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A TEST OF TEMPORAL INDEPENDENCE FOR PARTIALLY HOMOGENEOUS GAUSSIAN RANDOM FIELDS

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ABSTRACT. A partially homogeneous random field is a family of random variables $\{X(P, t)\}$ where the following two conditions are true: First, for a given time t, the $X(P, t)$ are observations on a nonhomogeneous random field where its second order properties are not a function of time, i.e., $EX(P, t)\overline{X(Q, t)} = R(P, Q)$. Second, that the family of random variables is weakly stationary over time, i.e., $EX(P, t) = 0$ and $EX(P, t)\overline{X(Q, s)} = R(P, Q, t - s)$. In this paper we consider the Karhunen-Loève representation of the random field, together with necessary and sufficient conditions for a Gaussian random field to be temporally independent. It is shown that a set of weakly stationary principal components can be used to construct a multivariate test of independence.

1. **Introduction.** A random field on the space S is a real, or possibly complex valued random function $X(P)$ defined on a space S. The random variables are defined by the existence of a multivariate function $F_{P_1,\ldots,P_n}$ of $\{X(P_1),\cdots, X(P_n)\}$ for every finite set $\{P_1, \cdots, P_n\} \subset S$, such that $F_{P_1,\ldots,P_n}$ satisfies the following symmetry and consistency conditions:

1. For every permutation $i_1, \cdots, i_n$ of $1, \cdots, n$, and real numbers $\alpha_1, \cdots, \alpha_n$,

$$F_{P_1,\ldots,P_n}(\alpha_1, \cdots, \alpha_n) = F_{P_{i_1},\ldots,P_{i_n}}(\alpha_{i_1}, \cdots, \alpha_{i_n}) \tag{1.1}$$

and

2. If $m < n$,

$$F_{P_1,\ldots,P_m}(\alpha_1, \cdots, \alpha_m) = F_{P_1,\ldots,P_n}(\alpha_1, \cdots, \alpha_m, \infty, \cdots, \infty). \tag{1.2}$$

Under these conditions and when a conditional distribution exists, it is possible to define a probability measure on an abstract space Ω (Doob [4, p. 639]), defined on a sufficiently large class of sets (which constitute a Borel Field in Ω) so that the joint distribution of $X(P_1, \omega), \cdots, X(P_n, \omega)$ defined on Ω will be the same as that of $X(P_1), \cdots, X(P_n)$.

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The space is assumed to have a topology T, a σ-finite measure μ defined on a σ-algebra of its subsets, Σ, and to be compact. Also the process is assumed to satisfy $EX(P) = 0$ and $E|X(P)|^2 < \infty$.

The covariance function of the random function is given by

$$R(P, Q) = EX(P)\overline{X(Q)} \quad \text{for any } P, Q \in S. \tag{1.3}$$

The process is said to be homogeneous if the covariance function depends only on the vector $r = \overline{PQ}$. In this case the covariance function is invariant under certain groups of transformations, which leads to a spectral decomposition of the covariance operator. For a detailed account of this subject, see Yaglom [9], [10] and Hannan [6], [7].

Natural data often does not possess this homogeneity condition, and techniques valid for nonhomogeneous random fields must be employed. A second order analysis is possible for certain nonhomogeneous random fields by using the Karhunen-Loève representation.

Since S is compact, it follows that $R(P, Q) \in L_2(\mu \times \mu)$, and that Mercer's theorem holds. Then

$$R(P, Q) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(P) \overline{\varphi_n(Q)} \tag{1.4}$$

where the convergence is uniform and in quadratic mean (q.m.). The $\{\varphi_n(P)\}$ are the eigenfunctions corresponding to the eigenvalues $\{\lambda_n\}$, and satisfy, where δ is the Kronecker δ,

$$\begin{aligned} &\int_S \varphi_n(P)\overline{\varphi_m(P)}\, d\mu(P) = \delta_{nm}, \\ &\int_S R(P, Q)\varphi(Q)\, d\mu(Q) = \lambda\varphi(P). \end{aligned} \tag{1.5}$$

Without loss of generality, we can assume that the set of eigenfunctions form a complete set in $L_2(\mu)$ since it is possible to add at most a countable set of functions belonging to $L_2(\mu)$ corresponding to the zero valued eigenvalues.

This, together with the conditions on $X(P)$, leads us to the Karhunen-Loève representation,

$$X(P) = \sum_{n=1}^{\infty} z_n \varphi_n(P) \quad \text{where } Ez_n\bar{z}_m = \delta_{nm}\lambda_n \tag{1.6}$$

and the first equality is taken to be in q.m. The orthogonality of the $\varphi_n(P)$ leads to the relation

$$z_n = \int_S X(P)\,\overline{\varphi_n(P)}\,d\mu(P). \tag{1.7}$$

In this paper, we are concerned with the time indexed class of random fields $\{X(P, t), P \in S\}$. It will be shown that even though the process is nonhomogeneous on S, if it is weakly stationary with respect to time (partially homogeneous), i.e.,

$$R(P, t, Q, s) = R(P, Q, t - s) \quad \text{for all } t, s, \tag{1.8}$$

the Karhunen-Loève representation will reduce to a weakly stationary multivariate vector process. While the vector is infinite dimensional, its elements are the principal components of $R(P, Q, t)$, and hence a finite subcollection will describe the process as closely as is desired. We will further show for a discrete time process, $X(P, t)$ is uncorrelated over time (white) if and only if the same is true for the vector process of principal components. Then for a Gaussian process, we develop a test of whiteness for $X(P, t)$.

2. **Space-time processes.** In this section, we will prove the several assertions made in the last part of the previous section, and some results needed for the test of independence given in the last section.

It is assumed that the covariance function for $X(P, t)$, given by $R(P, t, Q, s) = EX(P, t)\overline{X(Q, s)}$, is for any given time independent of that time. That is, $R(P, t, Q, t) = R(P, Q)$ for all t.

For any instant of time, then, we have that

$$X(P, t) = \sum_{n=1}^{\infty} z_n(t)\varphi_n(P, t) \tag{2.1}$$

and that

$$\int_S R(P, t, Q, t)\varphi_n(Q, t)\,d\mu(Q) = \lambda_n(t)\varphi_n(P, t). \tag{2.2}$$

But since $R(P, t, Q, t) = R(P, Q)$, and the eigenfunctions and eigenvalues are the solutions to the same Fredholm equation, we get that $\varphi_n(P, t) = \varphi_n(P)$ and $\lambda_n(t) = \lambda_n$ for all P and n. Hence it follows that

$$X(P, t) = \sum_{n=1}^{\infty} z_n(t)\varphi_n(P), \qquad P \in S \text{ and for all time } t, \tag{2.3}$$

a very valuable separation of time and space.

If we let $f_{nm}(t, s) = Ez_n(t)\overline{z_m(s)}$, then using

$$z_n(t) = \int_S X(P, t)\overline{\varphi_n(P)}\,d\mu(P) \tag{2.4}$$

and Fubini's theorem, it follows that

$$(2.5) \qquad f_{nm}(t, s) = \iint_S R(P, t, Q, s)\overline{\varphi_n(P)}\varphi_m(Q)\, d\mu(P)\, d\mu(Q).$$

If $X(P, t)$ is weakly stationary, then $R(P, t, Q, s)$ is a function of P, Q and $t - s$. It follows from (2.5) that $f_{nm}(t, s) = f_{nm}(t - s)$, that is that $z_n(t)$ is a weakly stationary process. Similarly, under conditions for the interchange of an integral with a summation, it can be shown that $z_n(t)$ weakly stationary implies that $X(P, t)$ is also weakly stationary.

An interesting sidelight at this point is that when the vector process $z(t)$ is weakly stationary, it has the usual Cramér representation. The Karhunen-Loève representation can then be combined with the Cramér representation to obtain a time-space decomposition of the process. This decomposition can be very important for prediction.

In the rest of this paper, we restrict ourselves to the discrete time case, that is where $t = \cdots, -1, 0, 1, \cdots$. The following lemma gives a condition for the "whiteness" of the discrete time process $X(P, t)$.

LEMMA. *If $X(P, t)$ is a discrete time random function admitting a Karhunen-Loève expansion on S, then $R(P, t, Q, s) = \delta_{ts}R(P, Q)$, where δ is the Kronecker delta, if and only if the representation process, $z_n(t)$, satisfies $f_{nm}(t, s) = \delta_{ts}\delta_{nm}\lambda_n$ for all n, m, s and t.*

PROOF. First, given that $R(P, t, Q, s) = \delta_{ts}R(P, Q)$, we use (2.5) to show that

$$f_{nm}(t, s) = \delta_{ts} \iint_S R(P, Q)\overline{\varphi_n(P)}\varphi_m(Q)\, d\mu(P)\, d\mu(Q).$$

By the uniform convergence of (1.4) and the orthogonality of the $\varphi_n(P)$ (see equation (1.5)), we get

$$(2.6) \qquad f_{nm}(t, s) = \delta_{ts}\delta_{nm}\lambda_n.$$

Conversely, given that $f_{nm}(t, s) = \delta_{ts}\delta_{nm}\lambda_n$, and using the fact that S is compact assures that the λ_n converge absolutely, we employ Lebesgue's dominated convergence theorem and (2.3) to get

$$\begin{aligned} R(P, t, Q, s) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm}(t, s)\varphi_n(P)\overline{\varphi_m(Q)} \\ (2.7) \qquad &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \delta_{ts}\delta_{nm}\lambda_n\varphi_n(P)\overline{\varphi_m(Q)} \\ &= \delta_{ts} \sum_{n=1}^{\infty} \lambda_n\varphi_n(P)\overline{\varphi_m(Q)} = \delta_{ts}R(P, Q). \end{aligned}$$

This completes the proof of the lemma.

We say that $X(P, t)$ has a normal distribution if for any finite collection of points $\{P_1, \cdots, P_n\} \subset S$ and any finite collection of times $\{t_1, \cdots, t_n\}$, $\{X(P_1, t_1), \cdots, X(P_n, t_n)\}$ has a joint normal distribution. We assume that $EX(P_i, t_i) = 0$ and $EX(P_i, t_i)\overline{X(P_j, t_j)} = R(P_i, P_j, t_i - t_j)$.

Recalling equation (2.4),

$$z_n(t) = \int_S X(P, t)\overline{\varphi_n(P)}\, d\mu(P),$$

we use the fact that S is compact and separable to construct a sequence of partial sums of $X(P_i, t)\varphi_n(P_i)\, d\mu(P_i)$ to converge to $z_n(t)$. Since normally distributed variates preserve their normality under linear combinations and passing to a limit, the $z(t)$ are also normally distributed.

3. **A test of independence.** In this section, we develop, for $X(P, t)$ Gaussian and having discrete time, a test of independence over time against the alternative that it is weakly stationary with its time component having a slowly varying spectral density. In the last section, we demonstrated that this is equivalent to testing that $f_{nm}(t, s) = \delta_{ts}\delta_{nm}\lambda_n$ against the alternative that the $z(t)$ process is weakly stationary with a slowly varying spectral density.

Since the $z_n(t)$ are the principal components of $R(P, Q)$, a finite vector of p elements can be used to describe the process as well as is desired by the selection of p. A technique for selecting and computing the desired $z_n(t)$ is given by Cohen and Jones [3].

Since the $z(t)$ selected is normally distributed, the complex Fourier transform of the p-dimensional vector $\mathbf{z}(t)$,

$$\tilde{a}(f) = \sum_{t=1}^{n} \tilde{z}(t) \exp(2\pi i f t), \qquad -\tfrac{1}{2} < f < \tfrac{1}{2}, \tag{3.1}$$

will have a complex normal distribution with zero mean and variance equal to the spectral density of $z(t)$. We are letting $\tilde{z}(t)$ be a finite realization of $z(t)$. The complex Gaussian distribution of $\tilde{z}(t)$ is given by Goodman [5] to be

$$p(\tilde{a}) = \frac{1}{\pi^p|\Sigma|} \exp(-\tilde{a}'\Sigma^{-1}\tilde{a}). \tag{3.2}$$

Taking the f to be $f_j = j/n$, $j = 1, \cdots, [(n-1)/2]$, that is, the integer part of $(n-1)/2$, the $\tilde{a}(f_j)$ will be exactly independent under the null hypothesis, and $\Sigma = D$, a diagonal matrix with the eigenvalues λ_n on the diagonal. Under the alternative hypothesis, the

$\tilde{a}(f_j)$ will be approximately independent, and $\Sigma = \Sigma(f)$. Since variance estimates cannot be based on a sample of one, the hypothesis is that $\Sigma(f)$ is a slowly varying function of f, so the $\tilde{a}(f)$ can be separated into frequency bands. Then the assumption is made that $\Sigma(f)$ can be approximated by a constant in each frequency band. Let the sample be composed of N frequency bands, each of width m (so $mN = [(n-1)/2]$).

It is now possible to obtain the approximate likelihood ratio test statistic (see Anderson [1, p. 264], for the real version of the same problem),

$$\lambda = \prod_{j=1}^{N} \left(\frac{e}{m}\right)^{pm} |\hat{\Sigma}_j D^{-1}|^m \exp(\mathrm{tr}(\hat{\Sigma}_j D^{-1})), \tag{3.3}$$

where $\hat{\Sigma}_j$ = sum over the jth band of $\tilde{a}(f)\tilde{a}(f)'$. the test then is composed of N simultaneous independent tests in each of the N frequency bands. The following theorem is the complex version of the theorem on p. 268 of Anderson [1], applicable to the test in each frequency band.

Theorem. *Given $\tilde{a}(f_1), \cdots, \tilde{a}(f_m)$ as observation vectors of p complex components from $N(0, \Sigma)$, the approximate likelihood ratio criterion for testing the hypothesis $\Sigma = D$, where D is specified, is*

$$\lambda = \left(\frac{e}{m}\right)^{pm} |\hat{\Sigma} D^{-1}|^m \exp(-\mathrm{tr}(\hat{\Sigma} D^{-1}))$$

where

$$\hat{\Sigma} = \mathrm{sum}_j\, \tilde{a}(f_j)\tilde{a}(f_j)'.$$

Also $-2\log(\lambda)$ *has asymptotically, as m becomes large, a chi-square distribution with p^2 degrees of freedom.*

Proof. The approximate likelihood ratio comes from using (3.2) and the usual maximizing technique to estimate $\hat{\Sigma}$. We now proceed in an identical fashion to Anderson [1] to obtain the moments of λ and the characteristic function of $-2\log\lambda$.

The estimate $\hat{\Sigma}$ has a complex Wishart distribution with m degrees of freedom, which is given by Goodman [5] to be

$$w(\hat{\Sigma}|\Sigma) = \frac{|\hat{\Sigma}|^{m-p}\exp(-\mathrm{tr}(\Sigma^{-1}\hat{\Sigma}))}{\pi^{1/2p(p-1)}\Gamma[m]\cdots\Gamma[m-p+1]\,|\Sigma|^n}. \tag{3.4}$$

We now find the moments of λ. The hth moment of λ is

$$
\begin{aligned}
E\lambda^h &= \left(\frac{e}{m}\right)^{pmh} \int \cdots \int |\hat{\Sigma} D^{-1}|^{mh} \exp\left(-h \operatorname{tr}(\Sigma D^{-1})\right) w(\hat{\Sigma} \mid \Sigma) \\
&= \left(\frac{e}{m}\right)^{pmh} \int \cdots \int \frac{\prod_{k=1}^{p} \Gamma[m + mh + 1 - k]\ |\Sigma D^{-1}|^{mh}}{|I + h\Sigma D^{-1}|^{m+mh} \prod_{k=1}^{p} \Gamma[m + 1 - k]} \\
&\qquad\qquad \cdot w(\hat{\Sigma} | \Sigma^{-1} + hD^{-1})\, d\hat{\Sigma} \qquad (3.5)\\
&= \frac{(e/m)^{mph} \prod_{k=1}^{p} \Gamma[m + mh + 1 - k]\ |\Sigma D^{-1}|^{mh}}{|I + h\Sigma D^{-1}|^{m+mh} \prod_{k=1}^{p} \Gamma[m + 1 - k]}.
\end{aligned}
$$

It can also be shown that the characteristic function of $-2 \log \lambda$ is

$$
\begin{aligned}
\varphi(t) &= Ee^{-2it \log \lambda} = E\lambda^{-2it} \\
&= \left(\frac{e}{m}\right)^{-2itmp} \frac{|\Sigma D^{-1}|^{-2imt}}{|I - 2it\Sigma D^{-1}|^{m-2imt}} \prod_{k=1}^{p} \frac{\Gamma[m + 1 - j - 2imt]}{\Gamma[m + 1 - k]}. \qquad (3.6)
\end{aligned}
$$

Under the null hypothesis, $\Sigma = D$,

$$
\begin{aligned}
\varphi(t) &= \prod_{k=1}^{p} e^{-2imt} m^{2imt} (1 - 2it)^{-m+2imt} \frac{\Gamma[m + 1 - k - 2imt]}{\Gamma[m + 1 - k]} \\
&= \prod_{k=1}^{p} \varphi_k(t). \qquad (3.7)
\end{aligned}
$$

Using Stirling's approximation as m becomes large,

$$
\begin{aligned}
\varphi_k(t) &\approx (1 - 2it)^{-k} \left(\frac{m + 1 - k - 2imt}{(1 - 2it)(m + 1 - k)} \right)^{m-k} \\
&\qquad \cdot \left(\frac{m + 1 - k - 2imt}{m(1 - 2it)} \right)^{-2imt} \cdot \left(1 - \frac{2imt}{m + 1 - k} \right)^{1/2} \\
&= (1 - 2it)^{-k} \left(1 - \frac{2it(k - 1)}{(1 - 2it)(m + 1 - k)} \right)^{m-k} \qquad (3.8)\\
&\qquad \cdot \left(1 - \frac{k - 1}{(1 - 2it)m} \right)^{-2imt} \left(1 - \frac{2imt}{m + 1 - k} \right)^{1/2} \\
&\approx (1 - 2it)^{-k+1/2}.
\end{aligned}
$$

Hence

$$
\varphi(t) = \prod_{k=1}^{p} \varphi_k(t) \approx (1 - 2it)^{p^2/2}, \qquad (3.9)
$$

which completes the proof of the theorem.

Now recalling equation (3.3), the test statistic for the original test of independence,

$$\lambda = \prod_{j=1}^{N} \left(\frac{e}{m}\right)^{pm} |\hat{\Sigma}_j D^{-1}|^m \exp(-\operatorname{tr}(\hat{\Sigma}_j D^{-1})),$$

it follows from the theorem that $-2 \log \lambda$ has approximately a chi-square distribution with Np^2 degrees of freedom.

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References

1. T. W. Anderson, *An introduction to multivariate statistical analysis*, Wiley Publ. in Statistics, Wiley, New York; Chapman and Hall, London, 1958. MR **19**, 992.

2. A. Cohen, *On random fields with no symmetry structure*, Technical Report, no. 72, Dept. of Statistics, Johns Hopkins University, Baltimore, Maryland, 1967.

3. A. Cohen and R. H. Jones, *Regression on a random field*, Technical Report, no. 61, Dept. of Statistics, Johns Hopkins University, Baltimore, Maryland, 1967.

4. J. L. Doob, *Stochastic processes*, Wiley, New York; Chapman and Hall, London, 1953. MR **15**, 445.

5. N. R. Goodman, *Statistical analysis based on a certain multivariate complex Gaussian distribution*, Ann. Math. Statist. **34** (1963), 152–177. MR **26** #3148a.

6. E. J. Hannan, *Group representations and applied probability*, J. Appl. Probability **2** (1965), 1–68. MR **35** #2307a.

7. ———, *Fourier methods and random processes*, Thirty-Sixth Session of the Institute for Scientific Information, Sydney, Australia, 1967.

8. R. H. Jones, *Stochastic processes on a sphere*, Ann. Math. Statist. **34** (1963), 213–218. MR **30** #616.

9. A. M. Jaglom, *Second-order homogeneous random fields*, Proc. Fourth Berkeley Sympos. Math. Statist. and Probability, vol. 2, Univ. California Press, Berkeley, Calif., 1961, pp. 593–622. MR **26** #4399.

10. ———, *An introduction to the theory of stationary random functions*, Uspehi Mat. Nauk **7** (1952), no. 5 (51), 3–168; English transl., rev. ed., Prentice-Hall, Englewood Cliffs, N. J., 1962. MR **14**, 485; MR **32** #1762.

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