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LAMBERT SERIES, FALSE THETA FUNCTIONS, AND PARTITIONS

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1. **Introduction.** One of the recent important results in the theory of partitions is the following theorem due to B. Gordon [5].

THEOREM. Let $A_{k,a}(N)$ *denote the number of partitions of N into parts* $\not\equiv 0, \pm a \pmod{2k+1}$. *Let* $B_{k,a}(N)$ *denote the number of partitions of N of the form* $N = \sum_{i=1}^{\infty} f_i i$ (f_i *denotes the number of times the summand i appears in the partition*) *where* $f_1 \leqq a-1$ *and* $f_i + f_{i+1} \leqq k-1$. *Then* $A_{k,a}(N) = B_{k,a}(N)$.

This theorem reduces to the Rogers-Ramanujan identities when $k = 2$.

In this paper we shall study a partition function $W_{k,i}(n; N)$ which is somewhat similar to $B_{k,i}(N)$. $W_{k,i}(n; N)$ denotes the number of partitions of N of the form $N = \sum_{i=1}^{n} f_i i$, with $f_1 = i$, $f_j \leqq k-1$, and $f_j + f_{j+1} = k$ or $k+1$ for $1 \leqq j \leqq n-1$. We let $w_{k,i}(n; q) = 1 + \sum_{N=1}^{\infty} W_{k,i}(n; N) q^N$. Our first result relates $w_{k,i}(n; q)$ to certain Lambert series.

THEOREM 1. *For* $|q| < 1$,

$$1 - \sum_{n=1}^{\infty} q^{(2k-1)n^2/2 + n/2 - (k-i)n} \frac{(1 - q^{2n(k-i)})}{1 + q^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n w_{k,i}(n; q)}{(1+q)(1+q^2) \cdots (1+q^n)}.$$

When $i = k-1$, we see that the left-hand series in Theorem 1 reduces to a false theta series. From Theorem 1 it is possible to prove results on partitions which we shall examine in §3.

2. **Proof of Theorem 1.** We define the function $f_{k,i}(x)$ as follows:

$$f_{k,i}(x) = \sum_{n=0}^{\infty} x^{kn} q^{(2k-1)n^2/2 + n/2 - in} (1 - x^i q^{2ni}) \frac{(-1)_n}{(-xq)_n}, \tag{2.1}$$

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where $(\alpha)_n = (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})$. It may be noted that

$$f_{k,i}(x) = C_{k,i}(-1, q; x; q)$$

in the notation of [**2**, equation (1.1), p. 433]. The results in [**2**] imply, therefore, that the $f_{k,i}(x)$ satisfy certain systems of homogeneous q-difference equations. The following lemma establishes that $f_{k,i}(x)$ also satisfy certain nonhomogeneous q-difference equations.

LEMMA 1.

$$f_{k,i}(x) = 1 - \frac{x^i}{1+xq} f_{k,k-i}(xq) - \frac{x^{i+1}q}{1+xq} f_{k,k-i-1}(xq).$$

PROOF.

$$\begin{aligned}
f_{k,i}(x) &= 1 + \sum_{n=1}^{\infty} x^{kn} q^{(2k-1)n^2/2+n/2-in} \frac{(-1)_n}{(-xq)_n} \\
&\quad - x^i \sum_{n=0}^{\infty} x^{kn} q^{(2k-1)n^2/2+n/2+in} \frac{(-1)_n}{(-xq)_n} \\
&= 1 + \frac{x^k q^{k-i}}{1+xq} \sum_{n=0}^{\infty} x^{kn} q^{(2k-1)n^2/2+n/2-in+(2k-1)n} \frac{(-1)_n(1+q^n)}{(-xq^2)_n} \\
&\quad - \frac{x^i}{1+xq} \sum_{n=0}^{\infty} x^{kn} q^{(2k-1)n^2/2+n/2+in} \frac{(-1)_n(1+xq^{n+1})}{(-xq^2)_n} \\
&= 1 - \frac{x^i}{1+xq} \sum_{n=0}^{\infty} (xq)^{kn} q^{(2k-1)n^2/2+n/2} \frac{(-1)_n}{(-xq^2)_n} \\
&\qquad \cdot \{q^{in-kn}(1+xq^{n+1}) - (xq)^{k-i} q^{-in+(k-1)n}(1+q^n)\} \\
&= 1 - \frac{x^i}{1+xq} \sum_{n=0}^{\infty} (xq)^{kn} q^{(2k-1)n^2/2+n/2-(k-i)n} \frac{(-1)_n}{(-xq^2)_n} \\
&\qquad \cdot (1 - (xq)^{k-i} q^{2n(k-i)}) \\
&\quad - \frac{x^{i+1}q}{1+xq} \sum_{n=0}^{\infty} (xq)^{kn} q^{(2k-1)n^2/2+n/2-(k-i-1)n} \frac{(-1)_n}{(-xq)_n} \\
&\qquad \cdot (1 - (xq)^{k-i-1} q^{2n(k-i-1)}) \\
&= 1 - \frac{x^i}{1+xq} f_{k,k-i}(xq) - \frac{x^{i+1}q}{1+xq} f_{k,k-i-1}(xq).
\end{aligned}$$

We now define

$$h_{k,i}(x) = (1 + x^{k-i} - f_{k,k-i}(x))/2x^{k-i}. \tag{2.2}$$

Since $f_{k,0}(x) = 0$, we see that $h_{k,k}(x) = 1$. Furthermore Lemma 1 may be rephrased in terms of $h_{k,i}(x)$.

LEMMA 2.

$$h_{k,i}(x) = 1 + \frac{(xq)^i}{1 + xq}(1 - h_{k,k-i}(xq) - h_{k,k-i+1}(xq)). \tag{2.3}$$

LEMMA 3. *If $\overset{*}{h}_{k,i}(x)$ is any function of x and q analytic around $x = 0$, $q = 0$, and*

$$\overset{*}{h}_{k,k}(x) = 1, \tag{2.4}$$

$$\overset{*}{h}_{k,i}(x) = 1 + \frac{(xq)^i}{1 + xq}(1 - \overset{*}{h}_{k,k-i}(xq) - \overset{*}{h}_{k,k-i-1}(xq)), \quad 1 \leqq i \leqq k - 1, \tag{2.5}$$

$$\overset{*}{h}_{k,i}(0) = 1, \qquad 1 \leqq i \leqq k, \tag{2.6}$$

then $h_{k,i}(x) = \overset{*}{h}_{k,i}(x)$ for $1 \leqq i \leqq k$.

PROOF. We let

$$h_{k,i}(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_i(m, n) x^m q^n,$$

$$\overset{*}{h}_{k,i}(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_i^*(m, n) x^m q^n.$$

Then clearly

$$\begin{aligned} a_k(m, n) = a_k^*(m, n) &= 1 \quad \text{if } m = n = 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

From (2.1) and (2.2), we see directly that $h_{k,i}(0) = 1$; this and (2.6) imply

$$\begin{aligned} a_i(0, n) = a_i^*(0, n) &= 1 \quad \text{if } n = 0, \\ &= 0 \quad \text{if } n > 0. \end{aligned} \tag{2.7}$$

(2.3) and (2.5) imply

$$
\begin{aligned}
(2.8)\qquad & a_i(m, n) + a_i(m-1, n-1) \\
& \quad = \epsilon_i(m, n) - a_{k-i}(m-i, n-m) - a_{k-i+1}(m-i, n-m)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.9)\qquad & a_i^*(m, n) + a_i^*(m-1, n-1) \\
& \quad = \epsilon_i(m, n) - a_{k-i}^*(m-i, n-m) - a_{k-i+1}^*(m-i, n-m),
\end{aligned}
$$

where $\epsilon_i(0,0) = \epsilon_i(1,1) = \epsilon_i(i,i) = 1$, $\epsilon_i(m,n) = 0$ otherwise, and any $a_i(m,n)$ or $a_i^*(m,n)$ with negative entries is zero.

Now we may proceed by mathematical induction on m to verify that $a_i(m,n) = a_i^*(m,n)$. (2.7) takes care of $m = 0$. If $a_i(m,n) = a_i^*(m,n)$ for $m < m_0$, then (2.8) and (2.9) imply that $a_i(m_0, n) = a_i^*(m_0, n)$. Thus Lemma 3 is established.

Lemma 4. *Let $\overline{W}_{k,i}(n; M, N)$ denote the number of partitions of the type enumerated by $W_{k,i}(n; N)$ with M parts. Then*

$$
(2.10)\qquad \overline{W}_{k,i}(0; M, N) = \begin{cases} 1 & \text{if } M = N = 0, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
(2.11)\qquad \overline{W}_{k,i}(1; M, N) = \begin{cases} 1 & \text{if } M = N = i, \\ 0 & \text{otherwise,} \end{cases}
$$

for $n > 1$,

$$
\begin{aligned}
(2.12)\qquad \overline{W}_{k,i}(n; M, N) &= \overline{W}_{k,k-i}(n-1; M-i, N-M) \\
&\quad + \overline{W}_{k,k-i+1}(n-1; M-i, N-M).
\end{aligned}
$$

Proof. (2.10) and (2.11) are directly from the definition of $\overline{W}_{k,i}(n; M, N)$.

To prove (2.12), we start with the partitions enumerated by the left-hand side. Let us consider two classes of such partitions: (1) those in which 2 appears $k - i$ times, and (2) those in which 2 appears $k - i + 1$ times. We now transform our partitions by deleting the i ones in each partition and subtracting 1 from all other summands. The number being partitioned now drops to $N - M$; there are now $M - i$ parts, and the largest part is $n - 1$. Indeed this procedure shows that there are $\overline{W}_{k,k-i}(n-1; M-i, N-M)$ partitions in the first class and $\overline{W}_{k,k-i+1}(n-1; M-i, N-M)$ elements of the second class. Thus we have (2.12).

We transform Lemma 4 into identities for the related generating functions.

LEMMA 5. *If*

$$\bar{w}_{k,i}(n; x; q) = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \overline{W}_{k,i}(n; M, N) x^M q^N,$$

then

$$\bar{w}_{k,i}(0; x; q) = 1; \tag{2.13}$$

$$\bar{w}_{k,i}(1; x; q) = (xq)^i; \tag{2.14}$$

and for $n > 1$,

$$\begin{aligned} &\bar{w}_{k,i}(n; x; q) \\ &= (xq)^i(\bar{w}_{k-i}(n-1; xq; q) + \bar{w}_{k,k-i+1}(n-1; xq; q)). \end{aligned} \tag{2.15}$$

PROOF. (2.13), (2.14), and (2.15) follow directly from (2.10), (2.11), and (2.12) respectively.

LEMMA 6. *If*

$$H_{k,i}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \bar{w}_{k,i}(n; x; q)}{(-xq)_n},$$

then $H_{k,i}(x)$ *is analytic around* $x = 0, q = 0$, *and*

$$H_{k,k}(x) = 1; \tag{2.16}$$

$$H_{k,i}(x) = 1 + \frac{(xq)^i}{1 + xq}(1 - H_{k,k-i}(xq) - H_{k,k-i+1}(xq)), \tag{2.17}$$

$$H_{k,i}(0) = 1, \qquad 1 \leqq i \leqq k. \tag{2.18}$$

PROOF. For $|q| < 1$, $|x| < 1$, we clearly have

$$\bar{w}_{k,i}(n; |x|; |q|) \leqq |q|^{\binom{n+1}{2}} |x|^n \prod_{j=1}^{n} (1 + |x|\,|q|^j + \cdots + |x|\,|q|^{j(k-2)}).$$

This estimate is sufficient to guarantee uniform convergence of the series for $H_{k,i}(x)$ around $x = q = 0$.

Now since all partitions of the type enumerated by $\overline{W}_{k,k}(n; M, N)$ must have $k \leqq f_1 \leqq k - 1$, we see that no partitions except the empty partition are counted. Thus $\overline{W}_{k,k}(n; M, N) = 1$ if $n = M = N = 0$ and equals 0 otherwise. Hence $\bar{w}_{k,i}(n; x; q) = 1$ if $n = 0$ and 0 if $n > 0$. Thus $H_{k,k}(x) = 1$.

Now by Lemma 5,

$$H_{k,i}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \bar{w}_{k,i}(n; x; q)}{(-xq)_n}$$

$$= 1 - \frac{(xq)^i}{1+xq} - \sum_{n=1}^{\infty} \frac{(-1)^n \bar{w}_{k,i}(n+1; x; q)}{(-xq)_{n+1}}$$

$$= 1 - \frac{(xq)^i}{1+xq} - \frac{(xq)^i}{1+xq}$$

$$\cdot \sum_{n=1}^{\infty} \frac{(-1)^n(\bar{w}_{k,k-i}(n; xq; q) + \bar{w}_{k,k-i+1}(n; xq; q))}{(-xq^2)_n}$$

$$= 1 - \frac{(xq)^i}{1+xq} - \frac{(xq)^i}{1+xq}(H_{k,k-i}(xq) + H_{k,k-i+1}(xq) - 2)$$

$$= 1 + \frac{(xq)^i}{1+xq}(1 - H_{k,k-i}(xq) - H_{k,k-i+1}(xq)).$$

Finally we note that $\bar{w}_{k,i}(n; 0, q) = 1$ if $n = 0$ and $= 0$ if $n > 0$. Hence $H_{k,i}(0) = 1$.

Thus we see that the lemma is established.

We are now prepared to prove Theorem 1. First Lemmas 3 and 6 imply that $H_{k,i}(x) = h_{k,i}(x)$. Consequently

$$1 - \sum_{n=1}^{\infty} q^{(2k-1)n^2/2+n/2-(k-i)n} \frac{(1-q^{2n(k-i)})}{1+q^n}$$

$$= 1 - \tfrac{1}{2} f_{k,k-i}(1) = h_{k,i}(1) = H_{k,i}(1)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \bar{w}_{k,i}(n; 1; q)}{(-q)_n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n w_{k,i}(n; q)}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

This concludes the proof of Theorem 1.

COROLLARY.

$$\sum_{n=1}^{\infty} q^{(2k-1)n^2/2-n/2}(1-q^n) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} w_{k,k-1}(n; q)}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

PROOF. Set $i = k - 1$ in Theorem 1 and simplify.

3. **Partition theorems.** In this section we shall prove some partition

theorems which follow from Theorem 1 and its corollary. First we remark that when $k = 2$, the corollary of Theorem 1 may be stated as

$$\sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)},$$

a result due to L. J. Rogers; to see this we note that the only partition counted by $W_{2,1}(n; N)$ is $N = 1 + 2 + \cdots + n$ since every part can appear at most once yet $f_j + f_{j+1} = 2$ or 3. As remarked in [**1**, p. 137] this identity may be used to prove a partition theorem of N. J. Fine [**4**, Theorem 2(iii)].

More generally in the notation of [**3**, p. 556] we have

THEOREM 2.

$$N\Big(s = \sum_{i=1}^{n} f_i \cdot i + \sum_{j=1}^{n} g_j \cdot j, f_1 = k - 1,$$
$$f_i + f_{i+1} = k \text{ or } k + 1, f_i \leqq k - 1; (-1)^{n-1+\Sigma g_j}\Big)$$
$$= 1 \quad \text{if } s = n((2k-1)n - 1)/2,$$
$$= -1 \quad \text{if } s = n((2k-1)n + 1)/2,$$
$$= 0 \quad \text{otherwise.}$$

PROOF.

$$\sum_{n=1}^{\infty} q^{(2k-1)n^2/2 - n/2}(1 - q^n) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} w_{k,k-1}(n; q)}{(-q)_n}$$
$$= \sum_{n=1}^{\infty} N\Big(s = \sum_{i=1}^{n} f_i i + \sum_{j=1}^{n} g_i j, f_1 = k - 1, f_i + f_{i+1}$$
$$= k \text{ or } k + 1, f_i \leqq k - 1; (-1)^{n-1+\Sigma g_j}\Big) q^n.$$

4. **Conclusion.** Other theorems of the nature discussed here are available for the false theta functions. In the notation of [**2**, equation (1.1), p. 433] if

$$f^{*}_{k,i}(x; d; q) = C_{k,i}(d, q; x; q),$$

then as in Lemma 1, we may prove

$$f^{*}_{k,i}(x; d; q) = 1 - \frac{x^i}{1 - xq/d} f_{k,k-i}(xq; d; q) + \frac{x^{i+1} q d^{-1}}{1 - xq/d} f_{k,k-i-1}(xq; d; q).$$

We note also that

$$f_{k,i}^{*}(1; -q; q^2) = \sum_{n=0}^{\infty} q^{(2k-1)n^2-2in}(1 - q^{4ni}).$$

Probably further results could be obtained by studying $C_{k,i}(q, a_2, \cdots, a_\lambda; x; q)$ in general.

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