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SOLUTION OF THE ALMOST COMPLEX SPHERES PROBLEM USING K-THEORY

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1. **Introduction.** Let $F(n)$ denote $SO(2n)/U(n)$. We shall abbreviate $K_C^*(X)$ to simply $K(X)$. Finally, $K(X; Q)$ represents $K(X) \otimes Q$, where Q is the field of rational numbers.

The two results of this paper are the following:

1.1. A description of $K(F(n); Q)$.

1.2. A new proof that the only almost complex spheres are S^2 and S^6.

The first proof that the only almost complex spheres are S^2 and S^6 was given by Borel and Serre in [5]; their proof used the Steenrod reduced power operations. Our proof uses 1.1 and the Chern character.

The contents of this paper are as follows: §2 contains background material. In §3 we calculate $K(F(n); Q)$. We also indicate a method for calculating $K(F(n))$. §4 is devoted to 1.2.

This material constitutes part of the author's doctoral thesis [2]. I wish to thank Professor Albert Lundell for his advice.

2. **Background.** A complete reference for this section is [8].

A $2n$-dimensional real manifold M is *almost complex* if its tangent sphere bundle

$$S^{2n-1} \to T(M) \to M$$

with structural group $O(2n)$ is equivalent in $O(2n)$ to a bundle with structural group $U(n)$. This happens if and only if the associated bundle with fibre $F(n)$ has a cross section.

For the sphere S^{2n}, the tangent sphere bundle is

$$S^{2n-1} \to SO(2n+1)/SO(2n-1) \to S^{2n}.$$

The associated principal bundle is

$$SO(2n) \to SO(2n+1) \to S^{2n},$$

and, since $SO(2n+1)/U(n) \approx F(n+1)$, the associated bundle with

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fibre $F(n)$ is

$$(2.1) \qquad F(n) \to F(n+1) \to S^{2n}.$$

We shall show in §4 that (2.1) has a section only if $n < 4$. This will complete 1.2, since it is well known that S^2 and S^6 are almost complex while S^4 is not.

3. **Description of $K(F(n); Q)$.** We begin with the easy task of describing the additive structure of $K(F(n))$.

Given a locally trivial fibration

$$F \xrightarrow{i} E \to S^p,$$

there is an exact *Wang sequence* for K-theory;

$$(3.1) \cdots \to K^{-n-1}(F) \xrightarrow{\theta} K^{-n-p}(F) \xrightarrow{\phi} K^{-n}(E) \xrightarrow{i^!} K^{-n}(F) \to \cdots.$$

This sequence can be constructed in the same way as the Wang sequence for ordinary cohomology. Moreover, the action of θ and ϕ on products is the same as the action on products of the corresponding maps in the Wang sequence for ordinary cohomology. Details can be found in [2] or [7].

Using the Wang sequence for the fibration $F(n) \to F(n+1) \to S^{2n}$ and induction on n, we easily arrive at

THEOREM 3.2. *Additively, $K(F(n)) = K^0(F(n))$, and $K(F(n))$ is the direct sum of 2^{n-1} copies of the integers.*

The Wang sequence provides scant information about products in the ring $K(F(n))$. For that we refer to representation theory.

Let $T(n) \subset U(n) \subset SO(2n)$ where $T(n)$ is a maximal torus. Any representation $r : U(n) \to U(m)$ together with the classifying map for the bundle $U(n) \to SO(2n) \to F(n)$ induces a principal $U(m)$-bundle over $F(n)$. This induces a ring homomorphism $\alpha : RU(U(n)) \to K(F(n))$ which we shall use to describe $K(F(n); Q)$.

Recall that $RU(T(n)) = Z[y_i, y_i^{-1}]$, $i = 1, \cdots, n$, where $y_j : T(n) \to U(1)$ is given by $y_j(t_1, \cdots, t_n) = \exp(2\pi i t_j)$. Here we are considering $T(n)$ as n-tuples of reals mod 1. Under the map $RU(U(n)) \to RU(T(n))$ induced by restriction of representations, $RU(U(n))$ is identified with the ring of finite symmetric Laurent series in $y_1, \cdots, y_n$ with integer coefficients. Details can be found in [6].

We introduce some notation. In $RU(U(n))$ we set $z_i = y_i - 1$, for $i = 1, \cdots, n$. The symbol $\sigma_j(z)$ will represent the jth elementary symmetric function in $z_1, \cdots, z_n$. The map $\eta : K(F(n)) \to K(F(n); Q)$ is the coefficient map. We can now prove

THEOREM 3.3. *$K(F(n); Q)$ is generated by* 1 *and the simple monomials in* $g_1, \cdots, g_{n-1}$, *where* $g_j = \eta \circ \alpha(\sigma_j(z))$.

PROOF. We need some well-known facts about the following commutative diagram; see [3] and [4].

$$\begin{array}{ccc}
RU(U(n)) & \to & RU(T(n)) \\
\downarrow\alpha & & \downarrow\alpha \\
K(F(n)) & \to & K(SO(2n)/T(n)) \\
\downarrow\eta & & \downarrow\eta \\
K(F(n); Q) & \to & K(SO(2n)/T(n); Q) \\
\downarrow ch & & \downarrow ch \\
H^*(F(n); Q) & \to & H^*(SO(2n)/T(n); Q)
\end{array}$$

The necessary facts are the following:

(1) The composition down the right-hand column is given by $ch \circ \eta \circ \alpha(y_i^{\pm 1}) = \exp(\pm x_i)$, $i = 1, \cdots, n$.

Here, $\exp(\pm x_i)$ represents a power series in $H^*(SO(2n)/T(n); Q)$. As usual, $H^*(SO(2n)/T(n); Q)$ is identified as a quotient of $H^*(BT(n); Q)$, and $x_1, \cdots, x_n$ are the generators of $H^2(BT(n); Z)$.

(2) $H^*(F(n); Z)$ is generated by 1 and the simple monomials in $a_1, \cdots, a_{n-1}$, where $a_j = (1/2)\sigma_j(x) = (1/2)$($j$th Chern class of $F(n)$).

From these one easily concludes that

$$\begin{aligned} ch(g_j) &= ch \circ \eta \circ \alpha(\sigma_j(z)) \\ &= \sigma_j(x) + (\text{higher terms}) \text{ in } H^*(F(n); Q). \end{aligned}$$

Now the theorem follows from 2.4 of [1].

REMARKS. (1) It is possible to describe products in $K(F(n); Q)$ using ch and knowledge of the product structure of $H^*(F(n); Q)$. (2) If we let $U'(n)$ be the 2-fold covering of $U(n)$, then $F(n) = \mathrm{Spin}(2n)/U'(n)$. One can then describe generators for $K(F(n))$ in the image of $\alpha' : RU(U'(n)) \to K(F(n))$. Details can be found in [2].

4. **S^{2n} is not almost complex if $n > 3$.** Suppose that $s : S^{2n} \to F(n+1)$ is a section of the fibration $F(n) \to F(n+1) \xrightarrow{\pi} S^{2n}$. Then s induces $s^! : K(F(n+1); Q) \to K(S^{2n}; Q)$. Let $s^!(g_i) = m_i g$, $i = 1, \cdots, n$, where $g_i = \eta \circ \alpha(\sigma_i(z))$ is described in Theorem 3.3, and g is an integral generator. Since g_i is an integral class, each m_i is an integer. We shall show that m_1 can only be an integer if $n < 4$.

Let θ_n generate $H^{2n}(S^{2n}; Z)$, and let a_n be one of the generators of $H^{2n}(F(n+1); Z)$, as described in §3. Then, since $s^*\pi^* =$ identity, we have $s^*(a_n) = \pm\,\theta_n$, and $\pi^*(\theta_n) = \pm a_n +$ (higher terms).

We now compute $s^* \circ ch(g_1)$.

$$s^* \circ ch(g_1) = ch \circ s^!(g_1) = ch(m_1 g) = \pm m_1\,\theta_n.$$

If we compute $ch(g_1)$ first, we get

$$\begin{aligned} s^* \circ ch(g_1) &= s^*(\exp(x_1) + \cdots + \exp(x_{n+1}) - (n+1)) \\ &= s^*(\textstyle\sum_1(x) + (1/2!)\sum_2(x) + \cdots + (1/n!)\sum_n(x) + \cdots), \end{aligned}$$

where $\sum_j(x) = x_1{}^j + \cdots + x_n{}^j$. Since $H^q(S^{2n}; Q) = 0$ if $q \neq 2n$, we have

$$\begin{aligned} s^* \circ ch(g_1) &= s^*((1/n!)\textstyle\sum_n(x)) \\ &= s^*\{(1/n!)[\sigma_1(x)\textstyle\sum_{n-1}(x) - \sigma_2(x)\sum_{n-2}(x) + \cdots + (-1)^{n-1} n\sigma_n(x)]\} \\ &= (\pm 1/n!)\{s^*(n\sigma_n(x))\}, \end{aligned}$$

since products are trivial in $H^*(S^{2n}; Q)$. Therefore,

$$s^* \circ ch(g_1) = (\pm 1/(n-1)!)(2\,\theta_n).$$

We conclude that the integer m_1 is $\pm 2/(n-1)!$, which implies that $n < 4$.

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