ B.C. under the usual maximum norm

$$\|u\|_1 = \max_{[0,\pi]} |u(x)| + \max_{[0,\pi]} |u'(x)|.$$

It is easy to see that $G: R \times E \to E$ is compact as is $L: E \to E$. Moreover the assumption on F near $(\xi, \eta) = (0, 0)$ implies $H = o(\|u\|_1)$ near $u = 0$ uniformly on bounded λ intervals. The eigenvalues of $\mathcal{L}$ are equal to the characteristic values of L. Hence all $\mu_k \in r(L)$ satisfy the hypotheses of Theorem 1.10 and accordingly there exists a component $\mathcal{C}_k$ of $\mathcal{S}$ which meets $(\mu_k, 0)$ and is either unbounded in $R \times E$ or meets $(\mu_j, 0)$ where $j \neq k$. Actually only the first alternative is possible as shall be shown next.

Let S_k^+ denote the set of $\varphi \in E$ such that φ has exactly $k - 1$ simple zeros in $(0, \pi)$, $\varphi > 0$ near $\lambda = 0$, and all zeros of φ in $[0, \pi]$ are simple. Let $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. Then the sets $S_k^\pm$, S_k are open in E and $v_k \in S_k$. The normalization $\|v_k\|_1 = 1$ and $v_k \in S_k^+$ makes v_k unique.

THEOREM 2.5. *$\mathcal{C}_k$ is unbounded in $R \times S_k$.*

Two lemmas are required to prove Theorem 2.5.

LEMMA 2.6. *If (λ, u) is a solution of (2.1) and u has a double zero (i.e., $u(\tau) = 0 = u'(\tau)$ for some $\tau \in [0, \pi]$), then $u \equiv 0$.*

PROOF. If $F(x, \xi, \eta)$ is Lipschitz continuous with respect to ξ, η, the result follows from the basic uniqueness theorem for the initial value problem for ordinary differential equation. The general case is also fairly simple and we omit the proof. See e.g. [9].

LEMMA 2.7. *For each $j > 0$ there exists a neighborhood $\mathcal{N}_j$ of $(\mu_j, 0)$ such that $(\lambda, u) \in \mathcal{N}_j \cap \mathcal{S}$ and $u \not\equiv 0$ implies $u \in S_j$.*

PROOF. If not, there exists a sequence $(\lambda_n, u_n) \in \mathcal{S}$ such that $0 \not\equiv u_n \in S_j$ and $(\lambda_n, u_n) \to (\mu_j, 0)$. Writing (2.4) as

$$w_n \equiv u_n/\|u_n\|_1 = \lambda_n L w_n + H(\lambda_n, u_n)/\|u_n\|_1, \tag{2.8}$$

it follows that the second term on the right $\to 0$ as $n \to \infty$. Since L is compact, a subsequence of Lw_n converges. Hence the left-hand side of (2.8) has a convergent subsequence $w_{n_i} \to w$ with $\|w\|_1 = 1$ and satisfying

$$w = \mu_j L w. \tag{2.9}$$

Consequently $w = v_j$ or $w = -v_j$. In any event $w \in S_j$. Since this set is open, w_{n_i} and therefore $u_{n_i} \in S_j$ for u_i large, contrary to assumption. The lemma is proved.

PROOF OF THEOREM 2.5. Suppose $\mathcal{C}_k \subset (\boldsymbol{R} \times S_k) \cup \{(\mu_k, 0)\}$. Then since $S_j \cap S_k = \emptyset$ for $j \neq k$, it follows from Lemma 2.7 and Theorem 1.10 that $\mathcal{C}_k$ must be unbounded in $\boldsymbol{R} \times S_k$. Hence Theorem 2.5 will be established once we show $\mathcal{C}_k \not\subset (\boldsymbol{R} \times S_k) \cup \{(\mu_k, 0)\}$ is impossible. By Lemma 2.7, $\mathcal{C}_k \cap \mathcal{N}_k \subset (\boldsymbol{R} \times \mathcal{C}_k) \cup \{(\mu_k, 0)\}$. Hence if $\mathcal{C}_k \not\subset (\boldsymbol{R} \times S_k) \cup \{(\mu_k, 0)\}$ there exists $(\lambda, u) \in \mathcal{C}_k \cap (\boldsymbol{R} \times \partial S_k)$ with $(\lambda, u) \neq (\mu_k, 0)$ and $(\lambda, u) = \lim_{n\to\infty}(\lambda_n, u_n)$, $u_n \in S_k$. If $u \in \partial S_k$, by Lemma 2.6, $u \equiv 0$. Hence $\lambda = \mu_j$, $j \neq k$. But then, by Lemma 2.7, $(\lambda_n, u_n) \in \mathcal{N}_j \cap (\boldsymbol{R} \times S_k)$ for n large which is impossible and the proof is complete.

REMARK. By using the stronger version of Theorem 1.10 mentioned after the proof of Theorem 1.10 and valid when $\mu \in r(L)$ is simple, Theorem 2.5 can be improved to read: $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$ where $\mathcal{C}_k^\pm$ is unbounded in $\boldsymbol{R} \times S_k^\pm$.

An instructive example of the possible behavior of the sets $\mathcal{C}_k$ is given by the equation

$$(2.10) \qquad -u'' = \lambda(1 + f(u^2 + (u')^2, \lambda))u, \; 0 < x < \pi, u(0) = 0 = u(\pi),$$

where f is continuous and $f(0, \lambda) \equiv 0$. The related linear problem is

$$(2.11) \qquad -v'' = \mu v, \qquad 0 < x < \pi, v(0) = 0 = v(\pi),$$

which possesses eigenvalues $\mu_k = k^2$ and v_k a multiple of $\sin kx$. Trying for a solution of the form $(\lambda, u) = (\lambda, c \sin x)$ leads to the relation

$$(2.12) \qquad 1 = \lambda(1 + f(c^2, \lambda))$$

which on varying f gives us an idea of the possible structure of $\mathcal{C}_1$.

As an application of Theorem 2.5 we will prove a generalization of a theorem of Nehari [17]. Consider the equation

$$(2.13) \qquad \mathcal{L}u = f(x, u)u, \qquad 0 < x < \pi, u(0) = 0 = u(\pi),$$

where $\mathcal{L}$ is as in (2.1) with $q \geqq 0$, f is continuous in its arguments, $f(x, 0) \equiv 0$, and $f(x, \xi) \to \infty$ uniformly in x as $|\xi| \to \infty$. Note that (2.13) is not of the form (2.1) since the right-hand side of (2.13) has Fréchet derivative 0 at $u = 0$. Under the above hypotheses we have

THEOREM 2.14. *For each $k \in N$, there exists $u_k \in S_k$ satisfying* (2.13).

PROOF. Since $f(x, \xi) \to \infty$ as $|\xi| \to \infty$ uniformly in x, there exists a constant $M > 0$ such that $M + f(x, \xi) \geqq 1$ for all $x \in [0, \pi]$ and $\xi \in \boldsymbol{R}$. Consider the equation

$$\mathcal{L}u + Mu = \lambda(M + f(x, u))u, \quad 0 < x < \pi, u(0) = 0 = u(\pi). \tag{2.15}$$

Theorem 2.14 is obtained by showing that for all $k \in \boldsymbol{N}$, (2.15) possesses a solution $(1, u_k)$, $u_k \in S_k$. Linearizing (2.15) about $u = 0$ yields the linear eigenvalue problem

$$\mathcal{L}v + Mv = \mu Mv, \qquad 0 < x < \pi, v(0) = 0 = v(\pi). \tag{2.16}$$

Since $q \geqq 0$, it is easy to see e.g. using the maximum principle or the variational characterization of the smallest eigenvalue μ_1 of (2.16) that $\mu_1 > 1$. Let μ_k denote the kth eigenvalue of (2.16). By Theorem 2.5, (2.15) possesses an unbounded component of solutions $\mathcal{C}_k \subset (\boldsymbol{R} \times S_k) \cup \{(\mu_k, 0)\}$. If $\mathcal{C}_k \cap (\{1\} \times E) \neq \emptyset$, then there exists $(1, u_k) \in \mathcal{C}_k \cap (\{1\} \times S_k)$ and Theorem 2.14 will be proved. The following two lemmas constrain $\mathcal{C}_k$ to intersect $\{1\} \times E$.

LEMMA 2.17. *There is a constant $\lambda_k > 0$ such that if (λ, u) is a solution of (2.15) and $\lambda > 0$, $u \in S_k$, then $\lambda \leqq \lambda_k$.*

PROOF. Since $u \in S_k$, λ is the kth eigenvalue and u a kth eigenfunction of the linear eigenvalue problem

$$(\mathcal{L} + M)w = \nu(M + f(x, u))w, \ 0 < x < \pi, w(0) = 0 = w(\pi). \tag{2.18}$$

Hence λ can be characterized as [18]:

$$\lambda = \inf_{w \in \mathcal{A}} \frac{\int_0^\pi (pw'^2 + \bar{q}w^2)\, dx}{\int_0^\pi \bar{f} w^2\, dx}, \tag{2.19}$$

where $\bar{q}(x) = q(x) + M$, $\bar{f}(x) = f(x, u(x)) + M$, and $\mathcal{A} = \{w \in E \mid w \not\equiv 0, \int_0^\pi \bar{f}(x) w w_i\, dx = 1, 1 \leqq i \leqq k - 1\}$ where w_i is an ith eigenfunction of (2.18), $1 \leqq i \leqq k - 1$. By our choice of M,

$$\lambda \leqq \inf_{w \in \mathcal{A}} \frac{\|p\| \int_0^\pi (w')^2\, dx + \|\bar{q}\| \int_0^\pi w^2\, dx}{\int_0^\pi w^2\, dx}, \tag{2.20}$$

where $\|\varphi\| = \max_{[0,\pi]} |\varphi(x)|$. Let λ_j denote the jth eigenvalue and φ_j a corresponding eigenfunction of

$$-\|p\|\varphi'' + \|\bar{q}\|\varphi = \lambda\varphi, \qquad 0 < x < \pi, \varphi(0) = 0 = \varphi(\pi), \tag{2.21}$$

and let $w = \sum_1^k c_j \varphi_j$. Then the c_r can be chosen so that $w \in \mathcal{A}$. Substituting w into (2.20) shows $\lambda \leqq \lambda_j$ and the lemma is proved.

LEMMA 2.22. *There is a constant* $M_k > 0$ *such that if* $(\lambda, u) \in [1, \lambda_k] \times S_k$ *is a solution of* (2.15), *then* $\|u\|_1 \leqq M_k$.

PROOF. If not, there is a sequence (ρ_n, w_n) of solutions of (2.15) with $\rho_n \in [1, \lambda_k]$, $w_n \in S_k$, and $\|w_n\|_1 \to \infty$ as $n \to \infty$. Let $z_{1,n}, \cdots, z_{k-1,n}$ denote the zeros of w_n in $(0, \pi)$. At least one subinterval $(z_{j,n}, z_{j+1,n}) \equiv I_n$ is of length at least π/k. The idea now is to show the following: (A) $\{\max_{I_n} |w_n|\}$ is an unbounded sequence. (B) There is a subinterval $J_n \subset I_n$, with J_n having length at least $\pi/2k$, in which $\min_{J_n} |w_n| \geqq K_n$ where $K_n \to \infty$ as $n \to \infty$. (C) For all n large enough, w_n must vanish at some point in J_n. Since (B) and (C) are incompatible, the lemma will be established.

(A) Suppose $\max_{I_n} |w_n|$ is bounded uniformly for some subsequence of n's. Clearly w_n' has at least one zero y_n in I_n. Integrating (2.15), for any $\zeta \in I_n$,

$$p(\zeta)w'(\zeta) = \int_y^\zeta (\bar{q} - \rho\bar{f})w\, dx, \tag{2.23}$$

where we have dropped subscripts. Hence (2.23) implies $\max_{I_n} |w_n'|$ is bounded uniformly in n along this subsequence. This implies $\{\|w_n\|_1\}$ is bounded uniformly in n along this subsequence contrary to our hypotheses on $\{\|w_n\|_1\}$. To see this, it suffices to show the uniform boundedness in n along this subsequence of $\max_{Q_n} |w_n|$ where $Q_n = (z_{j-1,n}, z_{j,n})$ or $(z_{j+1,n}, z_{j+2,n})$. (In $k-1$ steps this procedure shows $\{\|w_n\|_1\}$ are uniformly bounded.) Suppose $w_n \leqq 0$ in I_n and Q_n lies to the right of I_n. Other cases are handled similarly.

Note that $w \geqq 0$ in Q_n. Dropping subscripts again, let z denote the left endpoint of Q and let y be any point in Q at which w achieves its maximum. If $w''(y) = 0$, then, from (2.15), $\rho\bar{f}(y, w(y)) - \bar{q}(y) = 0$ which by the properties of f implies that $w(y) = \max_Q w \leqq K$ where K is a constant independent of n. Thus assume $w''(y) \neq 0$ so w has an isolated maximum at y. Let (i) $\tilde{y}$ denote the closest zero of w' in Q to y which lies to the left of y if such a point exists or (ii) $\tilde{y} = z$ if w' has no zeros in (z, y).

If (ii) occurs, $w' > 0$ in (z, y). From (2.15) for $\zeta \in [z, y]$,

$$\begin{aligned} p(\zeta)w'(\zeta) &\leqq p(z)w'(z) + \int_z^\zeta \bar{q}w\, dx \\ &\leqq \|p\|w'(z) + (\zeta - z)\|\bar{q}\| \max_{[z,\zeta]} w. \end{aligned} \tag{2.24}$$

Since $\max_{[z,\zeta]} w \leqq (\zeta - z) \max_{[z,\zeta]} w'$, by choosing ζ such that $(\zeta - z)^2\|q\| \leqq 2^{-1}$, we obtain from (1.24) an upper bound for

$\max_{[z,\zeta]} w'$ and $\max_{[z,\zeta]} w$ in terms of $\|p\|$ and $w'(z)$ (which is known in terms of $\max_I |w|$). Continuing this argument (slightly modified since w will no longer vanish at the left boundary) for finitely many steps, an estimate can be obtained for $\|w\|_1$ over Q_n.

If (i) occurs, a similar argument to (ii) is carried out using (2.24) except that the $\rho \bar{f}$ term will enter into the estimates but only in the region in which $\bar{f} \leqq 0$ and therefore w is bounded independently of n. We omit the details. Thus we can assume $\{\max_{I_n} |w_n|\}$ is an unbounded sequence.

(B) Dropping subscripts again, consider w in $I \equiv [a, b]$. We modify an argument used by Wolkowisky [19] in a related context. For convenience suppose $w \geqq 0$ in I; $w \leqq 0$ is handled similarly. From (2.15), $(\mathcal{L} + M)w \geqq 0$ in I and $w = 0$ on ∂I. Let $z \in (a, b)$ and define a new function $v_z \equiv v$ satisfying

$$(2.25) \qquad \begin{aligned} (\mathcal{L} + M)v &= 0, \qquad a < x < z, z < x < b, \\ v(a) = 0 &= v(b), \qquad v(z) = w(z). \end{aligned}$$

Let $U = u - v$. Then $(\mathcal{L} + M)U \geqq 0$ in (a, z) and (z, b). Since U cannot have a negative minimum, $w \geqq v$ in these intervals. The same argument shows $v \geqq 0$ in (a, z), (z, b) and the maximum of v occurs at z.

Choose z such that $\max_I w = w(z) = v(z)$. We can write $v(x) = w(z)\hat{v}(x)$ where $\hat{v}$ satisfies the equation (2.25) in (a, z), $\hat{v}(a) = 0$ and $\hat{v}(z) = 1$. Similarly $v(x) = w(z)\check{v}(x)$ in (z, b) where $\check{v}$ satisfies the equation (2.25) in (z, b) and $\check{v}(b) = 0$, $\check{v}(z) = 1$. Note that $\hat{v}$, $\check{v}$ are independent of w but do depend on the subinterval I and on $z \in I$. However since the length of I is not less than π/k and the coefficients of $\mathcal{L}$ are smooth, it is not difficult to see that $\hat{v}'(a)$, $\check{v}'(b)$ are bounded away from 0 and ∞ uniformly over all such subintervals $I \subset [0, \pi]$ and all $z \in I$.

The equation (2.25) implies pv' is an increasing function of x for $x \in (a, z), (z, b)$. Hence $v'(x) \geqq \alpha w(z)\hat{v}'(a)$ for $x \in [a, z]$ and $v'(x) \leqq \alpha w(z)\check{v}'(b)$ for $x \in [z, b]$ where $\alpha = \|p\|^{-1} \min_{[0,\pi]} p$. It therefore follows that the triangle formed by the lines $y = 0$, $y = \alpha w(z)\hat{v}'(a)(x - a)$, $y = \alpha w(z)\check{v}'(b)(x - b)$ lies beneath $w(x)$ for $x \in I$. The ordinate of the point on the triangle with largest ordinate is linearly proportional to $w(z)$ and $\to \infty$ as $w(z) \to \infty$ independently of I and z. The x interval J_n obtained by projecting the line joining the midpoints of the triangle (not on the x axis) on the x axis has length not less than $\pi/2k$. For $x \in J_n$, $w_n(x) \to \infty$ as $n \to \infty$ uniformly in n and x.

(C) Since the length of J_n is not less than $\pi/2k$, it is easy to see, e.g. by its variational characterization, that the smallest eigenvalue β_1 of

$$(2.26) \qquad -(pu')' = \nu u, \qquad x \in J_n; \qquad u = 0, \qquad x \in \partial J_n,$$

is bounded from above independent of n and the position of J_n. By (B) for n large enough, $\rho_n \bar{f}(x, w_n) - \bar{q} > \beta_1$ on J_n. The Sturm comparison theorem thus implies w_n vanishes in J_n which is impossible. Thus Lemma 2.22 is proved.

Remark. Lemma 2.22 can be restated to permit $\lambda \in (0, \lambda_k)$ but then M_k will have to depend on λ. Also in Theorem 2.14 at the expense of some obvious qualifications in the statement of the theorem, the condition that $q \geqq 0$ can be dropped. Likewise f can be replaced by λf. A simpler version of Theorem 2.14 is contained in [20]. A more general version has been obtained by R. Turner [21] who also permits some u' dependence for f.

The methods used to obtain Theorem 2.5 can also be applied to nonlinear eigenvalue problems for a family of nonlinear integral equations possessing oscillation kernels [8]. The properties of these kernels imply the related linear eigenvalue problem possesses simple characteristic values and the corresponding eigenfunctions are characterized by the number of nodes they possess. Rather than pursue this however we conclude this section with some applications to quasilinear elliptic partial differential equations [8]. Here the only analogue of the nodal properties used earlier is positivity and this will be exploited below.

Consider

$$(2.27) \qquad \begin{cases} Nu \equiv -\displaystyle\sum_{i,j=1}^{n} a_{ij}(x, u, Du) u_{x_i x_j} & \\ \quad = \lambda(a(x)u + F(x, u, \lambda)), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^n with smooth boundary; $x = (x_1, \cdots, x_n) \in R^n$; Du denotes first derivatives of u; a_{ij}, a, and F are continuously differentiable functions of their arguments; $a(x) \geqq a_0$, a positive constant;

$$(2.28) \qquad \sum_{i,j=1}^{n} a_{ij}(x, u, p)\xi_i \xi_j \geqq \beta |\xi|^2, \qquad x \in \Omega,\ \xi, p \in R^n, u \in R,$$

with $\beta > 0$ a constant; and $F(x, u, \lambda) = o(|u|)$ near $u = 0$ uniformly on bounded λ intervals. Condition (2.28) means that the differential operator in (2.27) is uniformly elliptic for all choices of its arguments.

N could also be permitted to depend on λ and lower order terms and F on Du.

Note that (2.27) possesses the line of trivial solutions. We will establish the existence of "positive" solutions of (2.27), i.e., (λ, u) such that $\lambda \in R^+$ and $u > 0$ (or $u < 0$) in Ω. To do this (2.27) is reformulated in a standard fashion as an equivalent operator equation in a Banach space. Let $\alpha \in (0,1)$ and let $C^{k+\alpha}(\Omega)$ denote the class of k times continuously differentiable functions in $\bar{\Omega}$ whose kth derivatives are Hölder continuous with exponent α. $C^{k+\alpha}(\Omega)$ is a Banach space under

$$\|u\|_{k+\alpha} = \sum_{|\sigma| \leq k} \max_{x \in \Omega} |D^\sigma u(x)| + \sum_{|\sigma| = k} \max_{x \neq y \in \Omega} \frac{|D^\sigma u(x) - D^\sigma u(y)|}{|x-y|^\alpha}$$

where the usual multi-index notation is being employed. According to the linear Schauder theory [18, Volume 2],

Lemma 2.29. *If* $\sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \geqq \beta|\xi|^2$ *for all* $x \in \Omega$, $\xi \in R^n$ *and if* $A_{ij}, f \in C^\alpha(\Omega)$, *then the equation*

$$(2.30) \qquad \begin{cases} -\displaystyle\sum_{i,j=1}^n A_{ij}(x) u_{x_i x_j} = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

possesses a unique solution $u \in C^{2+\alpha}(\Omega)$ *and*

$$(2.31) \qquad \|u\|_{2+\alpha} \leqq c\|f\|_\alpha$$

where c is a constant independent of u and f.

With the aid of Lemma 2.29, (2.27) can be converted to an operator equation in $E = C^{1+\alpha} \cap$ B.C. Let $(\lambda, u) \in R \times E$ and let $v = G(\lambda, u) \in C^{2+\alpha}(\Omega)$ denote the unique solution of

$$(2.32) \qquad \begin{cases} -\displaystyle\sum_{i,j=1}^n a_{ij}(x, u, Du) v_{x_i x_j} = \lambda(a(x)u + F(x,u)), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

Any solution of (2.27) satisfies $u = G(\lambda, u)$ and conversely. Lemma 2.29 implies that G is compact.

Let $w = Lu$ denote the unique solution of

$$(2.33)\qquad \begin{aligned} -\sum_{i,j=1}^{n} a_{ij}(x,0,0)w_{x_i x_j} &= a(x)u, \qquad && x \in \Omega, \\ w &= 0, && x \in \partial\Omega. \end{aligned}$$

Then $L : E \to E$ is linear and by Lemma 2.29 is compact. It is not too difficult to verify that $H(\lambda, u) \equiv G(\lambda, u) - \lambda Lu = o(\|u\|_{1+\alpha})$ near $u = 0$ uniformly on bounded λ intervals [9]. With the aid of the maximum principle for uniformly elliptic partial differential equations [18, Volume 2] and the Krein-Rutman theorem [22], it follows that the smallest in magnitude characteristic value μ_1 of L is positive and simple and possesses a corresponding eigenvector $v_1 > 0$ in Ω. Moreover v_1 normalized so that $\|v_1\|_{1+\alpha} = 1$ is the unique nonnegative eigenvector of L. Hence the hypotheses of Theorem 1.10 are satisfied here and there exists a component $\mathcal{C}_1$ of $\mathcal{S}$ satisfying the alternatives of that theorem.

As was the case with (2.1), a better result obtains here. Let $\partial/\partial\nu$ denote differentiation in the direction of the outward pointing normal to $\partial\Omega$; let $P^+ = \{u \in E \mid u > 0 \text{ in } \Omega,\ \partial u/\partial\nu < 0 \text{ on } \partial\Omega\}$, $P^- = -P^+$ and $P = P^+ \cup P^-$. Then P is open in E.

THEOREM 2.34. *$\mathcal{C}_1$ lies in $(\mathbf{R}^+ \times P) \cup \{(\mu_1, 0)\}$ and is unbounded.*

The proof of Theorem 2.34 follows the same lines as that of Theorem 2.5. Two preliminary lemmas are required.

LEMMA 2.35. *There exists a neighborhood, $\mathcal{N}_1$ of $(\mu_1, 0)$ such that $(\lambda, u) \in \mathcal{N}_1 \cap \mathcal{S}$ and $u \not\equiv 0$ implies that $u \in P$. Moreover if $\hat{\mu} \in r(L)$, $\hat{\mu} \neq \mu_1$, there exists a neighborhood $\hat{\mathcal{N}}$ of $(\hat{\mu}, 0)$ such that $(\hat{\lambda}, \hat{u}) \in \hat{\mathcal{N}} \cap \mathcal{S}$ and $\hat{u} \not\equiv 0$ implies $\hat{u} \in P$.*

PROOF. The proof of the first statement is the same as that of Lemma 2.7. If the second assertion is not true, as in Lemma 2.7 there exists a sequence $(\lambda_n, u_n) \to (\hat{\mu}, 0)$ such that $u_n \in P$ and $u_n\|u_n\|_{1+\alpha}^{-1} \to w$ satisfying

$$(2.36)\qquad w = \hat{\mu} L w.$$

Since $w \in \bar{P}$ and $\|w\|_{1+\alpha} = 1$, w or $-w = v$. But this is impossible since $\hat{\mu} \neq \mu$.

LEMMA 2.37. *Suppose $(\lambda, u) \in \mathcal{C}_1$, $(\lambda, u) \neq (\mu_1, 0)$ and $(\lambda, u) = \lim_{n\to\infty}(\lambda_n, u_n)$ where $(\lambda_n, u_n) \in (\mathbf{R}^+ \times P) \cap \mathcal{S}$. Then $(\lambda, u) \in (\mathbf{R}^+ \times P)$.*

PROOF. Observe first that $(\lambda, u) \in \mathcal{C}_1$ implies that $\lambda > 0$ for otherwise since $\mathcal{C}_1$ is connected, $(0, w) \in \mathcal{C}_1$ for some $w \in E$. Then w

satisfies (2.27) with the right-hand side equal to 0. Considering the coefficients $a_{ij}(x, w, Dw)$ to be known and invoking the uniqueness assertion of Lemma 2.29 it follows that $w \equiv 0$. But this is impossible since $(0, 0)$ is not a bifurcation point for an equation of the form (1.1). Hence $\lambda > 0$.

Next suppose that (λ, u) is as in the statement of Lemma 2.37 and $u \notin P$. Then $\lambda > 0$ and $u \in \partial P = \partial P^+ \cup \partial P^-$. The argument is the same for $u \in \partial P^+, \partial P^-$ so we assume the former. The definition of P^+ implies either (i) there exists $\xi \in \Omega$ such that $u(\xi) = 0$ or (ii) $\partial u(\eta)/\partial \nu = 0$ for some $\eta \in \partial\Omega$. Suppose that (i) occurs. Because of the form of F, there exists a neighborhood U of ξ such that $|F(x, u(x))| \leqq a_0 u(x)/2$ in U. Therefore, from (2.27),

$$(2.38) \qquad \begin{aligned} -\sum_{i,j=1}^{n} a_{ij}(x, u, Du)u_{x_i x_j} &\geqq 0, \qquad x \in U, \\ u &\geqq 0, \qquad x \in \partial U. \end{aligned}$$

Since u has a local minimum at $x = \xi$, the elliptic maximum principle [18] implies $u \equiv 0$ in U. A continuation argument then shows $u \equiv 0$ in Ω. Since $u = \lim_{n\to\infty} u_n$ with $u_n \in P^+$, by Lemma 2.35, $(\lambda, u) = (\mu_1, 0)$ contrary to hypothesis.

If (ii) occurs we argue similarly. There exists a neighborhood U of $\eta \in \partial\Omega$ in which (2.38) holds. Since u has a local minimum at η and $\partial u(\eta)/\partial\nu = 0$, the strong form of the maximum principle implies $u \equiv 0$ in U. A contradiction is then obtained as in (i) and the lemma is proved.

Proof of Theorem 2.34. If $\mathcal{C}_1$ does not lie in $(\boldsymbol{R}^+ \times P) \cup \{(\mu_1, 0)\}$, the connectedness of $\mathcal{C}_1$ and Lemma 2.35 imply there exists $(\lambda, u) \in \boldsymbol{R}^+ \times \partial P$ and satisfying the hypotheses of Lemma 2.37. But the conclusion of that lemma shows $(\lambda, u) \in \boldsymbol{R}^+ \times P$, a contradiction. Thus $\mathcal{C}_1 \subset (\boldsymbol{R}^+ \times P) \cup \{(\mu_1, 0)\}$ and, by Theorem 1.10, $\mathcal{C}_1$ is unbounded in this set.

3. **Genus and its properties.** Variational arguments have proved to be a powerful tool to treat a class of nonlinear eigenvalue problems which involve potential operators. In this context, solutions of (0.1) are obtained by extremizing a real valued functional on an appropriate manifold, u corresponding to a critical point of the functional and λ an associated 'Lagrange multiplier'.

Our goal here is to display in a very simple context the ideas that go into these variational results. Accordingly we will first prove a finite-dimensional version of a theorem of Ljusternik and Schnirelmann that was stated in the Introduction. Then a Galerkin argument is used to get an infinite-dimensional version of the result. It is possible to obtain the infinite-dimensional case directly but, by working first in finite dimensions, technicalities are minimized and the underlying ideas become more transparent.

The underlying topological basis for the arguments given here is the notion of genus due to Krasnoselski [2] rather than the more general notion of Ljusternik-Schnirelmann category which was employed by Professor Browder in his lectures. The approach taken here is an amalgam of those of Browder [25], Krasnoselski [2], Ljusternik [23] and Palais [24].

To pave the way for the finite- and infinite-dimensional results which will be given in §§4–5, the notion of genus is introduced in this section and its properties developed. The definition of genus given here is essentially that of Coffman [26].

Let E be a real Banach space and $\Sigma(E) \equiv \Sigma$ the set of closed (in E) subsets A of $E - \{0\}$ which are symmetric with respect to the origin, i.e., $x \in A$ implies $-x \in A$. A map $\gamma : \Sigma \to N$ is defined as follows: For $A \in \Sigma$, $\gamma(A)$, the *genus* of A, is the smallest integer n such that there exists an odd continuous map $\varphi \in C(A, \boldsymbol{R}^n - \{0\})$. By definition $\gamma(\emptyset) = 0$ and if there is no such n, $\gamma(A) = \infty$. It is easy to see that if $\gamma(A) = 1$ A is not connected. The important sets for later applications will be spheres or spherelike sets. To see how to calculate the genus of such sets, let S^{n-1} denote the unit sphere in $\boldsymbol{R}^n$.

LEMMA 3.1. *Let $A \in \Sigma$ be homeomorphic to S^{n-1} by an odd homeomorphism. Then $\gamma(A) = n$.*

To prove Lemma 3.1, a useful result from finite-dimensional degree theory is needed, namely the Borsuk Antipodensatz [4].

LEMMA 3.2. *Let $0 \in \Omega \subset \boldsymbol{R}^n$ where Ω is bounded, open, and symmetric. Suppose $\varphi \in C(\overline{\Omega}, \boldsymbol{R}^n)$ where φ is odd and nonzero on $\partial\Omega$. Then $d(\varphi, \Omega, 0)$ is an odd integer.*

REMARK. If Ω is as above in the real Banach space E and φ is as above with $\varphi = I - T$ and T compact, the same conclusion obtains.

COROLLARY 3.3. *Under the hypotheses of Lemma 3.2, $\varphi(\Omega)$ contains a neighborhood of 0.*

PROOF. Lemma 3.2 implies $0 \in \varphi(\Omega)$. The continuity of $d(\varphi, \Omega, b)$ in b then implies $b \in \varphi(\Omega)$ for all small b.

COROLLARY 3.4. *Let Ω be as in Lemma 3.2 and suppose $\psi \in$*

$C(\partial\Omega, \mathbf{R}^n)$ where ψ is odd and $\psi(\partial\Omega)$ lies in a proper subspace of $\mathbf{R}^n$. Then there exists $\xi \in \partial\Omega$ such that $\psi(\xi) = 0$.

PROOF. We can assume $\psi(\partial\Omega) \subset \mathbf{R}^{n-1}$. Suppose $\psi \not\equiv 0$ on $\partial\Omega$. By the Tietze Extension Theorem, ψ can be extended to $\hat{\psi} \in C(\overline{\Omega}, \mathbf{R}^{n-1})$. By Corollary 3.3, $\hat{\psi}(\overline{\Omega})$ contains a neighborhood of 0 in $\mathbf{R}^n$ but since $\hat{\psi}(\overline{\Omega}) \subset \mathbf{R}^{n-1}$ this is impossible. Hence there exists $\xi \in \partial\Omega$ such that $\psi(\xi) = 0$.

PROOF OF LEMMA 3.1. Let h denote the odd homeomorphism, $h : A \to S^{n-1}$. Clearly $\gamma(A) \leqq n$. If $\gamma(A) < n$, there exists an odd $\varphi \in C(A, \mathbf{R}^j - \{0\})$ where $j < n$. Therefore the odd map $\varphi \circ h^{-1} \in C(S^{n-1}, \mathbf{R}^j - \{0\})$. But this violates Corollary 3.4. Thus $\gamma(A) = n$.

For the remainder of this section some of the properties of genus will be developed. Ljusternik-Schnirelmann category possesses similar properties which are generally somewhat more difficult to prove.

LEMMA 3.5. *Let $A, B \in \Sigma$.*

1°. *If there is an odd $f \in C(A, B)$, then $\gamma(A) \leqq \gamma(B)$.*

PROOF. The result is trivial if $\gamma(B) = \infty$. Hence we assume here and for the same reason in the following results that $\gamma(A), \gamma(B) < \infty$. Suppose $\gamma(B) = n$. Then there exists an odd $\varphi \in C(B, \mathbf{R}^n - \{0\})$. Since $\varphi \circ f \in C(A, \mathbf{R}^n - \{0\})$ and is odd, $\gamma(A) \leqq n = \gamma(B)$.

2° (MONOTONICITY). *If $A \subset B$, then $\gamma(A) \leqq \gamma(B)$.*

PROOF. Immediate from 1° with $f = I$.

3°. *If h is an odd homeomorphism of A onto B, then $\gamma(A) = \gamma(B)$.*

PROOF. Immediate from 1° by interchanging the roles of A and B.

4° (SUBADDITIVITY). $\gamma(A \cup B) \leqq \gamma(A) + \gamma(B)$.

PROOF. Suppose $\gamma(A) = m, \gamma(B) = n$. Therefore there exist odd maps $\varphi \in C(A, \mathbf{R}^n - \{0\})$, $\psi \in C(B, \mathbf{R}^m - \{0\})$. By the Tietze Extension Theorem, φ, ψ can be extended to $\hat{\varphi} \in C(E, \mathbf{R}^m)$, $\hat{\psi} \in C(E, \mathbf{R}^n)$. Moreover by replacement if necessary by $\frac{1}{2}(\hat{\varphi}(x) - \hat{\varphi}(-x))$, it can be assumed that $\hat{\varphi}, \hat{\psi}$ are odd. Let $f = (\hat{\varphi}, \hat{\psi})$. Then $f \in C(A \cup B, \mathbf{R}^{m+n} - \{0\})$ and f is odd. Hence $\gamma(A \cup B) \leqq m + n = \gamma(A) + \gamma(B)$.

5°. *If $\gamma(B) < \infty$, $\gamma(\overline{A - B}) \geqq \gamma(A) - \gamma(B)$.*

PROOF. $A \subset \overline{A - B} \cup B$. Hence the result follows from 2° and 4°.

6°. *If A is compact, $\gamma(A) < \infty$.*

PROOF. For $x \neq 0$ and $r < \|x\|$, $B_r(x) \cap B_r(-x) = \emptyset$. Therefore $\gamma(B_r(x) \cup B_r(-x)) = 1$. Since A is compact, it can be covered by finitely many such pairs of balls. Hence $\gamma(A) < \infty$.

7°. *If A is compact, there exists a uniform neighborhood $N_\delta(A)$*

(i.e., all points whose distance from A is not greater than δ) such that $\gamma(N_\delta(A)) = \gamma(A)$.

PROOF. Suppose $\gamma(A) = n$. Then there is an odd $\varphi \in C(A, \boldsymbol{R}^n - \{0\})$. Extend φ oddly to $\hat{\varphi} \in C(E, \boldsymbol{R}^n)$ as in 4°. Since $\varphi \neq 0$ on A which is compact, for some $\delta > 0$, $\hat{\varphi} \neq 0$ on $N_\delta(A)$. Therefore $\gamma(N_\delta(A)) \leqq n = \gamma(A)$. Since 2° implies the reverse inequality, $\gamma(N_\delta(A)) = \gamma(A)$.

We conclude this section with some properties of genus which we will not use later. First an observation about S^{n-1}: There exist n closed antipodal sets $B_i = C_i \cup (-C_i)$, $1 \leqq i \leqq n$, such that $C_i \cap (-C_i) = \emptyset$, $B_i \subset S^{n-1}$, and $\bigcup_1^n B_i = S^{n-1}$. To see this, e.g. let B_j be the complement (in S^{n-1}) of a small symmetric neighborhood of $\{x_j = 0\} \cap S^{n-1}$. It is easy to verify that the B_j possess the above properties.

The next lemma provides us with an equivalent characterization of genus which is essentially that of Krasnoselski [2].

LEMMA 3.6. *Let $A \in \Sigma$. $\gamma(A) = n$ if and only if there exist $A_1, \cdots, A_n \in \Sigma$ such that $\gamma(A_r) = 1$, $1 \leqq r \leqq n$, and $A \subset \bigcup_1^n A_r$ with n being the smallest integer having this property.*

PROOF. If there exist sets A_r as above, then, by 4° of Lemma 3.5, $\gamma(A) \leqq n$. Hence to complete the proof of this part of the lemma and likewise the converse statement, it suffices to show $\gamma(A) = j$ implies there exist $D_1, \cdots, D_j \in \Sigma$ such that $\gamma(D_r) = 1$, $1 \leqq r \leqq j$, and $A \subset \bigcup_1^j D_i$. Since $\gamma(A) = j$, there is an odd $\varphi \in C(A, \boldsymbol{R}^j - \{0\})$. By the remarks preceding this lemma, there exist j closed sets $B_1, \cdots, B_j \subset S^{j-1}$ such that $B_i = C_i \cup (-C_i)$, $C_i \cap -C_i = \emptyset$ and $S^{j-1} = \bigcup_1^j B_i$. For $x \in \boldsymbol{R}^j - \{0\}$, let $p(x) = x/\|x\|$. Define $D_r \equiv \varphi^{-1} \circ p^{-1}(B_r)$, $1 \leqq r \leqq j$. These sets form a covering of A by closed sets. Moreover $D_r = \varphi^{-1} \circ p^{-1}(C_r) \cup \varphi^{-1} \circ p^{-1}(-C_r)$ with $\varphi^{-1} \circ p^{-1}(C_r) \cap \varphi^{-1} \circ p^{-1}(-C_r) = \emptyset$. Therefore $\gamma(D_r) = 1$ or 0 depending on whether or not $D_r = \emptyset$. But if any $D_r = \emptyset$, by 4° of Lemma 3.5, $\gamma(A) < j$, contrary to the hypotheses. Hence $\gamma(D_r) = 1$, $1 \leqq r \leqq j$, and the lemma is proved.

REMARK. Lemma 3.6 implies S^{n-1} cannot be decomposed into fewer than n B_i's as above.

As our last result we show a relationship between genus and Ljusternik-Schnirelmann category. Recall that if M is a topological space and $A \subset M$ is closed, A is said to have *Ljusternik-Schnirelmann category* 1 (denoted by $\mathrm{cat}_M A = 1$) if A is deformable in M to a point, i.e., if the identity map of A into M is homotopic as a map into M

to a constant map. If A is closed in M, $\mathrm{cat}_M A = n$ if there exist n closed sets $A_i \subset M$, $1 \leqq i \leqq n$, such that $A \subset \bigcup_1^n A_r$, $\mathrm{cat}_M A_r = 1$, and n is the smallest integer with these properties.

For $A \subset E$, let $A^* = \{(x, -x) \mid x \in A\}$. In a natural fashion A^* can be made a topological space.

THEOREM 3.7. *If $A \in \Sigma$ is compact, $\gamma(A) = \mathrm{cat}_{(E-\{0\})^*} A^*$.*

PROOF. The compactness of A implies that A^* is compact in $(E - \{0\})^* \equiv M$. A slight modification of the proof of 6° of Lemma 3.5 shows $\mathrm{cat}_M A^*$ is finite. Suppose $\mathrm{cat}_M A^* = n$. Therefore $A^* = \bigcup_1^n A_r^*$ where A_r^* can be assumed to be compact in M and $\mathrm{cat}_M A_r^* = 1$. Let $A_r = \{x \in E \mid (x, -x) \in A_r^*\}$. Then A_r is compact, $A_r \in \Sigma$, and $A \subset \bigcup_1^n A_r$. We will show $\gamma(A_r) = 1$ and hence, by 4° of Lemma 3.5, $\gamma(A) \leqq n = \mathrm{cat}_M A^*$.

Since $\mathrm{cat}_M A_r^* = 1$, there is a homotopy $h_r^* \in C(A_r^* \times [0, 1], M)$ such that $h_r^*(\,\cdot\,, 0) = I$ and $h_r^*(\,\cdot\,, 1) = (\xi_r, -\xi_r)$. This implies the existence of a homotopy $h_r \in C(A_r \times [0, 1], E - \{0\})$ such that $h_r(\,\cdot\,, 0) = I$, $h_r(\,\cdot\,, 1) = \xi_r$ or $-\xi_r$ and $h_r(x, t) = -h_r(-x, t)$. Then, by 1° of Lemma 3.5, $\gamma(A_r) \leqq \gamma(h_r(A_r, 1)) = 1$ so $\gamma(A_r) = 1$.

To construct h_r, define $h_r(\,\cdot\,, 0) = I$. Let $\epsilon/2 = \min_{x \in A_r} \|x\|$. For each $(x, -x) \in A_r^*$, the continuity of h_r^* implies that $h_r^*((x, -x), t) = (y_x(t), -y_x(t))$ and $\|y_x(t) - x\| < \epsilon$ for $0 < t \leqq t_x$. Since A_r^* is compact, $t_x \equiv t_0 > 0$ can be chosen uniformly in $(x, -x) \in A_r^*$. Hence h_r can be extended to the interval $(0, t_0)$ by $h_r(x, t) = y_x(t)$, $h_r(-x, t) = -y_x(t) = -h_r(x, t)$. Using the uniform continuity of h_r^* on the compact set $A_r^* \times [0, 1]$ and the connectedness of $[0, 1]$, h_r can be extended to $t = 1$ satisfying the above properties.

To prove that $\gamma(A) \geqq \mathrm{cat}_M A^*$, suppose $\gamma(A) = m$. By Lemma 3.6 there exist m closed sets $A_r \in \Sigma$ such that $\gamma(A_r) = 1$ and $A \subset \bigcup_1^m A_r$. It can also be assumed that A_r is compact. Thus $A^* \subset \bigcup_1^m A_r^*$ and by the subadditivity property of category (which is easy to prove—see e.g. [24]), $\mathrm{cat}_M A^* \leqq \sum_1^m \mathrm{cat}_M A_r^*$. Hence to show $m = n$, it suffices to prove that $\gamma(A_r) = 1$ implies that $\mathrm{cat}_M A_r^* = 1$.

We give a proof essentially due to E. Fadell. For convenience the subscripts on the A_r will be dropped. Suppose first that E is infinite dimensional. Here no compactness is needed for A. Since $\gamma(A) = 1$, there exists an odd $\varphi \in C(A, \mathbf{R} - \{0\})$. Let $A^+ = \varphi^{-1}(\mathbf{R}^+)$, $A^- = \varphi^{-1}(\mathbf{R}^-)$. To show $\mathrm{cat}_M A^* = 1$, it suffices to define a homotopy of A^+ to a point in $E - \{0\}$ and then extend it to A^- by oddness. For $u \in E - \{0\}$, let $p(u) = u/\|u\|$. Then $H_1(u, t) = (1 - t)u + tp(u)$ defines a homotopy of A^+ to $\partial B_1(0)$. By a theorem of Dugundji [27], there exists a homotopy H_2 of $\partial B_1(0)$ to a point in itself. Thus following the homotopy H_1 by the homotopy H_2 gives the desired homotopy here.

Next suppose that E is finite dimensional. We can assume $E = \boldsymbol{R}^n$, $n > 1$ ($n = 1$ is trivial). Since A is compact, by 7° of Lemma 3.5, A possesses a uniform neighborhood with $\gamma(N_\delta(A)) = 1$. It can be assumed that $N_\delta(A)$ contains finitely many components each of which are arcwise connected as is $E - \{0\} - N_\delta(A)$. Let $K = K^+ \cup K^-$ be any pair of components of $N_\delta(A)$ where $K^+ = -K^-$. It suffices to find a homotopy of K^+ (or K^-) to a point in $E - \{0\}$ as in the infinite-dimensional case. The set $K \subset B_R(0)$ for some $R > 0$. Suppose there exists e.g. a piecewise linear arc l joining 0 to $\partial B_{2R}(0)$ in $E - K$. Then by a homotopy consisting of a series of translations opposite to l (equivalent to moving the origin along l to $\partial B_{2R}(0)$) we can assume K^+ lies exterior to $B_{2R}(0)$. Then K^+ can easily be deformed by a homotopy in $E - B_R(0)$ to e.g. any point in its convex hull and the proof will be complete.

It remains to prove the existence of l. If such an l does not exist, 0 belongs to bounded components Q^+ of $\boldsymbol{R}^n - K^+$ and $Q^- = -Q^+$ of $\boldsymbol{R}^n - K^-$. Then $Q = Q^+ \cup Q^-$ is a neighborhood of 0. Since $\partial Q \subset K$, $\gamma(\partial Q) = 1$ and there is an odd $\varphi \in C(\partial Q, \boldsymbol{R} - \{0\})$. But this is impossible since it violates Corollary 3.4.

For a more elementary proof of the existence of l which bypasses Corollary 3.4, suppose there exists Q^+, Q^- as above. Consider any half line L through 0. Let $a \in K^+$ be its first intersection with ∂Q^+ and $b \in K^+$ the first intersection of $-L$ with ∂Q^+. Then $b \neq -a$ since K^+ contains no antipodal points. Suppose $\|b\| > \|a\|$. Therefore $-a$ belongs to the same component, Q^+ of $\boldsymbol{R}^n - K^+$ as does 0. Hence $K^- \subset Q^+$. By symmetry, $K^+ \subset Q^-$. Let c be any point in K^+ at a maximal distance from the origin. Then the radial half line joining c to ∞ and not passing through the origin lies in $\boldsymbol{R}^n - K^+$. This implies K^+ lies in the unbounded component of $\boldsymbol{R}^n - K^-$ contrary to $K^+ \subset Q^-$.

4. **A finite-dimensional Ljusternik-Schnirelmann theorem.** This section is devoted to a proof of a theorem of Ljusternik and Schnirelmann. Below $C^1(A, B)$ denotes the space of continuously Fréchet differentiable maps from A to B.

THEOREM 4.1. *Let $f \in C^1(\boldsymbol{R}^n, \boldsymbol{R})$ and f be even. Then, for each $r > 0$, f possesses at least $2n$ distinct critical points on rS^{n-1}.*

Note that critical points of f occur in pairs because of the evenness of f. A good model case of Theorem 4.1 where one is on familiar grounds is $f(u) = (Mu, u)$ where M is a symmetric $n \times n$ matrix and $(\cdot,\cdot)$ denotes the usual inner product in $\boldsymbol{R}^n$. It is not difficult to extend Theorem 4.1 to more general symmetric manifolds than spheres or to restrict $f \in C^1(rS^{n-1}, \boldsymbol{R})$ but we prefer to minimize technicalities and work in the simplest case.

By a *critical point of f on rS^{n-1}* is meant a point at which the gradient of f relative to rS^{n-1} vanishes, i.e., where

$$\Gamma u \equiv \nabla f(u) - (\nabla f(u), u)u/(u, u) = 0. \tag{4.2}$$

Thus critical points of f are solutions of a nonlinear eigenvalue problem of the form $Tu = \lambda u$ where $T = \nabla f$ and $\lambda = (Tu, u)/(u, u)$ and, for $\lambda \neq 0$, (λ^{-1}, u) satisfies an equation of the form (0.1).

PROOF OF THEOREM 4.1. The proof of Theorem 4.1 will be accomplished in a series of five steps. Using the notion of genus, a minimax (or more properly maximin) characterization will be given for n critical values of f in (A). In (B) a 'gradient' mapping will be constructed and its properties studied. The mapping is then used in (C) to prove a deformation lemma. These preliminaries are then used in (D) and (E) to show the minimax numbers are critical values of f and we get enough critical points despite possible degeneracies.

For convenience we take $r = 1$.

(A) *Characterization of critical values of f.* By a *critical value of f* (restricted to S^{n-1}) is meant a real number c such that there exists $u \in S^{n-1}$ with $f(u) = c$ and u satisfies (4.2). A set of n numbers is defined as follows:

$$c_m(f) \equiv c_m = \sup_{A \subset S^{n-1};\, \gamma(A) \geqq m} \; \min_{u \in A} f(u), \qquad 1 \leqq m \leqq n. \tag{4.3}$$

Implicit in (4.3) is that $A \in \Sigma(\boldsymbol{R}^n)$. Since f is continuous and $\gamma(S^{n-1}) = n$, the numbers c_m are well defined, $1 \leqq m \leqq n$. Since c_{m+1} is a supremum over a smaller family of sets than c_m, $c_1 \geqq c_2 \geqq \cdots \geqq c_n$. Sets of the form $A = \{x, -x\}$, $x \in S^{n-1}$, are admissible for the computation of c_1 and $\min f(u) = f(x)$ for u in such an A. Hence $c_1 = \max_{S^{n-1}} f$. In a similar fashion $c_n = \min_{S^{n-1}} f$. To prove this it suffices to show that S^{n-1} is the only subset of S^{n-1} in Σ having genus n. Suppose $A \in \Sigma$ and $A \subset S^{n-1}$ with inclusion being proper. It can be assumed that A does not contain $(0, \cdots, 0, \pm 1)$. The projection mapping $p(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1}, 0)$ is odd and $p \in C(A, \boldsymbol{R}^{n-1} - (0))$. Hence, by 1° of Lemma 3.5, $\gamma(A) \leqq n - 1$. Consequently, $c_n = \min_{S^{n-1}} f$.

We will show the numbers c_m are critical values of f, $1 \leqq m \leqq n$. This in itself is not sufficient to prove Theorem 4.1 since the theorem asserts the existence of n pairs of distinct critical points and a priori it may happen that the c_m's are not all distinct. Thus a multiplicity lemma is also needed which guarantees that f has enough critical points even in the degenerate case $c_{j+1} = \cdots = c_{j+p} \equiv c, p > c$, $p > 1$. For the latter case it will turn out that f has infinitely many distinct critical points corresponding to the critical value c. That this is so in the matrix example (Mu, u) mentioned above is clear.

Another set of critical values of f can be characterized by

$$(4.4) \qquad b_k(f) \equiv b_k = \inf_{A \subset S^{n-1};\, \gamma(A) \geqq k} \max_{u \in A} f(u), \qquad 1 \leqq k \leqq n.$$

As above it readily follows that $b_1 \leqq \cdots \leqq b_n$, $b_1 = \min_{S^{n-1}} f = c_n$, $b_n = \max_{S^{n-1}} f = c_1$. In general as simple examples show, $c_j \neq b_{n+1-j}$ for $j \neq 1, n$. That the b_k's are critical values of f follows from the c_m case on replacing f by $-f$.

Another useful characterization of the critical values c_j can be given. Let $A_c = \{u \in S^{n-1} \mid f(u) \geqq c\}$. Then

$$(4.5) \qquad c_j = \sup \{r \in \boldsymbol{R} \mid \gamma(A_r) \geqq j\}, \qquad 1 \leqq j \leqq n.$$

To see this, denote the right-hand side of (4.5) by r_j. Since $\gamma(A_r) \geqq j$ implies $\min_{A_r} f(u) \leqq c_j$, therefore $r_j \leqq c_j$. If $r_j < c_j$, there exists $A \subset S^{n-1}$ with $\gamma(A) \geqq j$ and $c_j > \min_A f(u) \equiv p > r_j$. Hence $A \subset A_p$ and $j \leqq \gamma(A) \leqq \gamma(A_p)$. But this contradicts the definition of r_j. Thus $r_j = c_j$.

As a consequence of (4.5), for any $\epsilon > 0, \gamma(A_{c_j - \epsilon}) \geqq j$.

(B) *A gradient mapping.* The next step in the proof of Theorem 4.1 is to construct a mapping in a 'gradient' direction. First some technicalities: Recall for $u \in S^{n-1}$, $\Gamma u = Tu - \lambda u$ with $T = \nabla f$ and $\lambda = (Tu, u)$. Note $(\Gamma u, u) = 0$. Let $\gamma_1 = \max_{u \in S^{n-1}} \|\Gamma u\| \leqq \gamma_2 = \max_{u \in S^{n-1}} \|Tu\| \geqq |\lambda|$, and choose γ_3 so that $(1 + x)^{-1/2} \geqq \frac{3}{4}$ and $|(1 + x)^{-1/2} - 1| \geqq \frac{3}{4} |x|$ for $|x| \leqq \gamma_3$.

Let $R(u, v) = f(u + v) - f(u) - (Tu, v)$. Since f is continuously differentiable, there exists a continuous function $\delta(\epsilon)$ (independent of u) such that for all u, $u + v \in S^{n-1}$, $|R(u, v)| \leqq \epsilon \|v\|$ if $\|v\| \leqq \delta(\epsilon)$. In addition $\delta(0) = 0$ and $\delta(\epsilon)$ is a nondecreasing function of ϵ. ($\delta(\epsilon)$ can be taken as $\sup \{\eta \in \boldsymbol{R} \,|\, |R(u, v)| \leqq \epsilon\eta$ for all $u \in S^{n-1}$ and $\|v\| \leqq \eta\}$.) For $u \in S^{n-1}$, let

$$(4.6) \qquad \alpha(u) = \min \frac{1}{\gamma_2} \left\{ \frac{\delta(\|\Gamma u\|/8)}{2}, \sqrt{\gamma_3}, \frac{1}{3} \right\}.$$

Then $\alpha(u)$ is a continuous nonnegative even function of u on S^{n-1}.

Finally we define a mapping $\chi : S^{n-1} \to S^{n-1}$ by $\chi(u) = (u + \alpha(u)\Gamma u)\|u + \alpha(u)\Gamma u\|^{-1}$. It is clear from its definition that χ is odd and continuous and therefore $\chi(A) \in \Sigma$ if $A \in \Sigma$.

LEMMA 4.7. *For* $u \in S^{n-1}$, $f(\chi(u)) \geqq f(u) + \alpha(u)\|\Gamma u\|^2/4$.

PROOF. $f(\chi(u)) = f(u) + (Tu, \chi(u) - u) + R(u, \chi(u) - u)$. Equation (4.6) and some easy estimates show

$$(4.8) \qquad \|\chi(u) - u\| \leqq \alpha\|\Gamma u\| + \alpha^2\|\Gamma u\|^2 \leqq 2\alpha\|\Gamma u\| \leqq \delta(\|\Gamma u\|/8).$$

Therefore, by (4.8) and the properties of δ,

$$(4.9) \qquad |R(u, \chi(u) - u)| \leqq \|\Gamma u\| \, \|\chi(u) - u\|/8 \leqq \alpha\|\Gamma u\|^2/4.$$

Hence once we show that

$$(4.10) \qquad (Tu, \chi(u) - u) \geqq \tfrac{1}{2}\, \alpha(u)\|\Gamma u\|^2,$$

the lemma obtains. To prove (4.10), note that

$$(4.11) \quad (Tu, \chi(u) - u) = \frac{\alpha(u)\|\Gamma u\|^2}{\|u + \alpha(u)\Gamma u\|} + \lambda\left(\frac{1}{\|u + \alpha(u)\Gamma u\|} - 1\right).$$

Using (4.6) and (4.11), (4.10) readily follows.

Lemma 4.7 has an infinite-dimensional analogue. Suppose E is a Hilbert space, f has a compact Fréchet derivative T (and therefore f is weakly continuous [**28**]), and f is uniformly differentiable on bounded sets, i.e., there exists a function $\delta(\epsilon)$ as above with δ independent of u on bounded subsets of E. Then the proof of Lemma 4.7 goes through essentially unchanged. See also [**2**].

An immediate consequence of Lemma 4.7 is that $\chi : A_c \to A_c$ where $A_c = \{u \in S^{n-1} \mid f(u) \geqq c\}$. In fact inclusion is proper unless c is a critical value of f. A more precise result is given below.

(C) *A deformation lemma.* Let $K_c = \{u \in S^{n-1} \mid f(u) = c, \Gamma u = 0\}$, i.e., K_c is the set of critical points having c as critical value.

LEMMA 4.12. *Let $c \in R$ and $\mathcal{O}$ be a symmetric open neighborhood of K_c in S^{n-1}. Then there exists $\epsilon > 0$ such that $\chi(A_{c-\epsilon} - \mathcal{O}) \subset A_{c+\epsilon}$. In particular if c is a regular value of f, i.e., $K_c = \emptyset$, $\chi(A_{c-\epsilon}) \subset A_{c+\epsilon}$.*

PROOF. Let $u \in f^{-1}(c) - \mathcal{O}$. Then $\Gamma u \neq 0$ and, by Lemma 4.7, there exists an $\eta_u > 0$ such that $f(\chi(u)) > c + 2\eta_u$. Therefore there exists a neighborhood N_u of u such that $f(\chi(v)) > c + \eta_u$ for all $v \in N_u$. Since $f^{-1}(c) - \mathcal{O}$ is compact with $\{N_u\}$ as an open covering, there exists a finite subcovering by $N_{u_1}, \cdots, N_{u_j}$. Note that $f(\chi(v)) > c + \eta_{u_i} \geqq c + \eta$ for all $v \in N_{u_i}$, $1 \leqq i \leqq j$, where $\eta = \min_{1 \leqq i \leqq j} \eta_{u_i}$.

Let $\hat{\mathcal{O}} = \mathcal{O} \cup N_{u_1} \cup \cdots \cup N_{u_j}$. Then $\hat{\mathcal{O}}$ is a neighborhood of $f^{-1}(c)$. This implies there exists $\epsilon > 0$ such that $\mathcal{O} \supset f^{-1}([c-\epsilon, c+\epsilon])$ for otherwise we can find a sequence $\epsilon_m \to 0$ and $u_m \in f^{-1}([c-\epsilon_m, c+\epsilon_m])$ such that $u_m \notin \hat{\mathcal{O}}$. By the compactness of S^{n-1}, a subsequence of (u_m) converges to $\hat{u}$ satisfying $f(\hat{u}) = c$ and $\hat{u} \notin \hat{\mathcal{O}}$ which is impossible. Hence

$$\hat{\mathcal{O}} \supset f^{-1}([c-\epsilon, c+\epsilon]).$$

It can further be assumed $\epsilon \leqq \eta$.

$$A_{c-\epsilon} = A_{c+\epsilon} \cup f^{-1}([c-\epsilon, c+\epsilon]) \subset A_{c+\epsilon} \cup \mathcal{O} \cup N_{u_1} \cup \cdots \cup N_{u_j}.$$

This implies $A_{c-\epsilon} - \mathcal{O} \subset A_{c+\epsilon} \cup N_{u_1} \cup \cdots \cup N_{u_j}$. Since $\chi(A_{c+\epsilon}) \subset A_{c+\epsilon}$ and $\chi(N_{u_i}) \subset A_{c+\eta} \subset A_{c+\epsilon}$, it follows that $\chi(A_{c-\epsilon} - \mathcal{O}) \subset A_{c+\epsilon}$.

(D) *c_m is a critical value of f.* With the aid of Lemma 4.12, it is an easy matter to prove that c_m is a critical value of f, $1 \leqq m \leqq n$. For if not, by Lemma 4.12, there is an $\epsilon > 0$ such that

$$\chi(A_{c_m-\epsilon}) \subset A_{c_m+\epsilon}. \tag{4.13}$$

By (4.5), $\gamma(A_{c_m-\epsilon}) \geqq m$ and, by 1° of Lemma 3.5, $\gamma(\chi(A_{c_m-\epsilon})) \geqq m$. The definition of c_m implies $c_m \geqq \min_{\chi(A_{c_m-\epsilon})} f(u)$ but by (4.13) this min is $\geqq c_m + \epsilon$, a contradiction.

(E) *A multiplicity lemma.* We now know each c_m, $1 \leqq m \leqq n$, is a critical value of f. As was noted in (A), it may occur that $c_{m+1} = \cdots = c_{m+p}$, $p > 1$. The following lemma then completes the proof of Theorem 4.1.

LEMMA 4.14. *Suppose $c_{m+1} = \cdots = c_{m+p} \equiv c$. Then $\gamma(K_c) \geqq p$.*

The lemma for $p > 1$ implies K_c contains infinitely many distinct critical points since any finite point set has genus 1.

PROOF OF LEMMA 4.14. Suppose $\gamma(K_c) \leqq p - 1$. By 7° of Lemma 3.5, for some $\delta > 0$, $\gamma(N_\delta(K_c)) \leqq p - 1$. Letting $\mathcal{O} = \operatorname{int} N_\delta(K_c)$, by Lemma 4.12, $\chi(A_{c-\epsilon} - \mathcal{O}) \subset A_{c+\epsilon}$. Note that

$$B \equiv \overline{A_{c-\epsilon} - N_\delta(K_c)} = A_{c-\epsilon} - \operatorname{int} N_\delta(K_c).$$

From (4.5) with $c = c_{m+p}$, $\gamma(A_{c-\epsilon}) \geqq m + p$. Therefore by 5° of Lemma 3.5, $\gamma(B) \geqq \gamma(A_{c-\epsilon}) - \gamma(N_\delta(K_c)) \geqq m + 1$. By 1° of Lemma 3.5, $\gamma(\chi(B)) \geqq m + 1$. The definition of $c = c_{m+1}$ then shows $\min_{\chi(B)} f(u) \leqq c$. However $\chi(B) \subset A_{c+\epsilon}$ which implies the $\min \geqq c + \epsilon$, a contradiction, and the lemma is proved.

REMARK. It is an easy exercise to combine parts (D) and (E) here.

5. **The infinite-dimensional case.** In this section an infinite-dimensional analogue of Theorem 4.1 will be proved. Let E be a real separable Hilbert space with inner product denoted by $(\cdot, \cdot)$ and let $f \in C^1(E, R)$ where f is even and T, the gradient of f, is compact. Since f is even, T is odd. The compactness of T implies that f is weakly continuous [28]. Suppose further that f is uniformly differentiable on bounded sets, i.e., for all bounded $B \subset E$, there exists $\delta_B \in C(R^+, R^+)$ with $\delta_B(0) = 0$, $\delta_B(\epsilon)$ is a nondecreasing function of ϵ, and if $u, u + v \in B$, $R(u, v) \equiv f(u + v) - f(u) - (T(u), v)$ satisfies $|R(u, v)| \leqq \epsilon \|v\|$ if $\|v\| \leqq \delta_B(\epsilon)$. The uniform differentiability of f and compactness of T imply that T maps weakly convergent sequences to strongly convergent ones [2]. Lastly we assume $f(u) \geqq 0$ and $f = 0$ if and only if $u = 0$, $T(u) = 0$ if and only if $u = 0$.

THEOREM 5.1. *Under the above hypotheses on E, f, T, for each $r > 0$, f possesses infinitely many distinct pairs of critical points on $\partial B_r(0)$.*

Again by a critical point of f on $\partial B_r(0)$ we mean a point at which the gradient of f relative to $\partial B_r(0)$ vanishes, i.e.,

$$\Gamma u \equiv Tu - \lambda u = 0 \tag{5.2}$$

where $\lambda(u) = (Tu, u)/r^2$. Thus critical points of f provide us with solutions of the nonlinear eigenvalue problem (5.2). Observe again the evenness of f implies that critical points occur in antipodal pairs.

For convenience we assume $r = 1$ and set $S = \partial B_1(0)$.

PROOF OF THEOREM 5.1. The proof is given in four steps. An approximate finite-dimensional problem is set up and solved in (A). Then in (B) a characterization of critical values of f is given and it is shown that the approximate critical values obtained in (A) converge to those defined in (B). The existence of infinitely many distinct critical values is proved in (C). Lastly in (D) the convergence of subsequences of approximate critical points of f obtained in (A) to actual critical points of f is shown.

(A) *The finite-dimensional approximation.* Let (e_n) be an orthonormal basis for E and let $E_n = \text{span}\,\{e_1, \cdots, e_n\}$. Consider the restriction of f to $E_n \cap S$. Thus $f(u) = f(x_1e_1 + \cdots + x_ne_n) \equiv F_n(x)$ where $u = \sum_1^n x_ie_i$ and $x = (x_1, \cdots, x_n)$ with $\sum_1^n x_r^2 = 1$, i.e., $x \in S^{n-1}$. By Theorem 4.1, $F_n(x)$ possesses at least n distinct pairs of critical points $\pm x_{n,k}$, $1 \leqq k \leqq n$, with corresponding critical values given by

$$(5.3) \qquad c_{n,k} = \sup_{B \subset S^{n-1};\, \gamma(B) \geqq k} \ \min_{x \in B} F_n(x), \qquad 1 \leqq k \leqq n,$$

or equivalently (since each set B is homeomorphic via an odd homeomorphism to $A \subset S$)

$$(5.4) \qquad c_{n,k} = \sup_{A \subset S \cap E_n;\, \gamma(A) \geqq k} \ \min_{u \in A} f(u).$$

At the critical points $x_{n,k}$,

$$(5.5) \qquad \partial F_n(x_{nk})/\partial x_j - \lambda_{nk} x_{nkj} = 0, \qquad 1 \leqq j, k \leqq n,$$

where $\lambda_{nk} = \sum_{j=1}^{n} (\partial F_n(u_{nk})/\partial x_j)\, x_{nkj}$. Let $u_{nk} = \sum_{j=1}^{n} x_{nkj} e_j$. Since $x_{nkj} = (u_{nk}, e_j)$, $\partial F_n(u_{nk})/\partial x_j = (T(u_{nk}), e_j)$ and $\lambda_{nk} = (T(u_{nk}), u_{nk})$, (5.5) can be rewritten as

$$(5.6) \qquad (T(u_{nk}), e_j) - \lambda_{nk}(u_{nk}, e_j) = 0, \qquad 1 \leqq j \leqq n,$$

or more succinctly as

$$(5.7) \qquad (T(u_{nk}) - \lambda_{nk} u_{nk}, \varphi) = 0 \quad \text{for all } \varphi \in E_n.$$

Note also that $f(u_{nk}) = c_{nk}$.

(B) *A characterization of critical values.* A set of numbers is defined in a (by now) natural fashion. These will be shown later to be critical values for f on S. Let

$$(5.8) \qquad c_k = \sup_{A \subset S;\, A \text{ compact};\, \gamma(A) \geqq k} \ \min_{u \in A} f(u).$$

The existence of such A's is clear and because of the weak continuity of f, the numbers c_k are well defined. The positivity of f on S implies $c_k > 0$ for all k. Since they involve suprema over successively larger sets, $c_{nk} \leqq c_{n+1,k} \leqq c_k$. Our goal is to show that $c_{nk} \to c_k$ and a subsequence of u_{nk} converges to u_k a critical point of f on S with critical value c_k. The first step is accomplished with

LEMMA 5.9. *$c_{nk} \to c_k$ as $n \to \infty$.*

PROOF. Since c_{nk} is a monotone nondecreasing sequence in n bounded above by c_k, it converges. To prove the lemma, we show, for all $\epsilon > 0$, there exists $m = m(\epsilon)$ such that $c_{nk} \geqq c_k - \epsilon$ for all $n \geqq m$. By the definition of c_k, there is a compact set $A \subset S$ such that $\gamma(A) \geqq k$ and $\min_A f(u) \geqq c_k - \epsilon/2$. Using the compactness of A and 7° of Lemma 3.5, it is easy to find a neighborhood $N_\delta(A) \subset S$ such that $\gamma(N_\delta(A)) = \gamma(A) \geqq k$ and $\min_{N_\delta(A)} f \geqq c_k - \epsilon$.

Let $\eta = \frac{1}{3} \min(1, \delta)$. Let P_n denote the orthogonal projector of E

onto E_n and $P_n^\perp = I - P_n$, the orthogonal projector of E onto $E_n^\perp$. For each $u \in A$ there exists $j = j(u) \in N$ such that $\|P_j^\perp u\| \leqq \eta/2$. The compactness of A implies there exists a $\bar{j} \in N$ such that, for all $n \geqq \bar{j}$ and all $u \in A$, $\|P_n^\perp u\| \leqq \eta$.

Let $n \geqq \bar{j}$ and set $\Psi_n(u) = P_n u/\|P_n u\|$ where $u \in A$. Then

$$\|\Psi_n(u) - u\| \leqq \frac{\|P_n u - u\|}{\|P_n u\|} + \left\| \frac{u}{\|P_n u\|} - u \right\| \tag{5.10}$$
$$\leqq \frac{\eta}{1-\eta} + \left(\frac{1}{1-\eta} - 1\right) \leqq \delta.$$

Hence $\Psi_n \in C(A, S \cap N_\delta(A) \cap E_n)$ and is odd. Therefore by 1° and 2° of Lemma 3.5,

$$\gamma(A) \leqq \gamma(\Psi_n(A)) \leqq \gamma(N_\delta(A)) = \gamma(A). \tag{5.11}$$

Thus $\gamma(\Psi_n(A)) \geqq k$ and $\Psi_n(A)$ is an admissible set for the calculation of c_{nk} and c_k. Moreover since $\Psi_n(A) \subset N_\delta(A)$, $c_{nk} \geqq \min_{\Psi_n(A)} f(u) \geqq c_k - \epsilon$ for all $n \geqq \bar{j}$. Hence the lemma is proved.

(C) *The existence of infinitely many distinct c_k's.* Even without establishing that they are critical values, we can show there are infinitely many distinct numbers c_k and each c_k is of finite "multiplicity", i.e., $c_{k+1} = \cdots = c_{k+p}$ for at most finitely many p. These assertions are an immediate consequence of

LEMMA 5.12. *$c_k \to 0$ as $k \to \infty$.*

Before proving Lemma 5.12 another technical lemma is needed [2].

LEMMA 5.13. *Let E be a separable Hilbert space and $f : E \to R$ be weakly continuous in $B_r(0)$. Then, for all $\epsilon > 0$, there exists $m = m(\epsilon)$ such that, for all $n \geqq m$ and for all $u, v \in B_r(0)$,*

$$|f(u + P_n v) - f(u + v)| < \epsilon.$$

PROOF. If not, there is an $\epsilon > 0$ and sequences (u_n), $(v_n) \subset B_r(0)$ such that

$$|f(u_n + P_n v_n) - f(u_n + v_n)| \geqq \epsilon. \tag{5.14}$$

Since $B_r(0)$ is weakly compact, subsequences of (u_n), (v_n), $(P_n v_n)$ converge weakly to u, v, w respectively. Since f is weakly continuous, we can pass to a limit in (5.14) to get $|f(u + w) - f(u + v)| \geqq \epsilon$. Hence $v \neq w$. However

$$\|v - w\|^2 = \lim_{n_i \to \infty} (v_{n_i} - P_{n_i} v_{n_i}, v - w) = \lim_{n_i \to \infty} (v_{n_i}, (I - P_{n_i})(v - w))$$

and $|(v_{n_i}, (I - P_{n_i})(v - w))| \leqq r\|(I - P_{n_i})(v - w)\| \to 0$ as $n_i \to \infty$ which implies $v = w$, a contradiction.

PROOF OF LEMMA 5.12. By Lemma 5.13 with $u = 0$, for all $\epsilon > 0$, there exists $m = m(\epsilon)$ such that if $n \geqq m$,

$$(5.15) \qquad |f(v) - f(P_n v)| < \epsilon/2, \qquad v \in B_1(0).$$

Since $f(0) = 0$, the continuity of f implies there exists $\rho > 0$ such that if $\|w\| < \rho$, $f(w) < \epsilon/2$. Hence if $v \in B_1(0)$ and $\|P_m v\| < \rho$, then

$$(5.16) \qquad f(v) \leqq f(P_m v) + |f(v) - f(P_m v)| < \epsilon.$$

Thus if $A \subset S$ is compact, $A \in \Sigma$, and $\min_A f \geqq \epsilon$, $P_m(A)$ must lie in the spherical shell σ in E_m, $\sigma = \{\varphi \in E_m \mid \rho \leqq \|\varphi\| \leqq 1\}$. The unit sphere S_{m-1} in E_m has genus m. It easily follows from 1° and 2° of Lemma 3.5 that $\gamma(\sigma) = m$. Since P_m is an odd continuous map, by 1° again, $\gamma(A) \leqq m$.

The above analysis shows $\min_A f \geqq \epsilon$ implies $\gamma(A) \leqq m$. Conversely $\gamma(A) > m$ implies $\min_A f < \epsilon$ and in particular $c_n < \epsilon$ for all $n \geqq m$.

(D) *The limit procedure.* We complete the proof of Theorem 5.1 by showing a subsequence of $u_{n,k}$ converges to u_k, a critical point of f with $f(u_n) = c_k$. Since $B_1(0)$ is weakly compact, for fixed k a subsequence of u_{nk} (which for notational convenience we take to be all of u_{nk}) converges weakly to $u_k \in B_1(0)$. Therefore using the weak continuity of f, $f(u_{nk}) = c_{nk} \to f(u_k) = c_k > 0$ as $n \to \infty$. Hence $u_k \neq 0$. Since T maps weakly convergent sequences to strongly convergent sequences, $T(u_{nk}) \to T(u_k) \neq 0$ and $\lambda_{nk} = (T(u_{nk}), u_{nk}) \to (T(u_k), u_k) = \lambda_k$. Equation (5.7) now implies $(T(u_k) - \lambda_k u_k, \varphi) = 0$ for all $\varphi \in E_n$, $n \geqq k$, and therefore for all $\varphi \in E$. Hence

$$(5.17) \qquad T(u_k) = \lambda_k u_k.$$

Taking the inner product of (5.17) with $T(u_k)$ gives

$$(5.18) \qquad \|T(u_k)\|^2 = \lambda_k^2 \neq 0.$$

Therefore $\lambda_k \neq 0$. Taking the inner product of (5.17) with u_k then shows

$$(5.19) \qquad \lambda_k = \lambda_k \|u_k\|^2$$

or $u_k \in S$. Thus u_k is a critical point of f on S. Moreover $u_{nk} \to u_k$

as $n \to \infty$. This follows from $\|u_{nk} - u_k\|^2 = 2 - 2(u_{nk}, u_k) \to 0$ as $n \to \infty$. Lemma 5.12 implies the existence of infinitely many u_k's. Hence the proof of Theorem 5.1 is complete.

As in §4, a multiplicity result can be obtained here. In fact this result can be used as part of a direct existence proof of the infinite-dimensional case.

LEMMA 5.20. *Under the hypotheses of Theorem* 5.1, *suppose* $c_{m+1} = \cdots = c_{m+p} \equiv c$. *Then if* $K_c = \{u \in S \mid f(u) = c,\ \Gamma u = 0\}$, $\gamma(K_c) \geqq p$.

PROOF. Note first that K_c is compact. This follows from the last part of the proof of Theorem 5.1. More precisely, let $(v_n) \subset K_c$. It can be assumed that $v_n \to v$ (weakly) as $n \to \infty$. Therefore $f(v) = c > 0$ so $v \neq 0$. Thus $T(v_n) \to T(v) \neq 0$ and, as in (5.18), $\lambda^2 = \|T(v)\|^2 \neq 0$. As in (5.19), $\|v\| = 1$ and $v_n \to v$. Hence the compactness of K_c.

Suppose $\gamma(K_c) \leqq p - 1$. Then, by 7° of Lemma 3.5, there is a neighborhood $N_\eta(K_c) \subset S$ such that $\gamma(N_\eta(K_c)) = \gamma(K_c) \leqq p - 1$. Another application of the argument used to prove K_c is compact shows there exists σ, $\tau > 0$ such that $u \in S - \operatorname{int} N_\eta(K_c)$ and $|f(u) - c| < \sigma$ implies $\|\Gamma u\| > \tau$.

The definition of $c = c_{m+p}$ implies there is $A \subset S$, A compact, $\gamma(A) \geqq m + p$ and $\min_A f > c - \beta\tau^2/4$ where $\beta\tau^2/4 < \sigma$ and

$$0 < \beta < \frac{1}{2\gamma_2} \min \left\{ \frac{1}{2}\, \delta\left(\frac{\|\Gamma u\|}{8}\right), \sqrt{\gamma_3}, \frac{1}{3} \right\},$$

the γ_r, $1 \leqq r \leqq 3$, being as in (B) of the proof of Theorem 4.1 (with S^{n-1} replaced by S).

Let $B = \{u \in A \mid f(u) < c + \sigma\}$. Then $B \in \Sigma(E)$. Moreover $\|\Gamma u\| > \tau$ on $B - \operatorname{int} N_\delta(K_c)$ and therefore $2\beta < \alpha(u)$ on this set (where $\alpha(u)$ is as in §4). Next consider $A - \operatorname{int} N_\eta(K_c) = A - N_\eta(K_c) \equiv C$. By 5° of Lemma 3.5, $\gamma(C) \geqq \gamma(A) - \gamma(N_\eta(K_c)) \geqq m + 1$. Letting χ be the infinite-dimensional analogue of the χ of Lemma 4.6 — see the remark following that lemma — by 1° of Lemma 3.5, $\gamma(\chi(C)) \geqq m + 1$. Hence, by the definition of $c = c_{m+1}$, $\min_{\chi(C)} f \leqq c$. But if $u \in B - \operatorname{int} N_\eta(K_c)$, by Lemma 4.6,

$$f(\chi(u)) \geqq f(u) + \tfrac{1}{4}\alpha(u)\, \|\Gamma u\|^2 > c - \tfrac{1}{4}\beta\tau^2 + \tfrac{1}{2}\beta\tau^2 = c + \tfrac{1}{4}\beta\tau^2$$

while if $u \in A - B$, $f(\chi(u)) \geqq f(u) \geqq c + \sigma$. Thus we have a contradiction and the lemma is proved.

We conclude this section with some remarks. Since under the hypotheses of Theorem 5.1, any solution of (5.2) on $\|u\| = r$ has $\lambda^2 = \|T(u)\|^2 \neq 0$, we can divide (5.2) by λ to obtain

$$u = \nu T(u) \tag{5.21}$$

where $\nu = \lambda^{-1}$. Suppose further that $T(u) = Lu + M(u)$ where L is a compact selfadjoint linear operator and $M(u) = o(\|u\|)$ near $u = 0$. Let $(\nu_k(\rho), u_k(\rho))$ denote the solutions of (5.21) on $\partial B_\rho(0)$ obtained from (5.8). Then one might suspect that $(\nu_k(\rho), u_k(\rho)) \to (\mu, 0)$ as $\rho \to 0$ for some $\mu \in r(L)$. This is indeed the case and in fact under appropriate hypotheses each $\mu \in r(L)$ is a bifurcation point for (5.21) [29]. A remarkable theorem of Krasnoselski [2] shows this under more general hypotheses. Namely Krasnoselski shows if T is a compact potential operator, $T(0) = 0$, $f(0) = 0$, f is uniformly Fréchet differentiable near $u = 0$, and $L = T'(0)$ is selfadjoint then all $\mu \in r(L)$ are bifurcation points for $u = \lambda T(u)$. Note that f need not be even here. For a proof of a slightly weaker result using Morse theory, see [30]. For these variational cases, nothing seems to be known about the structure of the solution set comparable to Theorem 1.10.

For applications of more general versions of Theorem 5.1 to partial differential equations, see e.g. Professor Browder's lectures in this symposium or [25].

Theorem 5.1 implies (5.2) possesses infinitely many solutions $(\lambda_n(\rho), u_n(\rho))$ on $\partial B_\rho(0)$. In some recent papers Coffman [26], Hempel [31], and at this symposium Clark [32] have used variational arguments to obtain some interesting lower bounds on the number of solutions of equations of the form (5.2) where λ is fixed rather than u being constrained to be on a sphere or more general manifold.

6. **More on continua.** In this final section some additional results about continua of solutions of functional equations will be presented for situations where bifurcation need not occur. Applications to elliptic and hyperbolic partial differential equations will be given. First we give a result essentially due to Leray and Schauder [5]. See also [33]. Consider

$$u = G(\lambda, u) \tag{6.1}$$

where $G : \boldsymbol{R} \times E \to E$ is compact and $G(0, u) = 0$. Then $(0, 0)$ is a solution of (6.1). Let $\mathcal{T}$ denote the closure of the set of solutions of (6.1) and $\Phi(\lambda, u) = u - G(\lambda, u)$.

THEOREM 6.2. *The component of* $\mathcal{J}$ *to which* $(0,0)$ *belongs is unbounded in* $R^+ \times E$ *and in* $R^- \times E$.

PROOF. Let $\mathcal{C}$ denote the component of $\mathcal{J}$ containing $(0,0)$. Note that $\mathcal{C} \cap (\{0\} \times E) = \{(0,0)\}$, i.e., $(0,0)$ is the unique solution of (6.1) having $\lambda = 0$. If the conclusion of Theorem 6.2 is not valid, essentially as in Lemma 1.9 there is a bounded open set $\mathcal{O} \subset R \times E$ such that $\mathcal{C} \subset \mathcal{O}$, $\mathcal{J} \cap \partial\mathcal{O} = \emptyset$, and $\partial\mathcal{O} \cap (\{0\} \times E) \subset (\{0\} \times B_1(0))$. Let $\mathcal{O}_\lambda = \{u \in E \mid (\lambda, u) \in \mathcal{O}\}$. By Lemma 1.8 with $\Lambda = R$,

$$d(\Phi(\lambda, \cdot\,), \mathcal{O}_\lambda) \equiv \text{constant} \equiv c, \qquad \lambda \in R. \tag{6.3}$$

Since $\mathcal{O}_\lambda = \emptyset$ for $|\lambda|$ large, $c = 0$. On the other hand, $\Phi(0, \cdot\,) = I$ so $c = 1$, a contradiction.

REMARK. If $G(\lambda, 0) \equiv 0$, then $\mathcal{J}$ contains the line of trivial solutions, $\{(\lambda, 0) \mid \lambda \in R\}$, so for this case Theorem 6.2 is of no interest.

Next some applications of Theorem 6.2 to partial differential equations will be given. Consider first the quasilinear elliptic boundary value problem

$$\begin{cases} -\sum_{i,j=1}^{n} a_{ij}(x, u, Du)u_{x_i x_j} = \lambda F(x, u, Du, \lambda), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6.4}$$

where $\Omega \subset R^n$ is a smooth bounded domain, a_{ij} and F are continuously differentiable functions of their arguments, the uniform ellipticity condition (2.28) holds, and $F(x, 0, 0, \lambda) > 0$ for $x \in \Omega$ and $\lambda \in R^+$. As in §2, (6.4) is converted to an operator equation in $E = C^{1+\alpha}(\Omega) \cap$ B.C. where $\alpha \in (0, 1)$. For $(\lambda, u) \in R \times E$, let $v = G(\lambda, u) \in C^{2+\alpha}(\Omega)$ denote the unique solution of

$$\begin{cases} -\sum_{i,j=1}^{n} a_{ij}(x, u, Du)v_{x_i x_j} = \lambda F(x, u, Du, \lambda), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{6.5}$$

Then G is compact and Theorem 6.2 applies here. However a better result obtains. Let P^+ be as in §2.

THEOREM 6.6. $\mathcal{C} \cap (R^+ \times E) \subset (R^+ \times P^+) \cup \{(0,0)\}$.

PROOF. For $(\lambda, u) \in \mathcal{S} \cap (R^+ \times E)$ and u near 0, the right-hand side of (6.4) is nonnegative. Hence, by the maximum principle, $u \in P^+$. If $\mathcal{C} \cap (R^+ \times E) \not\subset (R^+ \times P^+) \cup \{0, 0\}$, there exists

$(\lambda, u) \in \mathcal{C} \cap (\boldsymbol{R} \times \partial P^+)$ with $\lambda > 0$. Therefore, $u(x_0) = 0$ for some $x_0 \in \Omega$ or $\partial u(x_0)/\partial \nu = 0$ for some $x_0 \in \partial\Omega$. Arguing as in Lemma 2.37 shows $u \equiv 0$ in Ω. Hence $(\lambda, u) = (\lambda, 0)$. But $(\lambda, 0)$ does not satisfy (6.4). The theorem is proved.

As a second application we treat the Dirichlet problem for the "H-equation" which arises in connection with the parametric form of the Plateau problem, i.e., the problem of spanning a given curve in $\boldsymbol{R}^3$ by a surface of prescribed mean curvature [34]. (We thank S. Hildebrandt for informing us of this question.) The equations involved are

$$(6.7)\qquad \begin{aligned} \Delta U &= 2HU_x \wedge U_y, && x^2 + y^2 < 1, \\ U &= h, && x^2 + y^2 = 1. \end{aligned}$$

Here Δ is the 2-dimensional Laplacian, $U(x, y) = (u(x, y),\ v(x, y),\ w(x, y))$, $h \in (C^\alpha(\partial\Omega))^3$, H is a constant, and $\wedge$ denotes vector product. We extend h to $\Omega = \{x^2 + y^2 < 1\}$, e.g. as a triple of harmonic functions, and let $V = U - h$. Substituting in (6.7) yields

$$(6.8)\qquad \begin{aligned} \Delta V &= 2H(h + V)_x \wedge (h + V)_y && \text{in } \Omega, \\ V &= 0 && \text{on } \partial\Omega. \end{aligned}$$

It is now a simple matter to convert (6.8) to an operator equation in $E = (C^{1+\alpha}(\Omega) \cap \text{B.C.})^3$ and of the form (6.1) with G satisfying the hypotheses of Theorem 6.2. We will not formalize this result. It would be interesting if the same sort of procedure could be applied to the Plateau problem itself.

Our last application of Theorem 5.2 will be to a hyperbolic partial differential equation. Consider the nonlinear wave equation

$$(6.9)\qquad \Box u \equiv u_{tt} - u_{xx} = \lambda F(x, t, u), \qquad 0 < x < \pi, 0 \leqq t \leqq 2\pi,$$

together with the periodicity and boundary conditions

$$(6.10)\qquad u(x, t + 2\pi) = u(x, t), \qquad u(0, t) = 0 = u(\pi, t).$$

The function F is assumed to be a twice continuously differentiable function of its arguments, 2π periodic in t. For convenience we introduce some notation. Let L^2 denote the closure of continuous function on $[0, \pi] \times [0, 2\pi]$, 2π periodic in t with respect to

$$|\varphi|^2 = \int_0^\pi \int_0^{2\pi} (\varphi(x, t))^2 \, dx \, dt$$

and let C^k denote the set of k times continuously differentiable functions of (x, t) on the above domain and 2π periodic with respect to t. As a norm in C^k we take $\| \cdot \|_n$ where

$$\|u\|_k = \sum_{|\sigma| \leqq k} \|D^\sigma u\|, \qquad \|\varphi\| = \max_{0 \leqq x \leqq \pi;\ 0 \leqq t \leqq 2\pi} |\varphi(x, t)|.$$

Finally set $C^0 \equiv C$ and $\| \cdot \|_0 \equiv \| \cdot \|$.

The problem (6.9)–(6.10) differs from those considered earlier in that $\Box$ together with (6.10) possesses a null space, in fact an infinite-dimensional one containing all $\varphi(x, t) = p(x + t) - p(-x + t)$ where p is 2π periodic in its argument and twice continuously differentiable. Let N denote the closure of this null space in L^2. Then $(0, u)$ satisfies (6.9)–(6.10) for all $u \in N \cap C^2$ (and in a generalized sense for all $u \in N$). In analogy with §1, these solutions will be called the *trivial solutions* of (6.9)–(6.10). A natural question to ask is what are the bifurcation points of (6.9)–(6.10) with respect to the space of trivial solutions. (Again in analogy with §1, $(0, u)$ is a *bifurcation point* of (6.9)–(6.10) if every neighborhood of $(0, u)$ contains nontrivial solutions.)

To find a necessary condition for $(0, v)$, $v \in N \cap C^2$, to be a bifurcation point, observe that if (λ, u) is a classical solution of (6.9)–(6.10), then because of the selfadjointness of $\Box$ with respect to the conditions (6.10),

$$\int_0^{2\pi} \int_0^\pi F(x, t, u)\varphi \, dx \, dt = 0 \quad \text{for all } \varphi \in N. \tag{6.11}$$

Letting $(\lambda, u) \to (0, v)$, convergence being in $\boldsymbol{R} \times C$, the necessary condition becomes $v \in N \cap C^2$, and v satisfies (6.11). Henceforth (6.11) will be denoted more briefly by $F(x, t, v) \perp N$. We pose a more general bifurcation question: Given $w \in C$, does there exist $v = V(w) \in N \cap C$ such that $F(x, t, v + w) \perp N$. An answer is provided by the following result:

LEMMA 6.12. *If $F(x, t, \xi)$ is strongly monotone in ξ, i.e. $\partial F(x, t, \xi)/\partial \xi \geqq \beta > 0$ for all $(x, t, \xi) \in [0, \pi] \times [0, 2\pi] \times \boldsymbol{R}$, then there exists a unique $v = V(w) \in N \cap C$ satisfying $F(x, t, v + w) \perp N$. The map $w \to V(w)$, $C \to N \cap C$ is continuous. Moreover if $w \in C^k$ and F is C^k in its arguments, then $V(w) \in C^k \cap N$.*

PROOF. The existence and regularity results are rather lengthy and will not be carried out here. See [35], [36]. To show the uniqueness of v, observe that, if v_1, v_2 are two solutions, (6.11) implies

$$
\begin{aligned}
0 &= \int_0^{2\pi} \int_0^{\pi} (F(x,t,v_1+w) - F(x,t,v_2+w))(v_1 - v_2)\,dx\,dt \\
&\geqq \beta \int_0^{2\pi} \int_0^{\pi} |v_1 - v_2|^2\,dx\,dt.
\end{aligned}
\tag{6.13}
$$

Hence $v_1 = v_2$.

To proceed we also need a simpler lemma which will be stated without proof. See [37]. Let $N^\perp$ denote the orthogonal complement of N in L^2.

Lemma 6.14. *If* $f \in N^\perp \cap C^k$, $k \geqq 1$, *there exists a unique* $w \in N^\perp \cap C^{k+1}$ *such that* $\Box w = f$, w *satisfies* (6.10), *and*

$$\|w\|_{k+1} \leqq c_k \|f\|_k. \tag{6.15}$$

With the aid of Lemmas 6.12 and 6.14, (6.9)–(6.10) can be converted to an operator equation of the form (6.1) in $E = N^\perp \cap C^2$ as follows: Let $w \in E$ and $\lambda \in \mathbf{R}$. Then by Lemma 6.12 there exists a unique $v = V(w) \in N \cap C^2$ such that $F(x, t, v + w) \perp N$. Hence by Lemma 6.14, there is a unique $W = G(\lambda, w) \in N^\perp \cap C^3$ satisfying

$$\Box W = \lambda F(x, t, V(w) + w), \qquad 0 < x < \pi, 0 \leqq t \leqq 2\pi, \tag{6.16}$$

and (6.10). It is easy to see that G is compact and $G(0, w) \equiv 0$. Hence Theorem 6.1 is applicable here and

Theorem 6.17. *There exists a component of the solution set of* (6.16), (6.10) *which meets* (0, 0) *and is unbounded in* $\mathbf{R} \times (C^2 \cap N^\perp)$.

It is then not too difficult to show that (5.9)–(5.10) possesses an unbounded component of nontrivial solutions which meets the bifurcation point $(0, V(0))$.

We conclude this section with a different sort of application to an elliptic problem. This was motivated by recent work of L. Nirenberg [38] which in turn generalizes a result of Landesman and Lazer [39]. Let $\mathcal{D} \subset \mathbf{R}^n$ be a smooth bounded domain. Consider the boundary value problem

$$
\begin{cases}
\mathcal{L}u \equiv \sum\limits_{|\sigma| \leqq 2m} a_\sigma(x) D^\sigma u = \lambda g(x, u), & x \in \mathcal{D}, \\
\mathcal{B}_j u \equiv \sum\limits_{|\gamma| \leqq m_j} b_{j\gamma}(x) D^\gamma u = 0, & x \in \partial\mathcal{D}, 1 \leqq j \leqq m,
\end{cases}
\tag{6.18}
$$

where the usual multi-index notation is being employed. $\mathcal{L}$ is a uniformly elliptic operator of order $2m$ with smooth coefficients and the boundary conditions are assumed to be complementary with $m_j \leqq m$,

$1 \leqq j \leqq m$ (see [40]). The function g is assumed to be continuously differentiable in its arguments. $\mathcal{L}$ together with the B.C. is a Fredholm map of $C^{2m+\alpha}(\mathcal{D}) \to C^{\alpha}(\mathcal{D})$ for any $\alpha \in (0, 1)$ [40].

As in the earlier examples in this section, (6.18) will be converted to an equivalent operator equation. However we will not in general be able to achieve the form (6.1). Let $E = C^{\alpha}(\mathcal{D})$. Let the kernel of $\mathcal{L}$ together with the B.C. be span $\{v_1, \cdots, v_k\} = V_k$ and let the cokernel be span $\{w_1, \cdots, w_j\} \equiv W_j$. We can identify V_k with $\boldsymbol{R}^k$, W_j with $\boldsymbol{R}^j$ and write $E = \boldsymbol{R}^k \times \hat{E} = \boldsymbol{R}^j \times \tilde{E}$ where $\hat{E}$, $\tilde{E}$ are respectively the subspaces of E orthogonal in an L^2 sense to $\boldsymbol{R}^k$, $\boldsymbol{R}^j$, i.e., $u \in E$ implies $u = \sum_1^k a_i v_i + \hat{u}$ where

$$\hat{u} \in \hat{E} = \left\{ w \in E \mid \int_{\mathcal{D}} w v_i \, dx = 0, 1 \leqq i \leqq k \right\}$$

and similarly for $\tilde{E}$. Thus if $u \in E$, $u = (a, \hat{u}) = (b, \tilde{u})$ where $a \in \boldsymbol{R}^k$, $b \in \boldsymbol{R}^j$, $\hat{u} \in \hat{E}$, $\tilde{u} \in \tilde{E}$. Let $\tilde{P}$ be the L^2 orthogonal projector on $\tilde{E}$, i.e., $\tilde{P}u = \tilde{u}$ for $u \in E$. Then if $f \in E$, there exists a unique $\hat{U} \in C^{2m+\alpha}(\Omega) \cap \hat{E}$ such that [40]

$$(6.19) \qquad \begin{cases} \mathcal{L}\hat{U} = \tilde{P}f, & x \in \mathcal{D}, \\ \mathcal{B}_j \hat{U} = 0, & x \in \partial\mathcal{D}, 1 \leqq j \leqq m. \end{cases}$$

With the aid of these preliminaries (6.18) can be converted to an operator equation in $E = \boldsymbol{R}^k \times \hat{E}$. Let $u = (a, \hat{u}) \in \boldsymbol{R}^k \times \hat{E}$ and let $G(\lambda, u) = (A, \hat{U})$ where

$$(6.20) \qquad \begin{cases} A_i = a_i - \lambda \displaystyle\int_{\mathcal{D}} g(x, u) w_i \, dx, & 1 \leqq i \leqq j, \\ \quad = a_i, & j + 1 \leqq i \leqq k \ (\text{if } k > j) \end{cases}$$

and

$$(6.21) \qquad \begin{cases} \mathcal{L}\hat{U} = \lambda \tilde{P} g(x, u), & x \in \mathcal{D}, \\ \mathcal{B}_i \hat{U} = 0, & x \in \partial\mathcal{D}, 1 \leqq j \leqq m, \end{cases}$$

where $\hat{U} \in C^{2m+\alpha}(\mathcal{D}) \cap \hat{E}$. By our above remarks it is easily seen that $G(\lambda, u)$ is compact. Note that if $\lambda \neq 0$, $G(\lambda, u) = u$ if and only if (λ, u) is a solution of (6.18). (For $\lambda = 0$, all $a \in \boldsymbol{R}^k$ are solutions.)

Let $\Phi(\lambda, u) = u - G(\lambda, u)$. Nirenberg [38] studied the case $j \leqq k$, i.e., the Fredholm index of $\mathcal{L}$ is $\geqq 0$. (He also worked in a Sobolev space framework rather than $C^{\alpha}(\Omega)$ so he could treat continuous g.) If $j < k$, the usual Leray-Schauder degree of $\Phi(\lambda, \cdot)$ must be zero since then $\Phi(\lambda, \cdot)$ maps $\boldsymbol{R}^k \times \hat{E}$ into a proper subset of itself as (6.20) shows. (More precisely if, for some $\Omega \subset \boldsymbol{R}^k \times \hat{E}$, $d(\Phi(\lambda, \cdot), \Omega, 0) \neq 0$, then $d(\Phi(\lambda, \cdot), \Omega, b) \neq 0$ for all b small. But the range of $\Phi(\lambda, \cdot)$ is a proper subspace of $R^k \times \hat{E}$ so this is impossible.) Nirenberg showed if $j < k$ and $\mathcal{L}$, g satisfy various technical conditions, then one can use an extension of the Leray-Schauder theory to conclude the existence of zeros of Φ.

Instead of using any generalized notion of degree we will apply the Leray-Schauder theory. Assume that g is an odd function of u. Then (6.18) has the solution $(\lambda, 0)$. The oddness of g implies that $G(\lambda, (a, \hat{u})) = -G(\lambda, (-a, -\hat{u}))$, i.e., G is odd in u. Let Ω be a symmetric bounded open neighborhood of 0 in $R^k \times \hat{E}$. Since by our above remarks, $d(\Phi(\lambda, \cdot), \Omega, 0) = 0$, the remark following Lemma 3.2 implies that $\Phi(\lambda, \cdot)$ has a pair of zeros $u, -u \in \partial\Omega$. Since this is true for all such Ω, it follows from the compactness of G and a lemma from point set topology [7] that $\Phi(\lambda, \cdot)$ has a symmetric unbounded component $\mathcal{C}$ ($\equiv \mathcal{C}^+ \cup \mathcal{C}^-$, $\mathcal{C}^- = -\mathcal{C}^+$) of zeros in $R^k \times \hat{E}$ which meet $u = 0$. With a bit more work it can be shown that

THEOREM 6.22. *Under the above hypotheses on $\mathcal{L}$ and g, for each $\lambda \in R$,* (6.18) *possesses a symmetric unbounded component of solutions in* $\{\lambda\} \times C^{2m+\alpha}(\mathcal{D})$.

REMARK. An analogous result can also be obtained if one works with $W^{2m,p}$ as does Nirenberg rather than $C^{2m+\alpha}$. Also provided that we have oddness, g can be permitted to depend on derivatives of u up to order $2m - 1$.

REFERENCES

1. J. B. Keller and S. Antman (Editors), *Bifurcation theory and nonlinear eigenvalue problems,* Benjamin, New York, 1969. MR **39** #2555.
2. M. A. Krasnosel'skiĭ, *Topological methods in the theory of nonlinear integral equations,* GITTL, Moscow, 1956; Egnlish transl., Macmillan, New York, 1964. MR **20** #3464; MR **28** #2414.
3. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory,* Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.
4. J. T. Schwartz, *Nonlinear functional analysis,* Lecture Notes, Courant Institute of Mathematical Sciences, New York University, 1965.
5. J. Leray and J. Schauder, *Topologie et équations fonctionelles,* Ann. Sci. Ecole Norm. Sup. (3) **51** (1934), 45–78. MR **9**, 606.
6. P. H. Rabinowitz, *Nonlinear Sturm-Liouville problems for second order ordinary differential equations,* Comm. Pure Appl. Math. **23** (1970), 939–961.
7. G. T. Whyburn, *Topological analysis,* Princeton Math. Series, no. 23, Princeton Univ. Press, Princeton, N. J., 1958, Chap. 1. MR **20** #6081.
8. P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems,* J. Functional Analysis **7** (1971), 487–513.
9. ——, *A global theorem for nonlinear eigenvalue problems and applications,* Contributions to Nonlinear Functional Analysis, edited by E. H. Zarantonello, Academic Press, New York, 1971.

10. R. E. L. Turner, *Transversality in nonlinear eigenvalue problems,* Contributions to Nonlinear Functional Analysis, edited by E. H. Zaratonello, Academic Press, New York, 1971.

11. L. Bauer, E. L. Riess and H. B. Keller, *Axisymmetric buckling of hollow spheres and hemispheres,* Comm. Pure Appl. Math. **23** (1970), 529–568. MR **43** #4335.

12. M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues,* J. Functional Analysis 8 (1971), 321–340.

13. N. Bazley and B. Zwahlen, *A branch of positive solutions of nonlinear eigenvalue problems,* Manuscripta Math. **2** (1970), 365–374. MR **42** #3628.

14. M. Reeken, *General theorem on bifurcation and its application to the Hartree equation of the helium atom,* J. Math. Phys. **11** (1970), 2505–2512. MR 43 #5369.

15. P. H. Rabinowitz, *A note on a nonlinear eigenvalue problem for a class of differential equations,* J. Differential Equations 9 (1971), 536–548. MR 42 #7980.

16. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations,* McGraw-Hill, New York, 1955. MR **16** #1022.

17. Z. Nehari, *Characteristic values associated with a class of nonlinear second-order differential equations,* Acta Math. **105** (1961), 141–175. MR **23** #A1097.

18. R. Courant and D. Hilbert, *Methods of mathematical physics.* Vols. 1, 2, Interscience, New York, 1953, 1962. MR **16**, 426; MR **25** #4216.

19. J. H. Wolkowisky, *Nonlinear Sturm-Liouville problems,* Arch. Rational Mech. Anal. **35** (1969), 299–320. MR **40** #4528.

20. P. H. Rabinowitz, *A nonlinear Sturm-Liouville theorem,* Proc. Japan-United States Seminar on Ordinary Differential and Functional Equations, Lecture Notes in Math., no. 243, Springer, New York, 1971.

21. R. E. L. Turner, *Superlinear Sturm-Liouville problems* (to appear).

22. M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space,* Uspehi Mat. Nauk 3 (1948), no. 1 (23), 3–95; English transl., Amer. Math. Soc. Transl. (1) **10** (1962), 199–325. MR **10**, 256.

23. L. A. Ljusternik, *The topology of the calculus of variations in the large,* Trudy Mat. Inst. Steklov. **19** (1947); English transl., Transl. Math. Monographs, vol. 16, Amer. Math. Soc., Providence, R. I., 1966. MR 9, 596; MR **36** #906.

24. R. S. Palais, *Critical point theory and the minimax principle,* Proc. Sympos. Pure Math., vol. 15, Amer. Math. Soc., Providence, R. I., 1970, pp. 185–212. MR 41 #9303.

25. F. E. Browder, *Existence theorems for nonlinear partial differential equations,* Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R. I., 1970, pp. 1–60. MR **42** #4855.

26. Charles V. Coffman, *A minimum-maximum principle for a class of nonlinear integral equations,* J. Analyse Math. 22 (1969), 391–419. MR **40** #3224.

27. J. Dugundji, *An extension of Tietze's theorem,* Pacific J. Math. 1 (1951), 353–367. MR **13**, 373.

28. M. M. Vaĭnberg, *Variational methods for the study of non-linear operators,* GITTL, Moscow, 1956; English transl., Holden-Day, San Francisco, Calif., 1964. MR **19**, 567; MR 31 #638.

29 M. Berger, *A bifurcation theory for nonlinear elliptic partial differential equations and related systems,* 113–190 in [1].

30. A. Marino and G. Prodi, *La teoria di Morse per gli spazi di Hilbert,* Rend. Sem. Mat. Univ. Padova 41 (1968), 43–68. MR 41 #2715.

31. J. A. Hempel, *Multiple solutions for a class of nonlinear boundary value problems,* Indiana Univ. Math. J. **20** (1970/71), 983-996. MR **43** #5145.

32. D. C. Clark, *A variant of the Ljusternik-Schnirelmann theory,* Indiana Univ. Math. J. **22** (1972).

33. F. E. Browder, *On continuity of fixed points under deformations of continuous mappings,* Summa Brasil. Mat. **4** (1960), 183-191. MR **24** #A543.

34. E. Heinz, *Über die Existinz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung,* Math. Ann. **127** (1954), 258-287. MR **16**, 1115.

35. P. H. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations,* Comm. Pure Appl. Math. **20** (1967), 145-205. MR **34** #6325.

36. ———, *Time periodic solutions of nonlinear wave equations,* Manuscripta Math. **5** (1971), 165-194.

37. H. Lovicarová, *Periodic solutions of a weakly nonlinear wave equation in one dimension,* Czechoslovak Math. J. **19** (94) (1969), 324-342. MR **40** #518.

38. L. Nirenberg, *An application of generalized degree to a class of nonlinear problems,* Proc. Conference on Functional Analysis, Liège, Sept. 1970.

39. E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance,* J. Math. Mech. **19** (1969/70), 609-623. MR **42** #2171.

40. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I., Comm. Pure Appl. Math. **12** (1959), 623-727. MR **23** #A2610.

UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706