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PERTURBATION AND APPROXIMATION THEORY FOR HIGHER-ORDER ABSTRACT CAUCHY PROBLEMS

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1. **Introduction and summary.** Given a (possibly unbounded) linear operator A on a Banach space B, and a polynomial $P(\lambda, s) = \sum_0^{i=m;j=t} c_{ij}\lambda^i s^j$, there is defined the abstract Cauchy problem

$$(0.1)\qquad \begin{aligned} P(d/dt, A)u &= 0 && \text{for } t > 0,\\ (d/dt)^j\,|_{t=0}\, u &= f_j && \text{for } 0 \leqq j \leqq m-1. \end{aligned}$$

It has been shown in [D] that if A generates a group $T(t)$ and if

$$(0.2)\qquad \int_{-\infty}^{\infty} \sum_{k=0}^{m-1} T(s) f_k \hat{g}_k(t, s)\, ds$$

converges, where $\hat{g}_k(t, s)$ is the solution of (0.1) in the special concrete case $A = -d/dx$, $f_j = \delta_{jk}\delta(x)$, then u is given by (0.2). If $\hat{g}_k$ is a generalized function, (0.2) is interpreted by integration by parts.

A number of concrete Cauchy problems may conveniently be studied in terms of (0.1). This was done in [M] for one-dimensional parabolic equations of arbitrary order whose coefficients are measurable functions of x; other applications are mentioned below.

In the present work we exploit formula (0.2) to study two types of perturbation problems for (0.1). In the last section, we replace the fixed generator A by an approximate generator A_ϵ. It is natural to suppose that if A_ϵ generates a group $T_\epsilon(t)$ and $T_\epsilon(t) \to T(t)$, then the corresponding solution u_ϵ converges to u. This is often true, but for some P we will see that it requires an extra restriction on the data f.

Most of our work is concerned with perturbations of the polynomial P; i.e., we keep A fixed and let the coefficients c_{ij} depend on ϵ. In particular, we allow the leading coefficients to vanish as $\epsilon \to 0$, so that *singular* as well as regular perturbations are included in our theorems.

For technical reasons, we find it convenient to treat several cases, depending on the "type" of P_ϵ and of P; see Friedman [C] or Gel'fand-Shilov [B] for definitions and properties. The details vary, depending on whether the approximating or limiting polynomial is hyperbolic,

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parabolic, or neither, but in each case our results have the following general form:

"If the fundamental solutions of Cauchy's problem for $P_\epsilon(d/dt, -d/dx)\hat{g}_\epsilon = 0$ converge to the fundamental solutions of Cauchy's problem for $P(d/dt, -d/dx)\hat{g} = 0$, then the solution of the abstract Cauchy's problem for $P_\epsilon(d/dt, A)u_\epsilon = 0$ converges to the corresponding solution of $P(d/dt, A)u = 0$." Thus variable-coefficient differential problems, for example, are brought into correspondence with constant-coefficient problems, which in principle can then be reduced to algebra by means of the Fourier transform. An example is

$$\epsilon u_{tt} + u_t = Bu \tag{0.3}$$

where B is a strongly elliptic operator on a domain $D \subset R^n$, with well-behaved space-dependent coefficients. It can be shown, using Theorem 23.9.5 of Hille-Phillips [Q], that if (0.3) is well posed, then B has a square root A which generates a group. Thus the singular perturbation problem for (0.3), with any appropriate homogeneous linear boundary conditions, is reduced to the special case $\epsilon u_{tt} + u_t = u_{xx}$ on R_1, which is treated in [E], [F]. (See Theorem 5 below.) We are indebted to Jerome Goldstein for pointing out that existence of a square root follows without extra hypotheses other than well-posedness of (0.3). Bobisud [F] may be cited as a reasonably general treatment of singular perturbations for the concrete Cauchy problem governed by a system of constant-coefficient partial differential equations.

It is explained in [D] how the representation (0.2) can be extended to systems of equations, to equations with time-dependent coefficients, and to equations involving several commuting operators A_i. Thus our singular perturbation results can be extended to those cases in a straightforward manner.

A forthcoming article will treat stability and convergence questions for finite-difference approximations to higher-order abstract Cauchy problems.

The literature on perturbations of higher-order abstract Cauchy problems is not as yet extensive. In 1966, Kato [P] wrote, "At present the abstract theory is not advanced enough to comprise singular perturbation theory for differential equations."

Since that time, several authors have studied abstract singular perturbation problems in Hilbert space. The most general results were obtained by Friedman [J] and by Bobisud and Calvert [K], who treated the case where A is selfadjoint and where $P_\epsilon(d/dt, A)$ has the special form $P_\epsilon(d/dt) + A$. (Friedman permits P_ϵ to have t-dependent coefficients.) These works depended on the spectral representation of A, as did the earlier work of Smoller [H], Latil [I], and Kisyński [G].

For nonselfadjoint operators, we are acquainted with only two earlier results. Schoene [O] used resolvent theory to treat $\epsilon u_{tt} + u_t = A^2 u$, where A generates a group; he obtained explicit representations of u and showed that for sufficiently regular data, convergence is $O(\epsilon)$. A weaker convergence result for this equation was obtained earlier by Griego and Hersh [N] as an application of the notion of a "random evolution."

2. **Degeneration of hyperbolic equations to hyperbolic equations.** Let $R(\lambda, s)$ be a polynomial of degree n_0 in λ and degree $m_0 \leqq n_0$ in s, let $Q(\lambda, s)$ be a polynomial of degree n in λ and degree $m \leqq N \equiv \max(n, n_0)$ in s, and set $P(\lambda, s; \epsilon) = \epsilon Q(\lambda, s) + R(\lambda, s)$, $L_\epsilon = P(\partial/\partial t,\ i\partial/\partial x;\ \epsilon)$, where $\epsilon \geqq 0$. For $\epsilon > 0$ we denote the roots of $P(\lambda, s; \epsilon)$ by $\lambda_1(s; \epsilon), \cdots, \lambda_N(s; \epsilon)$ and set

$$\Lambda(s; \epsilon) = \max_{1 \leqq j \leqq N} \operatorname{Re} \lambda_j(s; \epsilon);$$

we denote the roots of $P(\lambda, s; 0) = R(\lambda, s)$ by $\tilde{\lambda}_1(s), \cdots, \tilde{\lambda}_{n_0}(s)$ and set

$$\Lambda(s; 0) = \max_{1 \leqq j \leqq n_0} \operatorname{Re} \tilde{\lambda}_j(s).$$

We call L_ϵ *uniformly hyperbolic for* $0 \leqq \epsilon \leqq \epsilon_0$ if the roots $\lambda_i(s; \epsilon)$ satisfy

$$\lim_{\epsilon \to 0+} \lambda_i(s; \epsilon) = \tilde{\lambda}_i(s) \qquad (i = 1, \cdots, n_0)$$

and if there exist *constants* a, b, c such that

i. $\Lambda(s; \epsilon) \leqq a|s| + b$ (for all complex s),

ii. $\Lambda(\sigma; \epsilon) \leqq c$ (for real $s = \sigma$)

hold for $\epsilon \in [0, \epsilon_0]$. The constants a, b, c will be called the hyperbolicity constants for $\{L_\epsilon : 0 \leqq \epsilon \leqq \epsilon_0\}$.

Two consequences of the uniform hyperbolicity of the family $\{L_\epsilon : 0 \leqq \epsilon \leqq \epsilon_0\}$ are [**B**, Chapter 3, §3]:

A. The initial-value problem for $L_\epsilon[u] = 0$ has a unique distributional solution, given as the regular distribution generated by a continuous function, if the initial data are $N(N-1)$ times continuously differentiable.

B. The solution $\hat{g}_{i,\epsilon}(x, t)$ (in general, a distribution) of

$$(1) \qquad \begin{gathered} L_\epsilon[\hat{g}_{i,\epsilon}] = 0 \qquad (t > 0, -\infty < x < \infty), \\ \left(\frac{\partial}{\partial t}\right)^k \hat{g}_{j,\epsilon} \bigg|_{t=0} = \delta_{jk}\delta(x) \quad \begin{pmatrix} k = 0, \cdots, N-1 \text{ for } \epsilon > 0 \\ k = 0, \cdots, n_0 - 1 \text{ for } \epsilon = 0 \end{pmatrix} \end{gathered}$$

has, for each $t \geqq 0$, support contained in the interval $|x| \leqq at$.

With $q \geqq N(N-1)+2$, we let $\tilde{g}_{j,\epsilon}$ be the solution of

$$
\begin{aligned}
&L_\epsilon[\tilde{g}_{j,\epsilon}] = 0 \qquad (t > 0, -\infty < x < \infty),\\
(2) \qquad &\left(\frac{d}{dt}\right)^k \tilde{g}_{j,\epsilon}\Big|_{t=0} = \delta_{jk} x_+{}^{q-1} \begin{pmatrix} k = 0, 1, \cdots, N-1 \text{ for } \epsilon > 0 \\ k = 0, 1, \cdots, n_0 - 1 \text{ for } \epsilon = 0 \end{pmatrix},
\end{aligned}
$$

where $x_+ = \max(x, 0)$. As noted in A above, $\tilde{g}_{j,\epsilon}$ is a regular distribution; we use $\tilde{g}_{j,\epsilon}$ interchangeably for the distribution and for the continuous function generating the distribution. Clearly, in the distributional sense we have that

$$(\partial/\partial x)^q \, \tilde{g}_{j,\epsilon}(t, x) = \hat{g}_{j,\epsilon}(t, x),$$

so $\tilde{g}_{j,\epsilon}$ is a q-fold primitive of $\hat{g}_{j,\epsilon}$. Note, however, that the support of $\tilde{g}$ is not equal to the support of $\hat{g}$.

Denote by $u_\epsilon(t)$ the solution constructed in **[D]**, and described in the introduction, of the problem

$$
\begin{aligned}
&P(d/dt, -iA, \epsilon) u_\epsilon = 0 \qquad (t > 0),\\
(3) \qquad &\left(\frac{d}{dt}\right)^k u_\epsilon \Big|_{t=0} = u_k \in D^\infty(A) \quad \begin{pmatrix} k = 0, \cdots, N-1 \text{ if } \epsilon > 0 \\ k = 0, \cdots, n_0 - 1 \text{ if } \epsilon = 0 \end{pmatrix}
\end{aligned}
$$

for $0 \leqq \epsilon \leqq \epsilon_0$; we have the representation

$$u_\epsilon(t) = \int_{-\infty}^{\infty} \sum_{j=0}^{l} \hat{g}_{j,\epsilon}(t, s) T_A(s) u_j \, ds,$$

where $l = N-1$ if $\epsilon > 0$, $l = n_0 - 1$ if $\epsilon = 0$.

THEOREM 1. *Let $\{L_\epsilon : 0 \leqq \epsilon \leqq \epsilon_0\}$ be uniformly hyperbolic with constants a, b, c, and suppose that for some $\delta > 0$ the solutions of* (2) *for some $q \geqq N(N-1)+2$ satisfy*

(4) $\quad \tilde{g}_{j,\epsilon}(t, x) \to \tilde{g}_{j,0}(t, x) \qquad (|x| \leqq at + \delta, j = 0, 1, \cdots, n_0 - 1)$,

(5) $\quad \tilde{g}_{j,\epsilon}(t, x) \to 0 \qquad (|x| \leqq at + \delta, j = n_0, \cdots, N-1)$,

(6) $\quad |\tilde{g}_{j,\epsilon}(t, x)| \leqq M_j(x) \qquad (j = 0, \cdots, N-1; 0 \leqq \epsilon \leqq \epsilon_0)$

for some t, where $M_j(x)$ is integrable over $|x| \leqq at + \delta$. Then for that t

$$\|u_\epsilon(t) - u_0(t)\| \to 0$$

as $\epsilon \to 0+$. If the hypotheses hold uniformly for $t \in [t_0, t_1]$, then the conclusion is valid uniformly for $t \in [t_0, t_1]$.

Proof. Now $\tilde{g}_{j,\epsilon}$ is a qth primitive of $\hat{g}_{j,\epsilon}$, but it does not have compact support. Following [A, p. 164], let $h_\delta(t, x)$ be a C^∞ function which is identically 1 in a neighborhood of $|x| \leqq at$ and identically 0 for $|x| \geqq at + \delta$. Then

$$\hat{g}_{j,\epsilon}(x, t) = \sum_{k=0}^{q} (-1)^k \binom{q}{k} \left(\frac{\partial}{\partial x}\right)^{q-k} \left[\frac{\partial^k h}{\partial x^k} \tilde{g}_{j,\epsilon}\right],$$

and the representation of u_ϵ becomes

$$u_\epsilon(t) = \sum_{j=0}^{l} \sum_{k=0}^{q} (-1)^q \binom{q}{k} \int_{-\infty}^{\infty} \frac{\partial^k h(t, s)}{\partial s^k} \tilde{g}_{j,\epsilon}(t, s) T_A(s) A^{q-k} u_j ds.$$

We thus have

$$\begin{aligned} \|u_\epsilon(t) - u_0(t)\| &\leqq \left\| \sum_{j=0}^{n_0-1} \int_{-\infty}^{\infty} [\hat{g}_{j,\epsilon}(t, s) - \hat{g}_{j,0}(t, s)]\, T_A(s) u_j ds \right\| \\ &\qquad + \left\| \sum_{j=n_0}^{N-1} \int_{-\infty}^{\infty} \hat{g}_{j,\epsilon}(t, s) T_A(s) u_j ds \right\| \\ &\leqq \sum_{j=0}^{n_0-1} \sum_{k=0}^{q} \binom{q}{k} \left\| \int_{-\infty}^{\infty} \frac{\partial^k h(t, s)}{\partial s^k} [\tilde{g}_{j,\epsilon}(t, s) - \tilde{g}_{j,0}(t, s)] \right. \\ &\qquad \left. \cdot\, T_A(s) A^{q-k} u_j ds \right\| \\ &\qquad + \sum_{j=n_0}^{N-1} \sum_{k=0}^{q} \binom{q}{k} \left\| \int_{-\infty}^{\infty} \frac{\partial^k h(t, s)}{\partial s^k} \tilde{g}_{j,\epsilon}(t, s) T_A(s) A^{q-k} u_j ds \right\| \\ &\leqq \left\{ \sup_{|s| \leqq at + \delta} \|T_A(s)\| \right\} \\ &\qquad \cdot \left\{ \sum_{j=0}^{n_0-1} \sum_{k=0}^{q} \binom{q}{k} \|A^{q-k} u_j\| C_k \int_{-at-\delta}^{at+\delta} |\tilde{g}_{j,\epsilon}(t, s) - \tilde{g}_{j,0}(t, s)| ds \right. \\ &\qquad \left. + \sum_{j=n_0}^{N-1} \sum_{k=0}^{q} \binom{q}{k} \|A^{q-k} u_j\| C_k \int_{-at-\delta}^{at+\delta} |\tilde{g}_{j,\epsilon}(t, s)| ds \right\}; \end{aligned}$$

here C_k is a bound on $|\partial^k h(t, s)/\partial s^k|$. The right-hand side tends to 0 as $\epsilon \to 0$ by the Lebesgue dominated convergence theorem, establishing the result.

Remark. The family of operators $\partial^2/\partial t^2 - \partial^2/\partial x^2 + \epsilon$ is uniformly hyperbolic for $0 \leqq \epsilon \leqq \epsilon_0$. Here $\lambda_\pm(s; \epsilon) = \pm i(s^2 + \epsilon)^{1/2}$,

$\tilde{\lambda}_{\pm}(s) = \pm is$, $a = 1$, $b = 0$, $c = 0$. This family is associated with a regular perturbation problem. An example of a uniformly hyperbolic family of operators involving a singular perturbation is

$$\epsilon(\partial^2/\partial t^2 - \partial^2/\partial x^2) + \partial/\partial t;$$

here $\lambda_{\pm}(s;\epsilon) = (1/2\epsilon)[-1 \pm (1 - 4s^2\epsilon^2)^{1/2}]$, $\tilde{\lambda} = 0$, $a = 1$, $b = c = 0$. Classical results (obtainable, for example, by Fourier transform in x; see also [**E**], [**F**]) for these two families of operators can thus be carried over at once to the more general setup of the present theorem.

3. **Degeneration of hyperbolic equations to parabolic equations.** Let $R(\lambda, s)$ be a polynomial of degree n_0 in λ and degree m_0 in s with roots $\lambda_j(s)$, $j = 1, \cdots, n_0$; set

$$\Lambda(s, 0) = \max_{1 \leqq j \leqq n_0} \operatorname{Re} \lambda_j(s).$$

We assume that the differential operator $R(\partial/\partial t, i\partial/\partial x)$ is parabolic; i.e., there exist constants $c_1 > 0$, c_2, $h > 0$ such that for all real $s = \sigma$, $\Lambda(\sigma, 0) \leqq -c_1|\sigma|^h + c_2$. For the moment we assume also that the *genus* [**B**, p. 114] of the operator $R(\partial/\partial t; i\partial/\partial x)$ is positive.

Let $Q(\lambda, s; \epsilon)$ be a polynomial (coefficients depending on ϵ) of degree $N > n_0$ in λ for $\epsilon_0 \geqq \epsilon > 0$ and degree $m \leqq N$ in s for $\epsilon_0 \geqq \epsilon > 0$ and such that

$$\lim_{\epsilon \to 0+} Q(\lambda, s; \epsilon) = 0$$

for every pair of complex numbers (λ, s). (If $q(\lambda, s)$ is a polynomial of degree $N > n_0$ in λ and degree $m \leqq N$ in s, then two important special cases are $Q(\lambda, s; \epsilon) = \epsilon q(\lambda, s)$ and, if q has no constant term, $Q(\lambda, s; \epsilon) = q(\epsilon\lambda, \epsilon s)$.) Define

$$P(\lambda, s; \epsilon) = Q(\lambda, s; \epsilon) + R(\lambda, s).$$

We require that $L_\epsilon \equiv P(\partial/\partial t, i\partial/\partial^x; \epsilon)$ be, for $\epsilon_0 \geqq \epsilon > 0$, a hyperbolic operator; that is, denoting the roots of $P(\lambda, s; \epsilon)$ by $\lambda_j(s, \epsilon)$, $j = 1, \cdots, N$, and setting

$$\Lambda(s, \epsilon) = \max_{1 \leqq j \leqq N} \operatorname{Re} \lambda_j(s, \epsilon),$$

there exist functions $a(\epsilon)$, $b(\epsilon)$, $c(\epsilon)$ such that

$$\Lambda(s, \epsilon) \leqq a(\epsilon)|s| + b(\epsilon) \qquad \text{(all complex } s, 0 < \epsilon \leqq \epsilon_0\text{)},$$

$$\Lambda(\sigma, \epsilon) \leqq c(\epsilon) \qquad \text{(all } s = \sigma \text{ real}, 0 < \epsilon \leqq \epsilon_0\text{)}.$$

Then the results A, B of the preceding section continue to hold with the constant a of B replaced by the function $a(\epsilon)$. In general, $a(\epsilon) \to \infty$ as $\epsilon \to 0$.

For any given $K \geqq 0$ let $h_K(x)$ be a C^∞ function which is identically 1 on some neighborhood of $[-K, K]$ and 0 outside the interval $[-K-1, K+1]$ and such that

$$0 \leqq h_K(x) \leqq 1, \qquad \sup_{-\infty<x<\infty} \left| \left(\frac{d}{dx}\right)^r h_K(x) \right| \leqq M_r,$$

for each $r = 0, 1, 2, \cdots$, where M_r is independent of K. Since

$$\delta(x) = \sum_{l=0}^{q} (-1)^{q+l} \binom{q}{l} \left(\frac{d}{dx}\right)^l \left(x_+^{q-1} \left(\frac{d}{dx}\right)^{q-l} h_0(x) \right)$$

(cf. [A, p. 164]), we have

$$\hat{g}_{j,\epsilon} = \sum_{l=0}^{q} (-1)^{q+l} \left(\frac{\partial}{\partial x}\right)^l \tilde{g}_{l,j,\epsilon},$$

where $\hat{g}_{j,\epsilon}$ is the (distributional) solution of

$$L_\epsilon[\hat{g}_{j,\epsilon}] = 0, \qquad \left(\frac{\partial}{\partial t}\right)^k \hat{g}_{j,\epsilon} \Big|_{t=0} = \delta_{jk}\delta(x)$$

and $\tilde{g}_{l,j,\epsilon}$ is the solution, in the distributional sense, of

$$L_\epsilon[\tilde{g}_{l,j,\epsilon}] = 0,$$

$$\left(\frac{\partial}{\partial t}\right)^k \tilde{g}_{l,j,\epsilon} \Big|_{t=0} = \delta_{jk} \binom{q}{l} x_+^{q-1} \left(\frac{d}{dx}\right)^{q-l} h_0(x). \tag{7}$$

For $q \geqq N(N-1) + 2$ the data for $\tilde{g}_{l,j,\epsilon}$ are at least $N(N-1)$ times continuously differentiable with compact support; as noted in the hyperbolic $\to$ hyperbolic case, this implies that for each fixed $\epsilon > 0$ the $\tilde{g}_{l,j,\epsilon}$ are continuous functions having compact support. For $\epsilon = 0$ the $\tilde{g}_{l,j,\epsilon}$ are also continuous [B, p. 117].

For each $\epsilon \geqq 0$ and any $K > 0$ we write

$$\hat{g}_{j,\epsilon}(t, x) = h_K(x)\hat{g}_{j,\epsilon}(t, x) + (1 - h_K(x))\hat{g}_{j,\epsilon}(t, x)$$

where, for $\epsilon > 0$, each term on the right has compact support, and, for $\epsilon = 0$, the first term on the right has compact support. For each $\epsilon \geqq 0$ we write [A, p. 164]

$$h_K\hat{g}_{j,\epsilon} = \sum_{l=0}^{q} \sum_{s=0}^{l} (-1)^{q+l+s} \left(\frac{\partial}{\partial x}\right)^{l-s} \left[\tilde{g}_{l,j,\epsilon} \left(\frac{\partial}{\partial x}\right)^s h_K \right]$$

and for $\epsilon > 0$,

$$(1 - h_K)\hat{g}_{j,\epsilon} = \sum_{l=0}^{q} \sum_{s=0}^{l} (-1)^{q+l+s} \left(\frac{\partial}{\partial x} \right)^{l-s} \left[\tilde{g}_{l,j,\epsilon} \left(\frac{\partial}{\partial x} \right)^{s} (1 - h_K) \right],$$

where each term on the right has compact support.

Let $u_\epsilon(t)$ be the solution constructed in [D] of the problem

$$P(d/dt, -iA, \epsilon)u_\epsilon = 0 \qquad (t > 0),$$

$$\left(\frac{d}{dt} \right)^k u_\epsilon \bigg|_{t=0} = u_k \in D^\infty(A) \qquad \begin{pmatrix} k = 0, \cdots, N-1 \text{ if } \epsilon > 0 \\ k = 0, \cdots, n_0 - 1 \text{ if } \epsilon = 0 \end{pmatrix}$$

for $0 \leqq \epsilon \leqq \epsilon_0$.

THEOREM 2. *Let L_ϵ be as described, let $\|T_A(s)\| \leqq Me^{\beta s}$ for constants M, $\beta \geqq 0$, and suppose that for some $q \geqq N(N-1) + 2$ we have*

(i) $|\tilde{g}_{l,j,\epsilon}(t, s)| \leqq Q_{l,j}(s)$ *for* $\epsilon_0 \geqq \epsilon > 0$, *where* $Q_{l,j}(s)e^{\beta s}$ *is integrable.*

(ii) $\lim_{\epsilon \to 0+} \tilde{g}_{l,j,\epsilon}(t, s) = 0$ *for* $j = n_0, \cdots, N-1$,

(iii) $\lim_{\epsilon \to 0+} \tilde{g}_{l,j,\epsilon}(t, s) = \tilde{g}_{l,j,0}(t, s)$ for $j = 0, 1, \cdots, n_0 - 1$.

Then $\|u_\epsilon(t) - u_0(t)\| \to 0$ as $\epsilon \to 0+$. If the hypotheses hold uniformly for $t \in [t_0, t_1]$, then the conclusion holds uniformly in $t \in [t_0, t_1]$.

PROOF. Using the representation of [D], we write

$$\begin{aligned} u_\epsilon(t) - u_0(t) = & \int_{-\infty}^{\infty} \sum_{j=0}^{n_0-1} h_K(s)[\hat{g}_{j,\epsilon}(t, s) - \hat{g}_{j,0}(t, s)]\, T_A(s)u_j ds \\ & + \int_{-\infty}^{\infty} \sum_{j=n_0}^{N-1} h_K(s)\hat{g}_{j,\epsilon}(t, s)T_A(s)u_j ds \\ & + \int_{-\infty}^{\infty} \sum_{j=0}^{N-1} (1 - h_K(s))\hat{g}_{j,\epsilon}(t, s)T_A(s)u_j ds \\ & - \int_{-\infty}^{\infty} \sum_{j=0}^{n_0-1} (1 - h_K(s))\hat{g}_{j,0}(t, s)T_A(s)u_j ds \end{aligned}$$

We estimate the norm of each of these integrals in turn.

$$\|I_4\| \leqq \sum_{j=0}^{n_0-1} \int_{|s|>K} |\hat{g}_{j,0}(t, s)|\, \|T_A(s)u_j\| ds$$

$$\leqq \sum_{j=0}^{n_0-1} M\|u_j\| \int_{|s|>K} |\hat{g}_{j,0}(t, s)| e^{\beta s} ds,$$

which can be made arbitrarily small by choosing K large since $\hat{g}_{j,0}$ decays, as $s \to \pm\infty$, like $\exp(-ks^\gamma)$, $k > 0, \gamma > 1$.

$$\|I_3\| = \left\| \sum_{j=0}^{N-1} \sum_{l=0}^{q} \sum_{r=0}^{l} (-1)^q \int_{|s|>K} \tilde{g}_{l,j,\epsilon} \frac{\partial^r(1-h_K)}{\partial x^r} T_A(s) A^s u_j ds \right\|$$

$$\leqq \sum_{j=0}^{N-1} \sum_{l=0}^{q} \sum_{r=0}^{l} MM_r \|A^r u_j\| \int_{|s|>K} Q_{l,j}(s) e^{\beta s} ds,$$

which can again be made arbitrarily small by taking K large. Having fixed K so that, for a given $\bar{\epsilon} > 0$, $\|I_3\| + \|I_4\| \leqq \bar{\epsilon}/2$, we proceed to estimate I_2 and I_1.

$$\|I_2\| \leqq \sum_{j=n_0}^{N-1} \sum_{l=0}^{q} \sum_{r=0}^{l} MM_r e^{\beta(K+1)} \|A^r u_j\| \int_{|s|<K+1} |\tilde{g}_{l,j,\epsilon}(t, s)| ds,$$

and for fixed K the right-hand side tends to zero as $\epsilon \to 0+$ by the Lebesgue dominated convergence theorem, using hypotheses (i), (ii). In the same manner we conclude that $\|I_1\| \to 0$ from the estimate

$$\|I_1\| < \sum_{j=0}^{n_0-1} \sum_{l=0}^{q} \sum_{r=0}^{l} MM_r e^{\beta(K+1)} \|A^r u_j\|$$

$$\cdot \int_{|s|\leqq K+1} |\tilde{g}_{l,j,\epsilon}(t, s) - \tilde{g}_{l,j,0}(t, s)| ds.$$

The statement regarding uniformity in t is also obvious.

Dropping the requirement that the genus of the parabolic equation be positive, we get

THEOREM 3. *Let L_ϵ be as above except for the restriction on the genus of L_0, let $\|T_A(s)\| \leqq M|s|^\beta$ for some constants M, $\beta \geqq 0$ and suppose that for some $q \geqq N(N-1) + 2$ we have*

(i) $|\tilde{g}_{l,j,\epsilon}(t, s)| \leqq Q_{l,j}(s)$ *for* $\epsilon_0 \geqq \epsilon > 0$, *where* $Q_{l,j}(s)|s|^\beta$ *is integrable,*
(ii) $\lim_{\epsilon \to 0+} \tilde{g}_{l,j,\epsilon}(t, s) = 0$ *for* $j = n_0, \cdots, N-1$,
(iii) $\lim_{\epsilon \to 0+} \tilde{g}_{l,j,\epsilon}(t, s) = \tilde{g}_{l,j,0}(t, s)$ *for* $j = 0, 1, \cdots, n_0 - 1$.

The proof is similar to the proof of Theorem 2, except that we now use the fact [**B**, p. 123] that $\hat{g}_{j,0}$ decays like $\exp(-ks^\gamma)$, $k > 0, \gamma > 0$.

The authors are indebted to E. Hille for pointing out that there are groups satisfying $\|T_A(s)\| \leqq M|s|^{\beta}$ but not $\|T_A(s)\| \leqq K$ for any K.

Hypothesis (i) of Theorem 2 is difficult to verify directly; we, therefore, present a result guaranteeing that this hypothesis is satisfied.

THEOREM 4. *In each region $|\tau| \leqq K$ of the $(\sigma + i\tau)$-plane let the inequality*

$$\Lambda(\sigma + i\tau, \epsilon) \leqq \bar{K}$$

hold for all $\epsilon \in (0, \epsilon_0]$; here $\bar{K}$ may depend on K. Then for q sufficiently large hypothesis (i) *of Theorem* 2 *is satisfied.*

PROOF. The above inequality assures that for $\epsilon \in (0, \epsilon_0]$ the hyperbolic problem (7) is Petrovsky-correct with nonnegative genus; we may thus take $\mu = 0$ for the genus. Since we assume that $\Lambda(s, \epsilon) \leqq a(\epsilon)|s| + b(\epsilon)$ it follows that we also have $\Lambda(s, \epsilon) \leqq a'(\epsilon)|s|^{p_0} + b(\epsilon)$ for any $p_0 > 1$. The proof of Theorem 15′ of [C, p. 200] for $\mu = 0$ is now easily modified to yield the conclusion $|\tilde{g}_{l,j,\epsilon}(t, s)| \leqq C_K e^{-K|s|}$, since the (compactly supported) data for $\tilde{g}_{l,j,\epsilon}$ have a bound of the form $C_K' e^{-K|s|}$ for every $K > 0$; here the constant C_K can be taken independent of ϵ. Choosing $K > \beta$, the conclusion follows.

THEOREM 5. *Let A generate a strongly continuous semigroup. Then the solution $u_\epsilon(t)$ of the generalized telegraphist's equation*

$$\epsilon \frac{d^2}{dt^2} u_\epsilon + \frac{d}{dt} u_\epsilon - A^2 u_\epsilon = 0, \quad u_\epsilon \Big|_{t=0} = u_0, \qquad \frac{d}{dt} u_\epsilon \Big|_{t=0} = u_1,$$

$u_0, u_1 \in D(A^\infty)$, converges as $\epsilon \to 0+$ to the solution $U(t)$ of the generalized heat equation

$$\frac{d}{dt} U - A^2 U = 0, \qquad U \Big|_{t=0} = u_0.$$

The convergence is uniform on any $[t_1, t_2] \subset [0, \infty)$.

PROOF. One easily computes that, with $s = \sigma + i\tau$,

$$\begin{aligned}
\Lambda(s, \epsilon) &= \operatorname{Re} \frac{1}{2\epsilon}\{-1 + (1 - 4\epsilon s^2)^{1/2}\} \\
&= -\frac{1}{2\epsilon} + \frac{1}{\sqrt{2}}\Bigg[\frac{1}{4\epsilon^2} - (\sigma^2 - \tau^2) \\
&\qquad\qquad + \left\{\left(\frac{1}{4\epsilon^2} - (\sigma^2 - \tau^2)\right)^2 + 4\sigma^2\tau^2\right\}^{1/2}\Bigg]^{1/2} \\
&\leqq -\frac{1}{2\epsilon} + \left(\frac{1}{4\epsilon^2} + \tau^2\right)^{1/2} \leqq \epsilon\tau^2 \leqq \epsilon_0\tau^2 \leqq \epsilon_0 K^2
\end{aligned}$$

provided $|\tau| < K$. Thus by Theorem 4 the first hypothesis of Theorem 2 is satisfied. The genus of the parabolic equation $u_t - u_{xx}$ is 1. That hypotheses (ii), (iii) of Theorem 2 are satisfied for sufficiently large q is established in [E], [F].

4. **Degeneration of Petrovsky-correct equations to Petrovsky-correct equations.** Let $R(\lambda, s)$ be a polynomial of degree n_0 in λ and degree m_0 in s, and let $Q(\lambda, s; \epsilon)$ be a polynomial (coefficients depending on ϵ) of degree N in λ for $\epsilon_0 \geqq \epsilon > 0$ and degree m in s for $\epsilon_0 \geqq \epsilon > 0$ and such that

$$\lim_{\epsilon \to 0+} Q(\lambda, s; \epsilon) = 0$$

for all complex λ, s. For $\epsilon_0 \geqq \epsilon > 0$ define

$$\Lambda(s, \epsilon) = \max_{1 \leqq j \leqq l} \operatorname{Re} \lambda_j(s, \epsilon),$$

where we set, once and for all, $l = \max(N, n_0)$ if $\epsilon > 0$, $l = n_0$ if $\epsilon = 0$; here the λ_j are the roots of $P(\lambda, s; \epsilon) \equiv Q(\lambda, s; \epsilon) + R(\lambda, s) = 0$. We assume that the operator $L_\epsilon \equiv P(\partial/\partial t,\ i\partial/\partial x;\ \epsilon)$ is Petrovsky-correct for $0 \leqq \epsilon \leqq \epsilon_0$, i.e., that there exists a *constant* C such that

$$\Lambda(\sigma, \epsilon) \leqq C \qquad (-\infty < \sigma < \infty).$$

For each $\epsilon \in [0, \epsilon_0]$ the *genus* $\mu(\epsilon)$ of the operator can be defined [**B**, p. 138] as the least upper bound of the exponents μ such that the function $\Lambda(s, \epsilon)$ remains bounded above in the region $|\tau| \leqq K(1 + |\sigma|)^{\mu(\epsilon)}$, where $s = \sigma + i\tau$. Since $\mu(\epsilon) \geqq 1 - p_0(\epsilon)$ [**B**, p. 137], where $p_0(\epsilon)$ is a constant for $\epsilon_0 \geqq \epsilon > 0$ and finite for $\epsilon = 0$, we see that $\mu(\epsilon)$ is bounded below for $\epsilon \in [0, \epsilon_0]$; let μ be such that $\mu \leqq \mu(\epsilon)$ for $0 \leqq \epsilon \leqq \epsilon_0$, so that $\Lambda(\sigma + i\tau, \epsilon)$ is bounded in the region $|\tau| \leqq K(1 + |\sigma|)^{\mu}$ for some K and all $\epsilon \in [0, \epsilon_0]$. We shall assume, as we may, that $\mu < 0$.

Note that the correctness exponent $h(\epsilon)$ [**B**, p. 135] is bounded above uniformly in ϵ and since it may be increased without harm to our argument, we may take $h \geqq h(\epsilon)$ for $\epsilon \in [0, \epsilon_0]$ as the correctness exponent for every $\epsilon \in [0, \epsilon_0]$.

Let $\hat{g}_{j,\epsilon}$ be the (distributional) solution of

$$L_\epsilon[\hat{g}_{j,\epsilon}] = 0, \qquad (\partial/\partial t)^k \hat{g}_{j,\epsilon}|_{t=0} = \delta_{jk}\delta(x), \qquad k = 0, 1, \cdots, l-1.$$

By a slight modification of the proof in [**B**, p. 148] or that in [**C**, p. 198] it is easily shown that

$$\hat{g}_{j,\epsilon}(t, x) = (1 + d/dx)^q f_{j,q}(t, x; \epsilon)$$

for $q \geqq q_0$ for a certain $q_0 > 0$ (depending on μ and h), where $f_{j,q}$

is continuous in x and t and, for each t, integrable in x uniformly in ϵ. (In fact, $f_{j,q} = O(x^{-n})$, $n \geqq 2$, uniformly in ϵ as $|x| \to \infty$.) Since

$$\delta(x) = \left(1 + \frac{d}{dx}\right)^q \left\{\frac{1}{(q-1)!} e^{-x} x_+^{q-1}\right\},$$

it is easily seen that $f_{j,q}$ satisfies the problem

$$L_\epsilon[f_{j,q}(t, x; \epsilon)] = 0,$$

$$\left(\frac{\partial}{\partial t}\right)^k f_{j,q}(t, x; \epsilon)\Big|_{t=0} = \delta_{jk} \frac{1}{(q-1)!} e^{-x} x_+^{q-1} \tag{8}$$

$$\begin{pmatrix} k = 0, 1, \cdots, l-1 \\ j = 0, 1, \cdots, l-1 \end{pmatrix}.$$

For sufficiently large q this classical problem will have a classical solution.

The representation of [**D**] for the solution of the abstract problem is (cf. [**B**], [**C**])

$$u_\epsilon(t) = \sum_{j=0}^{l} \int_{-\infty}^{\infty} f_j(t, -s; \epsilon)\left(1 + \frac{d}{ds}\right)^q T_A(s) u_j ds$$

$$= \sum_{j=0}^{l} \int_{-\infty}^{\infty} f_j(t, -s; \epsilon) T_A(s)(1 + A)^q u_j ds.$$

THEOREM 6. *Let L_ϵ be as described and let A generate the equicontinuous group $T_A(s)$ ($\|T_A(s)\| \leqq M$). Suppose that for all large q the solutions $f_{j,q}$ of the classical problems* (8) *satisfy*

(i) $\lim_{\epsilon \to 0+} f_{j,q}(t, s; \epsilon) = f_{j,q}(t, s; 0)$ *for* $j = 0, 1, \cdots, n_0 - 1$,

(ii) $\lim_{\epsilon \to 0+} f_{j,q}(t, s; \epsilon) = 0$ *for* $j = n_0, \cdots, \max(N, n_0) - 1$ *if* $N > n_0$.

Then $\|u_\epsilon(t) - u_0(t)\| \to 0$ *as* $\epsilon \to 0+$. *If the hypotheses hold uniformly in* $t \in [t_0, t_1] \subset [0, \infty]$, *then the conclusion holds uniformly in* $t \in [t_0, t_1]$.

PROOF. We have

$$\|u_\epsilon(t) - u_0(t)\|$$

$$\leqq \sum_{j=0}^{n_0-1} \int_{-\infty}^{\infty} |f_{j,q}(t, -s; \epsilon) - f_{j,q}(t, -s; 0)| \, \|T_A(s)(I + A)^q u_j\| ds$$

$$+ \sum_{j=n_0}^{\max(N,n_0)-1} \int_{-\infty}^{\infty} |f_{j,q}(t, -s; \epsilon)| \, \|T_A(s)(I + A)^q u_j\| ds$$

$$\leqq \sum_{j=0}^{n_0-1} M\|(I+A)^q u_j\| \int_{-\infty}^{\infty} |f_{j,q}(t,-s;\epsilon)-f_{j,q}(t,-s;0)|ds$$

$$+ \sum_{j=n_0}^{\max(N,n_0)-1} M\|(I+A)^q u_j\| \int_{-\infty}^{\infty} |f_{j,q}(t,-s;\epsilon)|ds,$$

which converges to zero as $\epsilon \to 0+$ by the Lebesgue dominated convergence theorem since the $f_{j,q}$ are integrable uniformly in ϵ. The statement regarding uniformity in t is clear since all the estimates preceding the statement of the theorem may be made uniform in t for t in a compact interval.

As an application of this theorem, we extend certain results of [G], [H], [I], [J], [K] valid for a selfadjoint operator in Hilbert space. Consider the problem

$$L_\epsilon[u_\epsilon] \equiv \sum_{j=1}^{m} \epsilon^j a_{n+j} u_\epsilon^{(n+j)} + \sum_{k=1}^{n} a_k u_\epsilon^{(k)} + Au_\epsilon = 0,$$

$$u_\epsilon^{(k)}(0) = u_k \in D^\infty(A) \qquad \begin{pmatrix} k = 0, 1, \cdots, n+m-1, \epsilon > 0 \\ k = 0, 1, \cdots, n-1, \epsilon = 0 \end{pmatrix}$$

where we assume that $a_n \neq 0$, $a_{n+m} \neq 0$. Denote the roots of the polynomial

$$\sum_{j=1}^{m} a_{n+j}\lambda^j + a_n$$

by $\bar{\nu}_1, \cdots, \bar{\nu}_m$, and denote the roots of

$$\sum_{k=1}^{n} a_k \lambda^k + is$$

by $\bar{\mu}_1(s), \cdots, \bar{\mu}_n(s)$. The roots of

$$\sum_{j=1}^{m} \epsilon^j a_{n+j}\lambda^{n+j} + \sum_{k=1}^{n} a_k\lambda^k + is$$

can be denoted $\mu_1(s,\epsilon), \cdots, \mu_n(s,\epsilon), \nu_1(s,\epsilon)/\epsilon, \cdots, \nu_n(s,\epsilon)/\epsilon$ in such a way that $\mu_i(s,\epsilon) \to \bar{\mu}_i(s)$ uniformly in s as $\epsilon \to 0+$ and $\nu_j(s,\epsilon) \to \bar{\nu}_j$ as $\epsilon \to 0+$ [K], [L].

THEOREM 7. *Let there exist a constant $d > 0$ such that* $\operatorname{Re} \nu_j(\sigma,\epsilon) \leqq -d$ *for all real σ and all $\epsilon \in [0,\epsilon_0]$, and suppose that* $\operatorname{Re}(\bar{\mu}_i(\sigma)) \leqq C$ *for some constant C. Let A generate an equicontinuous group.*

Then

$$\lim_{\epsilon \to 0+} \|u_\epsilon(t) - u_0(t)\| = 0 \quad \textit{uniformly for } t \in [0, T], \textit{any } T > 0.$$

PROOF. By the foregoing, the real parts of the roots μ_i, ν_j/ϵ are bounded above, so the operators L_ϵ are uniformly correct. That hypotheses (i), (ii) are satisfied for sufficiently large q uniformly in $t \in [0, T]$ follows from a result of [**F**] on writing the classical problem for the $f_{j,q}$ as a system in the standard way.

It is well known that if A is selfadjoint in a Hilbert space, then iA generates an equicontinuous, uniformly bounded group; thus, Theorem 7 does indeed contain a number of the results of [**G**], [**H**], [**I**], [**J**], [**K**].

REMARK. On formulating the equivalent of Theorem 6 of this section for systems of the type studied in [**F**], the results of that work can be brought to the present context.

5. **Perturbation of the generator** A. We conclude by turning to a different kind of perturbation. We will now let P be fixed, and let A be approximated by A_ϵ as $\epsilon \to 0+$.

Since A may be unbounded, there is some question as to what we should mean by saying A_ϵ converges to A. We will short-circuit this difficulty by taking as a hypothesis that A_ϵ generates a group of bounded operators $T_\epsilon(t)$, and that $T_\epsilon(t) \to T_A(t)$ strongly for each t, uniformly on bounded t-intervals.

Kato [**P**] gives a variety of conditions on A, A_ϵ, and their resolvents which are sufficient for convergence of their groups. In particular, convergence of resolvents, $(\lambda - A_\epsilon)^{-1} \to (\lambda - A)^{-1}$, is necessary and sufficient, given that $|T_\epsilon(t)| \leqq m e^{\omega |t|}$ for m, ω independent of ϵ. Stronger conditions have been given by Trotter and by Lax; see Strang [**R**] for a careful comparison of these conditions.

We prepare the notation for our theorem.

Let $\hat{g}_j$ again be the solution of

$$L[\hat{g}_j] = 0 \qquad (t > 0, -\infty < x < \infty),$$
$$(\partial/\partial t)^k \hat{g}_j \,|_{t=0} = \delta_{jk}\delta(x) \qquad (k = 0, \cdots, n_0 - 1).$$

Let $\tilde{g}_j$ be a q-fold primitive of $\hat{g}$, and let $\tilde{g}_j$ be a signed measure, of finite absolute mass and compact singular support, such that, for large s, $|\tilde{g}_j(t, s)| \leqq K(t) e^{-\omega' |s|}$. Let A_ϵ generate a group $T_\epsilon(t)$, such that $|T_\epsilon(t)| \leqq M e^{\omega |t|}$, where $\omega < \omega'$. (If the A_ϵ are difference operators, this condition is usually called "stability".) Let $T_\epsilon(t)$

converge strongly to $T_A(t)$ for each t, and let $f \in B$ be such that

$$A_\epsilon{}^q f \text{ converges,} \quad \text{as } \epsilon \to 0^+, \text{ to } A^q f.$$

(If L is strictly hyperbolic, or parabolic of positive genus, then we can take $q = 0$, and the last condition is vacuous.)

Let u_ϵ, $0 \leqq \epsilon$, be the unique solution of

$$P(d/dt, A_\epsilon)u_\epsilon = 0 \quad \text{for } t > 0,$$
$$(d/dt)^k u_\epsilon \,|_{t=0} = \delta_{jk} f.$$

THEOREM 8. *Under the stated hypotheses u_ϵ converges to u_0 as $\epsilon \to 0+$.*

PROOF. Integration by parts gives:

$$\begin{aligned} |u_\epsilon - u| &= \left| \int_{-\infty}^{\infty} \hat{g}_j(t, s)(T_\epsilon(s) - T_A(s))f ds \right| \\ &= \left| \int_{-\infty}^{\infty} \tilde{g}_j(t, s)(A_\epsilon{}^q T_\epsilon(s) - A^q T_A(s))f ds \right| \\ &\leqq \left| \int_{-\infty}^{\infty} \tilde{g}_j(t, s) T_A(s)(A^q - A_\epsilon{}^q) f ds \right| \\ &\quad + \left| \int_{-\infty}^{\infty} \tilde{g}_j(t, s)(T_A(s) - T_\epsilon(s))A_\epsilon{}^q f ds \right|. \end{aligned}$$

The first term on the right can be estimated by

$$2MK(t)\left(\int_0^{\infty} e^{(\omega - \omega')s} ds \right) |(A^q - A_\epsilon{}^q)f|,$$

which goes to zero. The second term can be estimated for ϵ sufficiently small and $|s| > S$, by

$$4MK(t)\left(\int_S^{\infty} e^{(\omega - \omega')s} ds \right) |2A^q f|,$$

which is arbitrarily small if S is large enough. For $|s| \leqq S$, it is estimated by

$$\left(\int_{-S}^{S} |\tilde{g}_j(t, s)| ds \right) \max_{|s| \leqq S} |(T_A(s) - T_\epsilon(s))A_\epsilon{}^q f|,$$

which, for any fixed S, goes to zero as ϵ goes to zero. The proof is complete.

REMARK. We wish to show by an example that $A_\epsilon \to A$ is not necessarily enough to guarantee $u_\epsilon \to u$. Consider $P = (d/dt + A)^2$. For $L = (d/dt - d/dx)^2$, we find $g_0 = \delta(x - t) + t\delta'(x - t)$. We take $A = d/dx$, $A_\epsilon = (1 + \epsilon)d/dx$. One finds $u_\epsilon \to u$ only if $f(x) = u_\epsilon(0, x)$ is in C^1.

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