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SINGULAR BOUNDARY PROBLEMS FOR THE DIFFERENTIAL EQUATION $Lu = \lambda\sigma u$

DAVID A. SHOTWELL

1. **Introduction.** The classical theory of Sturm-Liouville boundary problems for second-order differential operators on finite intervals has served as the point of departure for a number of modern generalizations. An extensive literature exists in connection with equations of the type $Lu = \lambda u$, where L is a linear differential operator of order $n \geqq 1$ on a finite or infinite interval I, and λ is a complex parameter. Boundary conditions imposed upon solutions of these equations lead to differential boundary problems which are termed "singular" if I is infinite, or if the coefficients of the differential operator have a singular behavior near the endpoints of a finite interval. One method of dealing with such problems, due originally to H. Weyl, has been used effectively for a wider class of problems, notably by N. Levinson, E. A. Coddington, and F. Brauer. It consists of the replacement of the given problem by a sequence of regular (i.e. nonsingular) problems on finite subintervals which tend to the original interval. Known results for these regular problems then yield information about the singular case through a limiting process. This procedure may be carried out even though the so-called "singular" problem is not explicitly defined at the outset; the results obtained in the limit as the finite subintervals tend to the original interval are then defined as constituting the solution of a singular problem associated with the differential operator in question. The merit of this approach is that it does not require one to know in advance what boundary conditions, if any, are appropriate for the direct definition of a singular problem. However, such direct definition has been given by M. H. Stone, E. A. Coddington, and others, for important cases involving formally self-adjoint L with associated operators which are symmetric or selfadjoint in the Hilbert space of functions square-integrable in the interval I.

A further generalization of problems of this type leads to the consideration of

$$Lu = \lambda Mu, \tag{i}$$

where L and M are differential operators defined on I. Just as in the

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above cases where M is the identity operator, certain boundary problems for equations of this type can be dealt with by means of the theory of operators in Hilbert spaces. The present study is concerned with a class of problems for the interval $I = (-\infty, \infty)$, with L formally selfadjoint and of even order, M the operation of multiplication by a function which is real, continuous, and bounded on I. The theorems obtained include a Parseval relation, and an expansion theorem valid for functions belonging to a linear manifold which is dense in $L^2(-\infty, \infty)$. Certain results are first derived for boundary problems for (i) on finite subintervals of I. The proofs of some of the results for the finite interval problems are identical or very similar to those given by F. Brauer [2]. Accordingly, these proofs will either be omitted or only briefly sketched. However, the procedures involved in the passage to the limit, and the results obtained for the singular problem, differ from those of Brauer because of alteration of the hypotheses regarding the operators L and M. For the class of problems treated by Brauer, L and M were required to be semibounded below, $(Lu, u) \geqq K(u, u)$, $(Mu, u) \geqq d(u, u)$, for functions u satisfying suitable boundary conditions. In addition, $d > 0$, and this requirement made possible the introduction of a new Hilbert space based upon the inner product $[u, v] = (Mu, v)$. Completeness and expansion theorems relative to this new space, rather than the original space L^2, were obtained.

In the present paper the property of positive definiteness belongs instead to L, while the multiplication operator M, although bounded, may be indefinite in $L^2(I)$. A new space based upon (Lu, v) is introduced, and the Parseval relation is obtained with respect to the metric of this new space. The final expansion theorem, however, involves convergence in $L^2(I)$.

2. **Statement of hypotheses and summary of results.** Let L be a formally selfadjoint differential operator of even order $n \geqq 2$ given by

$$Lu(t) = \sum_{i=0}^{n} P_i(t)u^{(n-i)}(t), \tag{1}$$

where $P_i^{(n-i)}(t)$ exists and is continuous, that is, $P_i \in C^{(n-i)}$ on $(-\infty, \infty)$, and $P_0(t) \neq 0$ on $(-\infty, \infty)$.

Let $D_0^{(n)}$ denote the set of all functions $f(t)$ such that $f^{(n-1)}(t)$ exists and is absolutely continuous on any finite interval, $f^{(n)}(t) \in L^2(-\infty, \infty)$, and $f(t) \equiv 0$ for large $|t|$. Let L_∞^0 be the operator in $L^2(-\infty, \infty)$ with domain $D_0^{(n)}$ which takes $x(t) \in D_0^{(n)}$ into

$Lx(t)$, where L is given by (1). Then L_∞^0 is symmetric in $L^2(-\infty, \infty)$ and it will be assumed that the differential operator L is such that there exists an $\epsilon > 0$ for which $(L_\infty^0 x, x) \geqq \epsilon(x, x)$ is satisfied for every x in the domain $D_{L_\infty^0}$ of the operator L_∞^0. Finally, let L be such that no solution of the differential equation $Lu = iu$, or of $Lu = -iu$, belongs to $L^2(-\infty, \infty)$. (It is easily shown that the class of differential operators meeting all of these specifications is nonvacuous; an example is $Lu = -u'' + u$.)

Let $\sigma(t)$ be a function which is real-valued, continuous, and bounded on $(-\infty, \infty)$, $|\sigma(t)| < B$ for a positive number B. Also let the set of all zeros of σ be of measure zero, and suppose that there exists a number c and a closed neighborhood $N_0 = [c - \eta, c + \eta]$, such that $\sigma(t) \neq 0$ and $\sigma(t) \in C^{(n)}$ for $t \in N_0$. By assigning initial conditions at $t = c$, we shall select a basis for the solutions of the differential equation

$$(2) \qquad Lu = \lambda\sigma u$$

on $(-\infty, \infty)$, where λ is a complex-valued parameter. Let $S_j(t, \lambda)$, $j = 1, 2, \cdots, n$, be solutions of (2) which satisfy

$$(3) \qquad S_j^{(k-1)}(c, \lambda) = \delta_{jk}, \qquad k = 1, 2, \cdots, n.$$

These functions, together with their t-derivatives up to order n, are continuous in t, λ and entire in λ for fixed t on $(-\infty, \infty)$; these properties are derived in [5].

Let L_∞ denote the closure in the space $L^2(-\infty, \infty)$ of L_∞^0. The operator L_∞ is symmetric, and $(L_\infty u, u) \geqq \epsilon(u, u)$ for $u \in D_{L_\infty}$; n must be an even integer in order that the latter requirement be satisfied. Under the conditions imposed above on L, it is known that L_∞ is selfadjoint, $L_\infty = L_\infty^*$ (see [4]). On the linear manifold D_{L_∞} we introduce the inner product $[u, v] = (L_\infty u, v)$, and put $|||u|||^2 = [u, u]$. Let S denote the completion of the domain D_{L_∞} with respect to the new inner product. S is a Hilbert space which can be identified with a linear manifold in $L^2(-\infty, \infty)$; this manifold is the domain $D_{L_\infty^{1/2}}$ of a positive selfadjoint operator such that $L_\infty^{1/2} \cdot L_\infty^{1/2} = L_\infty$. (For a detailed discussion see [6], [7].) For $u, v \in S$, $[u, v] = (L_\infty^{1/2} u, L_\infty^{1/2} v)$. Thus there are two inner products on the manifold S, which is complete relative to [,], but is not complete in the original L^2 inner product.

From the properties specified for σ, the multiplication operator $M = \sigma$ takes $L^2(-\infty, \infty)$ into itself one-to-one, and is symmetric with respect to the L^2 inner product (,). $L_\infty \geqq \epsilon > 0$ implies that zero belongs to the resolvent set of L_∞, hence L_∞^{-1} is defined on

$L^2(-\infty, \infty)$ and bounded. For $s \in S$, let $Ts = L_\infty^{-1}\sigma s$. T is defined on S and takes S into $D_{L_\infty} \subset S$.

The main results to be proved can now be summarized in the statement of the following theorems.

THEOREM 1. *There exists an $n \times n$ matrix function $\rho(\lambda)$ defined on $(-\infty, \infty)$ whose elements are complex-valued functions of bounded variation on any finite λ-interval; $\rho(\lambda)$ is Hermitian, and $\rho(\lambda_2) - \rho(\lambda_1)$ is positive semidefinite if $\lambda_2 > \lambda_1$. Let h_1, h_2 be complex n-vector functions of λ and put*

$$(4) \qquad (h_1, h_2)_\rho = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{h_{2,j}(\lambda)} h_{1,k}(\lambda) d\rho_{jk}(\lambda)$$

for functions h_1, h_2, which are measurable with respect to ρ and for which $(h, h)_\rho = \|h\|_\rho^2$ exists; let H_ρ be the Hilbert space thus defined. For $f_s \in S$ let $g_s = Tf_s$ and let $u_{s,(a,b)}$ be the vector function with components given by

$$(5) \qquad \begin{aligned} u_{s,(a,b)k}(\lambda) &= \frac{1}{\lambda}\int_a^b L_\infty g_s(t)\overline{S_k(t,\lambda)}dt \qquad && \text{if } |\lambda| \geqq \epsilon_1, \\ &= 0 && \text{if } |\lambda| < \epsilon_1, \end{aligned}$$

for $k = 1, 2, \cdots, n$. Here, ϵ_1 is a fixed positive number less than ϵ/B (see above). Then $u_{s,(a,b)}$ belongs to H_ρ, and as $(a, b) \to (-\infty, \infty)$, $u_{s,(a,b)}$ converges in H_ρ to a limit h_s such that $|||g_s||| = \|h_s\|_\rho$. The range of T is dense in S, and the correspondence $g_s \to h_s$ extends uniquely to an isometric mapping V defined on S which takes S into H_ρ.

THEOREM 2. *Let $s(t)$ be any function in S, and put $u = Vs$. Then there is a sequence of intervals (μ_1, μ_2) tending to $(-\infty, \infty)$ such that the functions*

$$(6) \qquad \left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right] \left\{\sum_{j,k=1}^{n} \lambda^{-1} u_k(\lambda) S_j(t,\lambda) d\rho_{jk}(\lambda)\right\}$$

which belong to $L^2(-\infty, \infty)$, converge in norm to s in $L^2(-\infty, \infty)$.

It will be shown in the sections which follow that the matrix $\rho(\lambda)$ is the limit of a sequence of matrices corresponding to a sequence of boundary problems on finite intervals associated with L.

We turn now to the formulation of the finite-interval problems and the derivation of their properties, which will be used to establish the stated theorems.

3. **Boundary problems on finite intervals.** On a finite interval $\delta = [a, b]$, let $B_\delta u = 0$ denote n independent boundary contitions, the α's and β's being complex constants,

$$\sum_{j=0}^{n-1} [\alpha_{ij} u^{(j)}(a) + \beta_{ij} u^{(j)}(b)] = 0, \qquad i = 1, 2, \cdots, n. \tag{7}$$

In $L^2(a, b)$ define a linear manifold

$$D_{L_\delta} = \left\{ u(t) \,\middle|\, \begin{matrix} u^{(n-1)}(t) \text{ absolutely continuous} \\ u^{(n)}(t) \in L^2(a, b), \ B_\delta u(t) = 0 \end{matrix} \right\}.$$

Let L_δ be the operator in $L^2(a, b)$ with domain D_{L_δ} which takes u into Lu, with L given by (1). Let the boundary conditions be chosen so that L_δ is selfadjoint, and $(L_\delta u, u)_\delta \geqq \epsilon(u, u)_\delta$ for $u \in D_{L_\delta}$. (Here the subscript δ denotes the inner product $(\ ,\ )_\delta$ in $L^2(\delta)$.) To show that such a set of boundary conditions exists, let $L_\delta{}^0$ be an operator which takes u into Lu, its domain consisting of those $u(t)$ in $L^2(a, b)$ for which $u^{(n-1)}(t)$ is absolutely continuous, $u^{(n)}(t) \in L^2(a, b)$, and $u(t) = 0$ for all t outside of some closed interval contained in the open interval (a, b). Then $L_\delta{}^0$ and its closure $\overline{L_\delta{}^0}$ are symmetric, $(\overline{L_\delta{}^0} u, u) \geqq \epsilon(u, u)$ for $u \in D_{\overline{L_\delta{}^0}}$, and there exists at least one selfadjoint extension L_δ of $\overline{L_\delta{}^0}$ which is also bounded below by ϵ, e.g., the Friedrichs extension [7]. Boundary conditions B_δ exist which define this extension; see [4], [5].

We consider the boundary problem P_δ on δ,

$$P_\delta : L_\delta u = \lambda \sigma u, \qquad u \in D_{L_\delta}. \tag{8}$$

For λ an eigenvalue, u a corresponding eigenfunction,

$$0 < (L_\delta u, u) = \lambda(\sigma u, u) = \lambda \int_\delta |u|^2 \sigma \, dt$$

hence $\lambda = \bar{\lambda} \neq 0$. For two distinct eigenvalues λ_1, λ_2 with corresponding eigenfunctions u_1, u_2,

$$(L_\delta u_1, u_2) = \lambda_1(\sigma u_1, u_2) = \lambda_1(u_1, \sigma u_2),$$
$$(L_\delta u_1, u_2) = (u_1, L_\delta u_2) = \lambda_2(u_1, \sigma u_2),$$

from which $(L_\delta u_1, u_2) = 0$. On D_{L_δ} we introduce $[u, v]_\delta = (L_\delta u, v)$. $|||u|||_\delta^2 = [u, u]_\delta$. The eigenfunctions u_1, u_2 are orthogonal with respect to this new inner product.

A complex number λ is an eigenvalue of P_δ if and only if there is a nontrivial linear combination

$$u(t) = \sum_{j=1}^{n} c_j S_j(t, \lambda)$$

of the independent functions (3) for which $B_\delta u(t) = 0$, i.e., if and only if

$$\sum_{j=1}^{n} c_j B_{\delta_i} S_j(t, \lambda) = 0, \qquad i = 1, 2, \cdots, n,$$

where B_{δ_i} is the ith boundary operator,

$$B_{\delta_i} f(t) = \sum_{k=0}^{n-1} [\alpha_{ik} f^{(k)}(a) + \beta_{ik} f^{(k)}(b)].$$

Thus λ is an eigenvalue if and only if the determinant $\Delta(\lambda)$ of the matrix with $B_{\delta_i} S_j(t, \lambda)$ in the ith row and jth column is zero. $\Delta(\lambda)$ is entire in λ and is not identically zero because the eigenvalues are real; therefore the eigenvalues are at most countable and have no finite limit point.

In order to obtain the Green's function of P_δ, we require a function $K_0(t, \tau, \lambda)$, the construction of which is given in [4]. This function is defined for t, τ on $(-\infty, \infty)$ and for all λ, and has the form

(9a) $$K_0(t, \tau, \lambda) = \frac{1}{2} \sum_{j,k=1}^{n} S_{j,k}^{-1} S_k(t, \lambda) \overline{S_j(\tau, \bar{\lambda})} \qquad (t \geqq \tau),$$

(9b) $$K_0(t, \tau, \lambda) = -\frac{1}{2} \sum_{j,k=1}^{n} S_{j,k}^{-1} S_k(t, \lambda) \overline{S_j(\tau, \bar{\lambda})} \qquad (t \leqq \tau),$$

where the $S_{j,k}^{-1}$ are constants, the elements of a matrix S^{-1} which is non-singular and skew-Hermitian. The functions $\partial^{(l)} K_0(t, \tau, \lambda)/\partial t^l$ are continuous in t, τ, λ for $l < n - 1$, and, for $l = n - 1$ or $l = n$, are continuous on each of the domains $t \leqq \tau$, $\tau \leqq t$, while for $l = n - 1$,

(10) $$\frac{\partial^{n-1} K_0(t+, t, \lambda)}{\partial t^{n-1}} - \frac{\partial^{n-1} K_0(t-, t, \lambda)}{\partial t^{n-1}} = \frac{1}{P_0(t)}.$$

From these properties it follows that if $f(t) \in L^2(a, b)$, then the function

(11) $$v(t) = \int_a^b K_0(t, \tau, \lambda) f(\tau) d\tau$$

satisfies $(L - \lambda\sigma)v = f$ a.e. on $[a, b]$. The Green's function for P_δ is now constructed as follows. Let $K_{1\delta}(t, \tau, \lambda) = \sum_{j=1}^{n} c_j S_j(t, \lambda)$, where the $c_j = c_j(\tau, \lambda)$ are to be determined so that $G_\delta(t, \tau, \lambda) = K_0(t, \tau, \lambda) + K_{1\delta}(t, \tau, \lambda)$ satisfies the boundary conditions $B_\delta G_\delta(t, \tau, \lambda) = 0$. Here $G_\delta(t, \tau, \lambda)$ denotes G_δ considered as a function of t alone, τ and λ fixed, $\tau \in (a, b)$. Thus,

$$(12) \qquad -B_{\delta_i}K_0(t, \tau, \lambda) = \sum_{j=1}^{n} c_j B_{\delta_i} S_j(t, \lambda), \qquad i = 1, 2, \cdots, n.$$

$B_{\delta_i}K_0(t, \tau, \lambda)$ may be extended by continuity so that it is defined for $\tau \in [a, b]$; it is then a linear combination of the $\overline{S_j(\tau, \bar{\lambda})}$ with coefficients which depend only upon λ and are entire in λ. The c_j's are determined by (12) for all λ such that $\Delta(\lambda) \neq 0$, i.e., for all λ other than the eigenvalues. $K_{1\delta}(t, \tau, \lambda)$ therefore takes the form

$$(13) \qquad K_{1\delta}(t, \tau, \lambda) = \sum_{j,k=1}^{n} \Psi_{jk}(\lambda) S_k(t, \lambda) \overline{S_j(\tau, \bar{\lambda})},$$

where the $\Psi_{jk}(\lambda)$ are meromorphic functions of λ, with λ_0 a pole only if it is an eigenvalue of P_δ. The Green's function $G_\delta = K_0 + K_{1\delta}$ is defined except at the eigenvalues, meromorphic in λ for fixed t, τ, and the transformation

$$(14) \qquad G_\delta f = \int_a^b G_\delta(t, \tau, \lambda) f(\tau) d\tau, \qquad f \in L^2(a, b),$$

takes $L^2(a, b)$ onto D_{L_δ} one-to-one, so that $v = G_\delta f$ satisfies $L_\delta v - \lambda \sigma v = f$.

The following three lemmas are proved by Brauer [2].

Lemma 1. *For* im $(\lambda) \neq 0$, $G_\delta(t, \tau, \lambda) = \overline{G_\delta(\tau, t, \bar{\lambda})}$.

Lemma 2. *The poles of G_δ, as a function of λ with t, τ fixed on $[a, b]$, are simple.*

Let λ_K be an eigenvalue of order l, with eigenfunctions $Y_1(t)$, $Y_2(t), \cdots, Y_l(t)$ orthonormal in the new inner product: $[Y_j, Y_k]_\delta = \delta_{jk}$. We then have

Lemma 3. *The residue of $G_\delta(t, \tau, \lambda)$ at $\lambda = \lambda_K$ is*

$$(15) \qquad R(t, \lambda) = -\lambda_K \sum_{i=1}^{l} \overline{Y_i(\tau)} Y_i(t).$$

4. **Spectral matrices.** The operator L_δ has a positive selfadjoint square root $L_\delta^{1/2}$ in $L^2(a, b)$, whose domain $D_{L_\delta^{1/2}} = S_\delta$ is the completion of D_{L_δ} in the new inner product $[\ ,\ ]_\delta$. For u, v in S_δ,

$[u, v]_\delta = (L_\delta^{1/2}u, L_\delta^{1/2}v)$ and $|||u|||_\delta^2 = [u, u]_\delta$. (See Theorem 1 and paragraphs preceding it.) Let $T_\delta u = L_\delta^{-1}\sigma \cdot u$ for $u \in S_\delta$; then,

LEMMA 4. *The operator T_δ in space S_δ has the following properties:*

(i) *T_δ^{-1} exists.*

(ii) *T_δ is bounded and symmetric (hence selfadjoint) on S_δ.*

(iii) *T_δ and P_δ have the same set of eigenfunctions, the associated eigenvalues being reciprocals of each other. The eigenfunctions are complete in S_δ, and T_δ is completely continuous.*

We observe first that if $[T_\delta u, T_\delta u]_\delta = 0$, $u \in S_\delta$, then $(L_\delta L_\delta^{-1}\sigma u, L_\delta^{-1}\sigma u) = 0$, and $L_\delta \geqq \epsilon > 0$ implies $L_\delta^{-1}\sigma u = 0$, $\sigma u = 0$ in $L^2(a, b)$; $\sigma \neq 0$ a.e., hence $u = 0$ in $L^2(a, b)$, and

$$|||u|||_\delta = (L_\delta^{1/2}u, L_\delta^{1/2}u)^{1/2} = 0.$$

This shows that T_δ^{-1} exists.

To prove (ii), let u, v belong to S_δ. Then,

$$[T_\delta u, v]_\delta = (\sigma u, v) = (u, \sigma v) = [u, T_\delta v]_\delta.$$

Thus T_δ is defined on S_δ and is symmetric, which implies that it is bounded.

Zero is not an eigenvalue of T_δ, or of the boundary problem P_δ because $L_\delta \geqq \epsilon > 0$. If $T_\delta u = \nu u$, $0 \neq u \in S_\delta$, then $L_\delta u = \lambda \sigma u$ where $\lambda = \nu^{-1}$; conversely, if $L_\delta u = \lambda\sigma u$ then $T_\delta u = \nu u$. From the condition $|\sigma| < B$ it follows that if $L_\delta u = \lambda\sigma u$, $u \neq 0$, then $\epsilon(u, u) < |\lambda| B(u, u)$, or

$$0 < \epsilon/B < |\lambda|. \tag{16}$$

It is shown by Brauer [2] that if h is a function of class C^n on $[a, b]$, $h \in D_{L_\delta}$, and if $[h, u]_\delta = 0$ for every eigenfunction u of P_δ, then $h \equiv 0$ on $[a, b]$. The completeness of the eigenfunctions follows from this result: let S_1 be the closure in S_δ of the linear span of the set of eigenfunctions, and suppose that $g \in S_\delta$, g orthogonal to S_1, i.e. $[g, u]_\delta = 0$ for every eigenfunction u. Then

$$[T_\delta^2 g, u]_\delta = [T_\delta g, T_\delta u]_\delta = \nu[T_\delta g, u]_\delta = \nu^2[g, u]_\delta = 0,$$

where ν is the eigenvalue associated with u. Thus $T_\delta^2 g \in D_{L_\delta}$, $T_\delta^2 g$ is orthogonal to every eigenfunction u, and $T_\delta^2 g = L_\delta^{-1}(\sigma \cdot T_\delta g)$ is of class C^n. It follows that $T_\delta^2 g = 0$, and $g = 0$, from (i); hence $S_1 = S_\delta$ and the eigenfunctions are complete. This result implies that there are infinitely many eigenfunctions, hence that the sequence of eigenvalues of P_δ (each of multiplicity $\leqq n$) is not bounded. If an orthonormal basis is chosen for the eigenspace at each eigenvalue, a complete orthonormal sequence of eigenfunctions is obtained. Correspondingly, the spectrum of T_δ is discrete, with zero its only limit point, and T_δ is completely continuous in S_δ [7, §93].

Let $\{y_k^{(t)}\}$ be a complete orthonormal sequence of eigenfunctions of P_δ, the corresponding eigenvalues $\{\lambda_k\}$ being ordered so that $0 < |\lambda_1| \leqq |\lambda_2| \leqq \cdots \leqq |\lambda_k| \leqq |\lambda_{k+1}| \leqq \cdots$. For a function $f \in S_\delta$, the Parseval equality and the expansion theorem take the forms:

$$(17) \qquad [f, f]_\delta = \sum_{k=1}^{\infty} [f, y_k]_\delta \overline{[f, y_k]_\delta} = \sum_{k=1}^{\infty} \lambda_k^2 (f, \sigma y_k) \overline{(f, \sigma y_k)},$$

$$(18) \qquad \lim_{n\to\infty} \left[f - \sum_{k=1}^{n} [f, Y_k]_\delta y_k, f - \sum_{k=1}^{n} [f, y_k]_\delta y_k \right]_\delta = 0.$$

It is easily shown that the inequality

$$(19) \qquad [g, g]_\delta \geqq \epsilon(g, g),$$

valid originally for elements $g \in D_{L_\delta}$, extends to all of S_δ; from this it follows that $\sum_{k=1}^{n} [f, y_k]_\delta y_k$ also converges to f in the space $L^2(a, b)$.

Each eigenfunction y_k is a linear combination of the independent solutions $\{S_j(t, \lambda)\}$ with $\lambda = \lambda_k$;

$$(20) \qquad y_k(t) = \sum_{j=1}^{n} r_{kj} S_j(t, \lambda_k).$$

Thus (17) may be written

$$(21) \qquad [f, f]_\delta = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{\Phi_j(\lambda)} \Phi_k(\lambda) d\rho_{\delta_{jk}}(\lambda),$$

where

$$(22) \qquad \phi_j(\lambda) = \int_a^b f(t)\sigma(t) \overline{S_j(t, \lambda)} dt,$$

and $\rho_\delta(\lambda) = (\rho_{\delta_{jk}}(\lambda))$ is a matrix of step functions with discontinuities only at the eigenvalues, continuous from the right:

$$(23) \qquad \begin{aligned} \rho_{\delta_{jk}}(\lambda_i + 0) - \rho_{\delta_{jk}}(\lambda_i - 0) &= \rho_{\delta_{jk}}(\lambda_i) - \rho_{\delta_{jk}}(\lambda_i - 0) \\ &= \sum_{m=1}^{l} \lambda_i^2 r_{mj} \overline{r_{mk}}, \end{aligned}$$

where the sum contains l terms for λ_i an eigenvalue of multiplicity l. We also require $\rho_\delta(0) = (0)$, which determines $\rho_\delta(\lambda)$ uniquely.

The properties obtained for finite interval problems of type P_δ will now be applied to the derivation of results in the singular case.

5. **Proof of Theorem 1.** Let a sequence $\{\delta_m\}$ of intervals be chosen $\delta_m = [a_m, b_m] \to (-\infty, \infty)$ as $m \to \infty$, and with each such interval let boundary conditions B_m be associated so that the above results hold, i.e. there is a complete orthonormal sequence of eigenfunctions in the space $S_{\delta_m} = S_m$, and an expansion involving the spectral matrix $\rho_m(\lambda) = \rho_{\delta_m}(\lambda)$. The operator $L_m = L_{\delta_m}$, selfadjoint in the space $L^2(a_m, b_m)$, satisfies $(L_m u, u)_m \geqq \epsilon_m (u, u)_m$ where $\epsilon_m \geqq \epsilon$, ϵ_m being the greatest lower bound of L_m.

We shall first prove

LEMMA 5. *There exists at least one matrix $\rho(\lambda)$ which is the limit of a sequence of matrices for finite interval problems, and which has the properties specified in the statement of Theorem* 1.

This lemma is proved by the method of [5, Chapter 10, Theorem 2.1], with modifications to fit the present case. We remark that the proof is valid under somewhat less restrictive conditions than those which have been imposed; the requirement that $\epsilon_m \geqq \epsilon > 0$ may be replaced by $\epsilon_m > 0$, and the condition that $|\sigma(t)|$ be bounded on $(-\infty, \infty)$ is not used.

From its definition $\rho_m(\lambda)$ is Hermitian, and it is easily shown that $\rho_m(\lambda_2) - \rho_m(\lambda_1)$ is positive semidefinite if $\lambda_2 > \lambda_1$. On any fixed finite interval $[-\mu, \mu]$, $\rho_{m_{jk}}(\lambda)$ is of bounded variation, and it will be shown that the bound is uniform, that is, independent of m. From (23),

$$(24)\quad |\rho_{m_{jk}}(\lambda_2) - \rho_{m_{jk}}(\lambda_1)| \leqq \tfrac{1}{2}\, |\rho_{m_{jj}}(\lambda_2) - \rho_{m_{jj}}(\lambda_1) + \rho_{m_{kk}}(\lambda_2) - \rho_{m_{kk}}(\lambda_1)|.$$

Thus it suffices to prove the uniform boundedness for the diagonal elements of $\rho_m(\lambda)$. Let m be sufficiently large so that c, at which $S_j^{(k-1)}(c, \lambda) = \delta_{jk}$, satisfies $a_m < c < b_m$, and consider the contribution of the eigenvalue λ_i of P_m to (21) with $-\mu < \lambda_i \leqq \mu$; it is

$$(25)\qquad \sum_{j,k=1}^{n} \overline{\Phi_j(\lambda_i)}\Phi_k(\lambda_i)\, \Delta\rho_{m_{jk}}(\lambda_i) = A(f, \lambda_i) \geqq 0,$$

where $\Delta\rho_{m_{jk}}(\lambda_i) = \rho_{m_{jk}}(\lambda_i) - \rho_{m_{jk}}(\lambda_i - 0)$. Recalling the properties specified for $\sigma(t)$ in the neighborhood $N_0 = [c - \eta, c + \eta]$, we choose h, $0 < h < \eta$, such that if $|\lambda| \leqq \mu$, $c \leqq t \leqq c + h$, then

$$(26)\qquad |S_j^{(k-1)}(t, \lambda) - \delta_{jk}| < 1/16n^2, \qquad j, k = 1, 2, \cdots, n.$$

Let $g(t)$ be a nonnegative function of class $C^{(2n)}$ on $(-\infty, \infty)$ such that $g(t) = 0$ for $t \notin (c, c + h)$, and,

$$(27) \qquad \int_c^{c+h} g(t)dt = 1.$$

For $\nu = 1, 2, \cdots, n$, let functions $f_\nu(t)$ be defined as follows:

$$(28) \qquad \begin{aligned} f_\nu(t) &= (-1)^{\nu-1} g^{(\nu-1)}(t) \cdot [\sigma(t)]^{-1} \quad \text{for } t \in (c, c+h), \\ &= 0 \quad \text{for } t \in (-\infty, \infty), t \notin (c, c+h). \end{aligned}$$

Then $f(t) \in D_{L_m}$, and with $f(t) = f_\nu(t)$ in (21), an estimate will be obtained for the corresponding expression (25). From (22),

$$(29) \qquad \begin{aligned} \Phi_j(\lambda_i) &= \int_{a_m}^{b_m} f_\nu(t)\sigma(t)\overline{S_j(t, \lambda_i)}dt \\ &= (-1)^{(\nu-1)} \int_c^{c+h} g^{(\nu-1)}(t)\overline{S_j(t, \lambda_i)}dt = \int_c^{c+h} g(t)\overline{S_j^{(\nu-1)}(t, \lambda_i)}dt. \end{aligned}$$

For $j = \nu$, $S_\nu^{(\nu-1)}(t, \lambda_i) = 1 + \mu_\nu(t)$, $c \leqq t \leqq c + h$, while for $j \neq \nu$, $S_j^{(\nu-1)}(t, \lambda_i) = \mu_j(t)$, $c \leqq t \leqq c + h$, where $|\mu_j(t)| < 1/16n^2$, $j = 1, 2, \cdots, n$. Thus,

$$(30) \qquad \Phi_\nu(\lambda_i) = 1 + W_\nu, \quad \Phi_j(\lambda_i) = W_j \qquad \text{for } j \neq \nu,$$

where $|W_j| < 1/16n^2$ for $j = 1, 2, \cdots, n$. From (25),

$$A(f_\nu, \lambda_i) \geqq \sum_{j=1}^n |\Phi_j(\lambda_i)|^2 \Delta\rho_{m_{jj}}(\lambda_i) - \left| \sum_{j \neq k}^n \overline{\Phi_j(\lambda_i)}\Phi_k(\lambda_i) \Delta\rho_{m_{jk}}(\lambda_i) \right|,$$

hence,

$$(31) \qquad \begin{aligned} A(f_\nu, \lambda_i) \geqq& \sum_{j=1}^n |\Phi_j(\lambda_i)|^2 \Delta\rho_{m_{jj}}(\lambda_i) - \tfrac{1}{2} \sum_{j \neq k}^n |\Phi_j(\lambda_i)| \cdot |\Phi_k(\lambda_i)| \\ &\cdot [\Delta\rho_{m_{jj}}(\lambda_i) + \Delta\rho_{m_{kk}}(\lambda_i)] \\ \geqq& |\Phi_\nu(\lambda_i)|^2 \Delta\rho_{m_{\nu\nu}}(\lambda_i) - \tfrac{1}{2} \sum_{j \neq \nu}^n |\Phi_j(\lambda_i)| \cdot |\Phi_\nu(\lambda_i)| \\ &\cdot [\Delta\rho_{m_{jj}}(\lambda_i) + \Delta\rho_{m_{\nu\nu}}(\lambda_i)] \\ &- \tfrac{1}{2} \sum_{k \neq \nu}^n |\Phi_\nu(\lambda_i)| \cdot |\Phi_k(\lambda_i)| [\Delta\rho_{m_{\nu\nu}}(\lambda_i) + \Delta\rho_{m_{kk}}(\lambda_i)] \\ &- \tfrac{1}{2} \sum_{j \neq k; j \neq \nu, k \neq \nu}^n |\Phi_j(\lambda_i)| \cdot |\Phi_k(\lambda_i)| \\ &\cdot [\Delta\rho_{m_{jj}}(\lambda_i) + \Delta\rho_{m_{kk}}(\lambda_i)]. \end{aligned}$$

From (30) and (31) follows

$$(32)\qquad A(f_\nu, \lambda_i) \geqq \frac{3}{4}\Delta\rho_{m_{\nu\nu}}(\lambda_i) - \frac{1}{8n^2}\sum_{j=1}^{n} \Delta\rho_{m_{jj}}(\lambda_i).$$

Thus, since $n \geqq 2$,

$$(33)\qquad \sum_{\nu=1}^{n} A(f_\nu, \lambda_i) > \tfrac{1}{2}\sum_{j=1}^{n} \Delta\rho_{m_{jj}}(\lambda_i),$$

and from this relation together with (21) and (25),

$$(34)\qquad \sum_{\nu=1}^{n} [f_\nu, f_\nu]_m > \tfrac{1}{2}\sum_{j=1}^{n} [\rho_{m_{jj}}(\mu) - \rho_{m_{jj}}(-\mu)].$$

The left member of (34) is independent of m, which establishes that the $\rho_{m_{jj}}(\lambda)$ are of uniformly dominated total variation on $[-\mu, \mu]$.

By the Helly selection theorem there exists a subsequence of $\{\delta_m\}$ such that the corresponding matrices $\rho_m(\lambda)$ converge to a limit matrix for $-\mu \leqq \lambda \leqq \mu$; a subsequence of the first then leads to convergence on $-\mu_1 \leqq \lambda \leqq \mu_1$, where $\mu < \mu_1$. Continuation of this process for a sequence of λ-intervals tending to $(-\infty, \infty)$, together with the diagonal process, shows that there exists a subsequence $\{\delta_{m_k}\}$ for which the $\rho_{m_k}(\lambda)$ converge to a matrix $\rho(\lambda)$ for $-\infty < \lambda < \infty$. To simplify notation we omit the extra subscript k, so that $\{\delta_m\}$ will henceforth denote a subsequence of the sequence of intervals originally chosen, for which the associated matrices converge as described. The matrices $\rho_m(\lambda)$ are Hermitian, and if $\lambda_2 > \lambda_1$ then $\rho_m(\lambda_2) - \rho_m(\lambda_1)$ is positive semidefinite; $\rho_m(\lambda) \to \rho(\lambda)$ implies that $\rho(\lambda)$ also has these properties. Each element $\rho_{jk}(\lambda)$ is the limit of a sequence of functions which are of uniformly bounded variation on any fixed finite λ-interval, from which it follows easily that $\rho_{jk}(\lambda)$ is also of bounded variation on any finite interval. Thus, Lemma 5 is proved.

We proceed to the special case of the Parseval relation expressed by

LEMMA 6. *If $f(t) \in D_0{}^{(n)}$ and $h(\lambda)$ has components given by* (49), *with $g = Tf$, then $h \in H_\rho$ and $|||g||| = \|h\|$.*

PROOF. $F(t) \in D_{L_m}$ for large m, and the completeness relation (21) yields

$$(35)\qquad [f, f]_m = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{\Phi_j(\lambda)}\Phi_k(\lambda) d\rho_{m_{jk}}(\lambda)$$

where

$$\Phi_j(\lambda) = \int_{-\infty}^{\infty} f(t)\sigma(t)\overline{S_j(t, \lambda)}dt. \tag{36}$$

Let $g_m(t)$ be defined on $(-\infty, \infty)$ by:

$$\begin{aligned} g_m(t) &= L_m{}^{-1}\sigma(t)f(t) = T_m f && \text{for } t \in [a_m, b_m] = \delta_m, \\ &= 0 && \text{for } t \notin [a_m, b_m]. \end{aligned} \tag{37}$$

The restriction $\tilde{g}_m(t)$ of $g_m(t)$ to δ_m belongs to $D_{L_m} \subset S_m$; the completeness relation applied to $\tilde{g}_m$ yields

$$\begin{aligned} [\tilde{g}_m, \tilde{g}_m]_m &= \sum_{k=1}^{\infty} |[\tilde{g}_m, y_k]_m|^2 = \sum_{k=1}^{\infty} (\sigma f, y_k)\overline{(\sigma f, y_k)} \\ &= \left[\int_{-\infty}^{-\epsilon_1} + \int_{\epsilon_1}^{\infty}\right]\frac{1}{\lambda^2}\sum_{j,k=1}^{n} \overline{\Phi_j(\lambda)}\Phi_k(\lambda)d\rho_{m_{jk}}(\lambda). \end{aligned} \tag{38}$$

Here $\epsilon_1 < \epsilon/B$ is a fixed positive number; from (16), no eigenvalue of P_m, hence no point of increase of $\rho_m(\lambda)$, is distant less than ϵ/B from the origin. With $\|g_m\|$ denoting the norm of g_m in $L^2(-\infty, \infty)$, $\|g_m\| = \|\tilde{g}_m\|_m$, and from (19),

$$\begin{aligned} \|\tilde{g}_m\|_m^2 &\leqq \frac{1}{\epsilon}[\tilde{g}_m, \tilde{g}_m]_m = \frac{1}{\epsilon}(\sigma f, \tilde{g}_m)_m \\ &= \frac{1}{\epsilon}(\sigma f, g_m) \leqq \frac{1}{\epsilon}\|\sigma f\| \cdot \|g_m\|. \end{aligned} \tag{39}$$

The condition $|\sigma(t)| < B$, with (39), implies

$$\|g_m\| \leqq (B/\epsilon)\|f\|, \tag{40}$$

i.e., the sequence of functions $\{g_m\}$ is bounded in norm in the space $L^2(-\infty, \infty)$. Therefore, there exists at least one subsequence of $\{g_m\}$ which converges weakly to a function $g \in L^2(-\infty, \infty)$; thus, as $m \to \infty$ through values for which there is weak convergence to g, (38) yields

$$\begin{aligned} \lim_{m\to\infty} [\tilde{g}_m, \tilde{g}_m]_m &= \lim_{m\to\infty} (\sigma f, g_m) = (\sigma f, g) \\ &= \lim_{m\to\infty} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon_1}^{\infty}\right] \cdot \frac{1}{\lambda^2}\sum_{j,k=1}^{n} \overline{\Phi_j(\lambda)}\Phi_k(\lambda)d\rho_{m_{jk}}(\lambda). \end{aligned} \tag{41}$$

Let $x(t)$ be any function belonging to $D_0{}^{(n)} = D_{L_\infty{}^0}$. Then

$(L_\infty x, g_m) \to (L_\infty x, g)$ as $m \to \infty$; for large m, x satisfies the boundary conditions for L_m on the finite interval δ_m and vanishes outside δ_m, hence $(L_\infty x, g_m) = (L_m x, g_m)_m = (x, \sigma f)$, and $(L_\infty x, g) = (x, \sigma f)$. This relation extends to all $x \in D_{L_\infty}$: using the fact that L_∞ is the closure of the operator $L_\infty{}^0$, let $\{x_n\}$, $x_n \in D_0{}^{(n)}$, be a sequence such that $x_n \to x$, $L_\infty{}^0 x_n \to L_\infty x$. Then the result follows from the continuity of the inner product in $L^2(-\infty, \infty)$ with respect to mean convergence. Thus, g belongs to the domain of the adjoint operator $L_\infty{}^*$, and $L_\infty{}^* g = \sigma f$. But L_∞ is selfadjoint, hence $L_\infty g = \sigma f$, or $g = L_\infty{}^{-1}\sigma f = Tf$. The norm of g in the new space S satisfies

(42) $$|||g|||^2 = [g, g] = (\sigma f, g).$$

Thus g is determined uniquely by f, independently of the choice of a weakly convergent subsequence of $\{g_m\}$. From (41) and (42),

(43) $$|||g|||^2 = \lim_{m\to\infty} \left[\int_{-\infty}^{-\epsilon_1} + \int_{\epsilon_1}^{\infty}\right] \sum_{j,k=1}^{n} \frac{\overline{\Phi_j(\lambda)}\Phi_k(\lambda)}{\lambda^2} d\rho_{m_{jk}}(\lambda).$$

For large positive μ,

(44) $$\begin{aligned}&\left[\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty}\right] \sum_{j,k=1}^{n} \frac{\overline{\Phi_j(\lambda)}\Phi_k(\lambda)}{\lambda^2} d\rho_{m_{jk}}(\lambda)\\ &\leqq \frac{1}{\mu^2}\left[\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty}\right] \sum_{j,k=1}^{n} \overline{\Phi_j(\lambda)}\Phi_k(\lambda) d\rho_{m_{jk}}(\lambda);\end{aligned}$$

the right member of (44) exists because of (35), and

(45) $$\begin{aligned}&\left[\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty}\right] \sum_{j,k=1}^{n} \frac{\overline{\Phi_j(\lambda)}\Phi_k(\lambda)}{\lambda^2} d\rho_{m_{jk}}(\lambda)\\ &\leqq (1/\mu^2)[f, f]_m = (1/\mu^2)(L_\infty f, f).\end{aligned}$$

Hence,

(46) $$\begin{aligned}&\left| \, |||g|||^2 - \left[\int_{-\mu}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu}\right] \sum_{j,k=1}^{n} \frac{\overline{\Phi_j(\lambda)}\Phi_k(\lambda)}{\lambda^2} d\rho_{m_{jk}}(\lambda) \right|\\ &\leqq \left| \, |||g|||^2 - \left[\int_{-\infty}^{-\epsilon_1} + \int_{\epsilon_1}^{\infty}\right] \left\{ \sum_{j,k=1}^{n} \frac{\overline{\Phi_j(\lambda)}\Phi_k(\lambda)}{\lambda^2} d\rho_{m_{jk}}(\lambda) \right\} \right|\\ &\quad + (1/\mu^2)(L_\infty f, f).\end{aligned}$$

By a well-known integration theorem, the limit of the left member of (46) exists and is equal to the expression obtained by replacing ρ_m by the limit matrix ρ; from (43) and (46),

$$(47)\quad \left| \, |||g|||^2 - \left[\int_{-\mu}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu}\right]\left\{\sum_{j,k=1}^{n} \frac{\overline{\Phi_j(\lambda)}\Phi_k(\lambda)}{\lambda^2} d\rho_{jk}(\lambda)\right\} \right| \leqq (1/\mu^2)(L_\infty f, f).$$

Let $h(\lambda)$ be the vector function of λ, of n components, whose jth component is $(\Phi_j(\lambda))/\lambda$ for $|\lambda| \geqq \epsilon_1$, zero for $|\lambda| < \epsilon_1$. Then, as $\mu \to \infty$, (47) implies that h belongs to H_ρ, and

$$(48)\quad |||g|||^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \overline{h_j(\lambda)} h_k(\lambda) d\rho_{jk}(\lambda) = \|h\|_\rho^2;$$

this completes the proof of Lemma 6.

LEMMA 7. *The range of T is dense in S, and there is an isometric V defined on S and taking S into H_ρ, such that if f, g and h are as in Lemma 6, then $h = Vg$.*

PROOF. The symmetry and boundedness of T in S, and the existence of T^{-1}, are established in the same manner as for the operator T_δ considered earlier. The linear manifold $D_0^{(n)}$ is dense in S, for if $s \in$ S satisfies $[s, x] = 0$ for all $x \in D_0^{(n)}$, then $(s, L_\infty x) = 0$; for $f \in D_{L_\infty}$ choose $\{x_n\}$, $x_n \in D_0^{(n)}$, such that $x_n \to f$, $L_\infty x_n \to L_\infty f$ in $L^2(-\infty, \infty)$. Then $(s, L_\infty f) = 0$ for all $f \in D_{L_\infty}$; the range of $L_\infty = L_\infty^*$ is all of $L^2(-\infty, \infty)$, hence s is zero in $L^2(-\infty, \infty)$, also in S. It now follows from the foregoing properties of T that the set $\{Tf\}$, $f \in D_0^{(n)}$, is dense in the space S. Thus, the correspondence $g \to h$ of Lemma 6 is a linear mapping V_0 of the set $\{g\} = \{Tf\}$, $f \in D_0^{(n)}$, into H_ρ, and V_0 extends uniquely to an isometric mapping V defined on S, taking S into H_ρ. For $g = Tf$, $f \in D_0^{(n)}$, it follows from (35) and the definitions of h that

$$(49)\quad \begin{aligned} h_j(\lambda) &= \frac{1}{\lambda}\int_{-\infty}^{\infty} f(t)\sigma(t)\overline{S_j(t, \lambda)}dt \\ &= \frac{1}{\lambda}\int_{\infty}^{\infty} [L_\infty g(t)]\, \overline{S_j(t, \lambda)}dt \quad \text{for } |\lambda| \geqq \epsilon_1, \\ &= 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \text{for } |\lambda| < \epsilon_1, \end{aligned}$$

$j = 1, 2, \cdots, n$. These integrals exist for all λ because f vanishes outside a finite interval. It remains to extend this representation for $h = Vg$ to the larger set $\{g\} = \{Ts\}$, $s \in S$.

LEMMA 8. *If* $f_s \in S$, $g_s = Tf_s$, *and* $u_{s,(a,b)}$ *has components given by* (5), *then* $u_{s,(a,b)} \in H_\rho$, *and as* $(a, b) \to (-\infty, \infty)$, $u_{s,(a,b)} \to h_s = Vg_s$ *in* H_ρ.

PROOF. Let $D^{(n)}_{(a,b)}$ denote the set of functions in $D_0{}^{(n)}$ which vanish for $t \notin (a, b)$, where $[a, b]$ is a finite interval, and let $u = u(\lambda)$ be an element of H_ρ. For $f \in D^{(n)}_{(a,b)}$, $g = Tf$, $h = Vg$,

$$\begin{aligned}(h, u)_\rho &= \lim_{(\mu_1, \mu_2)\to(-\infty,\infty)} \int_{\mu_1}^{\mu_2} \sum_{j,k=1}^{n} \overline{u_j(\lambda)} h_k(\lambda) d\rho_{jk}(\lambda) \\ &= \lim_{(\mu_1, \mu_2)\to(-\infty,\infty)} \left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right] \sum_{j,k=1}^{n} \frac{1}{\lambda} \overline{u_j(\lambda)} \\ &\quad \cdot \int_a^b \sigma(t) f(t) \overline{S_k(t, \lambda)} dt d\rho_{jk}(\lambda) \\ &= \lim_{(\mu_1, \mu_2)\to(-\infty,\infty)} \int_a^b \left\{ \left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right] \right. \\ &\quad \left. \cdot \sum_{j,k=1}^{n} \lambda^{-1} \overline{u_j(\lambda)}\ \overline{S_k(t, \lambda)} d\rho_{jk}(\lambda) \right\} \sigma(t) f(t) dt\end{aligned} \tag{50}$$

where $\mu_1 \to -\infty$ through a sequence of values for which the matrix $\rho(\lambda)$ is continuous, similarly for $\mu_2 \to \infty$. The interchange in the order of integration is valid for continuous u; (50) is established for an arbitrary u through the employment of a sequence $\{u_n\}$ of continuous vector functions such that $u_n \to u$ in H_ρ.

With μ_1, μ_2 held fixed, the expression

$$\begin{aligned}F_{(\mu_1, \mu_2)}(\sigma f) = \int_a^b & \left\{ \left[\int_{\mu_1'}^{-\epsilon} + \int_{\epsilon_1}^{\mu_2}\right] \right. \\ & \left. \cdot \sum_{j,k=1}^{n} \frac{1}{\lambda} \overline{u_j(\lambda)}\ \overline{S_k(t, \lambda)} d\rho_{jk}(\lambda) \right\} \sigma(t) f(t) dt\end{aligned} \tag{51}$$

is a linear function on the set $\{\sigma \cdot f\}$ of functions in $L^2(a, b)$, $f \in D^{(n)}_{(a,b)}$. The set $D^{(n)}_{(a,b)}$ is dense in $L^2(a, b)$, as is its image under multiplication by σ. (Multiplication by $\sigma \neq 0$ a.e. yields a bounded selfadjoint operator in $L^2(a, b)$ which has an inverse.) By the Schwarz inequality,

$$|F_{(\mu_1, \mu_2)}(\sigma f)| \leqq \|w_{(\mu_1, \mu_2)}\|_{[a,b]} \cdot \|\sigma f\|_{[a,b]}, \tag{52}$$

where $w_{(\mu_1,\mu_2)}$ is the function, defined and continuous for $-\infty < t < \infty$, given by

$$(53)\qquad w_{(\mu_1,\mu_2)}(t) = \left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right]\sum_{j,k=1}^{n}\frac{1}{\lambda}\overline{u_j(\lambda)}\,\overline{S_k(t,\lambda)}d\rho_{jk}(\lambda).$$

Thus $F_{(\mu_1,\mu_2)}(\sigma f)$ is a bounded functional on the dense set $\{\sigma f\}$, which extends uniquely to all of $L^2(a,b)$ with preservation of norm. From the Riesz representation theorem for bounded functionals, and the density of $\{\sigma f\}$, it follows that the extended functional is given by

$$(54)\qquad F_{(\mu_1,\mu_2)}(x) = (x, w_{(\mu_1,\mu_2)}),\qquad x \in L^2(a,b)$$

and the norm of $F_{(\mu_1,\mu_2)}$ is $\|w_{(\mu_1,\mu_2)}\|_{[a,b]}$. From (51)

$$(55)\qquad F_{(\mu_1,\mu_2)}(\sigma f) = \left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right] \cdot\left\{\sum_{j,k=1}^{n}\frac{1}{\lambda}\overline{u_j(\lambda)}\int_a^b \sigma(t)f(t)S_k(t,\lambda)dt\right\}d\rho_{jk}(\lambda).$$

Applying the Schwarz inequality on the finite λ-interval $[\mu_1, \mu_2]$ yields

$$|F_{(\mu_1,\mu_2)}(\sigma f)|^2 \leqq \left\{\left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right]\sum_{j,k=1}^{n}\overline{u_j(\lambda)}\,u_k(\lambda)d\rho_{jk}(\lambda)\right\}$$

$$\cdot\left\{\left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right]\sum_{j,k=1}^{n}\overline{h_j(\lambda)}h_k(\lambda)d\rho_{jk}(\lambda)\right\},$$

hence

$$(56)\qquad |F_{(\mu_1,\mu_2)}(\sigma f)| \leqq \|u\|_\rho \cdot \|h\|_\rho.$$

From (42)

$$|||g|||^2 \leqq \|\sigma f\| \cdot \|g\| \leqq \|\sigma f\| \cdot (1/\sqrt{\epsilon})\,|||g|||,$$

or since $|\sigma| < B$,

$$(57)\qquad |||g||| \leqq (1/\sqrt{\epsilon})\|\sigma f\| \leqq (B/\sqrt{\epsilon})\|f\|.$$

From $|||g||| = \|h\|_\rho$, together with (56) and (57),

$$(58)\qquad |F_{(\mu_1,\mu_2)}(\sigma f)| \leqq (\|u\|/\sqrt{\epsilon}) \cdot \|\sigma f\|.$$

Thus the norm of $F_{(\mu_1,\mu_2)}$ on $\{\sigma f\}$, and of its extension to $L^2(a,b)$ is less than or equal to $\|u\|_\rho/\sqrt{\epsilon}$, i.e.,

$$(59)\qquad \|w_{(\mu_1,\mu_2)}(t)\|_{[a,b]} \leqq \|u\|_\rho/\sqrt{\epsilon}.$$

The right member of (59) is independent of a, b, and of μ_1, μ_2; hence for each pair (μ_1, μ_2) of the sequence, $w_{(\mu_1, \mu_2)}(t) \in L^2(-\infty, \infty)$ and $\|w_{(\mu_1, \mu_2)}\| \leqq \|u\|_\rho/\sqrt{\epsilon}$. It follows that there exists a subsequence of the sequence of intervals (μ_1, μ_2) originally chosen for which $w_{(\mu_1, \mu_2)}(t)$ converges weakly to a function $w(t) \in L^2(-\infty, \infty)$, and $\|w(t)\| \leqq \|u\|_\rho/\sqrt{\epsilon}$. In (50) let $(\mu_1, \mu_2) \to (-\infty, \infty)$ through this subsequence to obtain

$$(h, u)_\rho = \int_a^b w(t)\sigma(t)f(t)dt. \tag{60}$$

the interval $[a, b]$ was arbitrarily chosen, hence (60) may be written

$$(h, u)_\rho = \int_{-\infty}^{\infty} w(t)\sigma(t)f(t)dt, \tag{61}$$

in which form it is valid for all $f \in D_0^{(n)}$. It is easily seen that $w(t)$ is uniquely determined; that is, it is independent of the particular weakly convergent subsequence of $\{w_{(\mu_1, \mu_2)}(t)\}$ which is chosen.

Now let $g_s = Tf_s$, for an arbitrary $f_s \in S$, and put $h_s = Vg_s$. Let $\{f_m\}$, $f_m \in D_0^{(n)}$, be a sequence such that $|||f_s - f_m||| \to 0$ as $m \to \infty$. Then $|||g_s - g_m||| \to 0$, where $g_m = Tf_m$, also $h_m = Vg_m \to h_s$ in H_ρ. From (61),

$$(h_m, u)_\rho = \int_{-\infty}^{\infty} w(t)\sigma(t)f_m(t)dt. \tag{62}$$

The inequality

$$[g, g] \geqq \epsilon(g, g), \qquad g \in S, \tag{63}$$

can be established in the same manner as (19), and it implies that the sequences $\{f_m\}$, $\{\sigma f_m\}$ converge in $L^2(-\infty, \infty)$ to f_s, σf_s, respectively; hence

$$(h_s, u)_\rho = \int_{-\infty}^{\infty} w(t)\sigma(t)f_s(t)dt, \tag{64}$$

$$(h_s, u)_\rho = \lim_{(\mu_1, \mu_2) \to (-\infty, \infty)} \int_{-\infty}^{\infty} w_{(\mu_1, \mu_2)}(t)\sigma(t)f_s(t)dt \tag{65}$$

where $(\mu_1, \mu_2) \to (-\infty, \infty)$ through a sequence of intervals for which $w_{(\mu_1, \mu_2)}$ converges weakly to w. With $a < b$, the expression

$$\int_a^b w_{(\mu_1, \mu_2)}(t)\sigma(t)f_s(t)dt \tag{66}$$

may be rewritten, using (53), as

$$(67)\qquad \left| \int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2} \right| \left\{ \sum_{j,k=1} \frac{1}{\lambda} \overline{u_j(\lambda)} \int_a^b \sigma(t) f_s(t) \overline{S_k(t,\lambda)} dt \right\} d\rho_{jk}(\lambda).$$

It will be shown below that the vector function $u_{s,(a,b)}$ with components given by

$$(68)\qquad \begin{aligned} U_{s,(a,b)k}(\lambda) &= \frac{1}{\lambda} \int_a^b \sigma(t) f_s(t) S_k(t,\lambda) dt \qquad && \text{if } |\lambda| \geqq \epsilon_1, \\ &= 0 && \text{if } |\lambda| < \epsilon_1, \end{aligned}$$

belongs to H_ρ, and that as $(a, b) \to (-\infty, \infty)$, $u_{s,(a,b)}$ tends in H_ρ to a limit vector u_s; thus,

$$(69)\qquad \begin{aligned} &\int_{-\infty}^{\infty} w_{(\mu_1,\mu_2)}(t) \sigma(t) f_s(t) dt \\ &= \left| \int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2} \right| \sum_{j,k=1}^{n} \overline{u_j(\lambda)} u_{s,k}(\lambda) d\rho_{jk}(\lambda). \end{aligned}$$

Using (65), it follows that for all $u \in H_\rho$,

$$(70)\qquad (h_s, u)_\rho = (u_s, u)_\rho,$$

hence $h_s = u_s$. Thus if $g \in \{Tf_s\}$, $f_s \in S$, then its image $h_s = Vg_s$ has an integral representation in the sense that (68) converges to it in H_ρ as $(a, b) \to (-\infty, \infty)$.

To prove the assertion accompanying (68), let a, b be fixed, and choose a sequence $\{f_m\}$ of functions belonging to $D_{(a,b)}^{(n)}$ and tending to $f_s(t)$ on $[a, b]$ in the space $L^2(a, b)$. Let $[\mu_1, \mu_2]$ be a finite λ-interval belonging to the sequence of intervals previously chosen. The sequence of vectors $h_m = VTf_m = Vg_m$ have kth components given on $(-\infty, \infty)$ by

$$(71)\qquad \begin{aligned} h_{m,k}(\lambda) &= \frac{1}{\lambda} \int_a^b \sigma(t) f_m(t) \overline{S_k(t,\lambda)} dt, \qquad && |\lambda| \geqq \epsilon_1, \\ &= 0, && |\lambda| < \epsilon_1. \end{aligned}$$

Comparing (68) and (71), it follows from the L^2 convergence of the f_m to f_s that $h_{m,k}(\lambda)$ tends to the kth component of $u_{s,(a,b)}$ uniformly in λ on $[\mu_1, \mu_2]$. Thus, as $m \to \infty$,

$$(72)\qquad \begin{aligned} &\int_{\mu_1}^{\mu_2} \sum_{j,k=1}^{n} \overline{h_{m,j}(\lambda)} h_{m,k}(\lambda) d\rho_{jk}(\lambda) \\ &\to \int_{\mu_1}^{\mu_2} \sum_{j,k=1}^{n} \overline{u_{s,(a,b)j}(\lambda)}\, u_{s,(a,b)k}(\lambda) d\rho_{jk}(\lambda). \end{aligned}$$

For each m, the left member of (72) is less than or equal to $(h_m, h_m)_\rho = |||g|||^2$, hence from (57),

$$(73) \qquad \int_{\mu_1}^{\mu_2} \sum_{j,k=1}^{n} \overline{h_{m,j}(\lambda)} h_{m,k}(\lambda) d\rho_{jk}(\lambda) \leqq \frac{B^2}{\epsilon} \|f_m\|^2.$$

But $\|f_m\| = \|f_m\|_{(a,b)} \to \|f\|_{(a,b)}$ in $L^2(a, b)$, hence (72) and (73) imply

$$(74) \qquad \begin{aligned} \int_{\mu_1}^{\mu_2} \sum_{j,k=1}^{n} \overline{u_{s,(a,b)j}(\lambda)} u_{s,(a,b)k}(\lambda) d\rho_{jk}(\lambda) &\leqq \frac{B^2}{\epsilon} \|f\|^2_{(a,b)} \\ &\leqq (B^2/\epsilon)\|f\|^2. \end{aligned}$$

This bound is independent of μ_1, μ_2, and of a, b, i.e., each of the vector functions $u_{s,(a,b)}$ belongs to H and has norm dominated by $(B/\sqrt{\epsilon})\|f\|$. Let $u_{s\Delta}$ denote the difference of the two vector functions $u_{s,(a_1,b_1)}$, $u_{s,(a_2,b_2)}$ corresponding to intervals (a_1, b_1), (a_2, b_2) with $a_2 < a_1$, $b_1 < b_2$. According to (68), the kth component of u_s is

$$(75) \qquad \begin{array}{ll} \dfrac{1}{\lambda} \displaystyle\int_\Delta \sigma(t) f_s(t) \overline{S_k(t, \lambda)} dt & \text{if } |\lambda| \geqq \epsilon_1, \\ 0 & \text{if } |\lambda| < \epsilon_1, \end{array}$$

where $\Delta = (a_2, b_2) - (a_1, b_1) = \Delta_1 \cup \Delta_2$, $\Delta_1 = (a_2, a_1]$, $\Delta_2 = [b_1, b_2)$. Thus (75) may be written

$$(76) \qquad \begin{array}{ll} \dfrac{1}{\lambda} \displaystyle\int_{\Delta_1} \sigma(t) f_s(t) \overline{S_k(t, \lambda)} dt + \dfrac{1}{\lambda} \displaystyle\int_{\Delta_2} \sigma(t) f_s(t) \overline{S_k(t, \lambda)} dt, & |\lambda| \geqq \epsilon_1, \\ 0, & |\lambda| < \epsilon_1. \end{array}$$

The same arguemnt which let to (74) and the conclusions based upon it may now be applied to each of the two integrals in (76), with the result that $u_{s\Delta} \in H_\rho$ satisfies

$$\|u_{s\Delta}\|_\rho \leqq (B/\sqrt{\epsilon})\|f\|_\Delta.$$

But $\|f\|_\Delta \to 0$ as $(a_1, b_1) \to (-\infty, \infty)$ because $f \in L^2(-\infty, \infty)$. Hence, $u_{s,(a,b)} \to h_s$ in H_ρ, and the proof of Theorem 1 is complete.

6. **Proof of Theorem 2.** Returning to relation (61) and the associated notation, let $s(t)$ be any function in S, and in (61) let $u = Vs$:

$$(77) \qquad (h, u)_\rho = (Vg, Vs)_\rho = [g, s] = (\sigma f, s) = \int_{-\infty}^{\infty} w(t)\sigma(t)f(t)dt;$$

this relation holds for all $f \in D_0^{(n)}$. It follows that $\overline{w(t)} = s(t)$, i.e., from (86), $s(t)$ is the weak limit in $L^2(-\infty, \infty)$ of a subsequence of

$$(78) \qquad \left[\int_{\mu_1}^{-\epsilon_1} + \int_{\epsilon_1}^{\mu_2}\right]\left\{\sum_{j,k=1}^{n} \frac{1}{\lambda} u_k(\lambda) S_j(t,\lambda)\right\} d\rho_{jk}(\lambda),$$

$$(\mu_1, \mu_2) \to (-\infty, \infty)$$

where $\{(\mu_1, \mu_2)\}$ is the sequence of intervals employed in (50). For two intervals (μ_1, μ_2), (μ_1', μ_2') belonging to this sequence with $\mu_1' < \mu_1$, $\mu_2 < \mu_2'$, let $\Delta = [\mu_1', \mu_1] \cup [\mu_2, \mu_2']$ and consider the difference of the corresponding functionals (51); it is

$$(79) \qquad F_\Delta(\sigma f) = \int_a^b \left\{\int_\Delta \sum_{j,k=1}^{n} \frac{1}{\lambda} \overline{u_j(\lambda)}\ \overline{S_k(t,\lambda)} d\rho_{jk}(\lambda)\right\}\sigma(t) f(t) dt.$$

By the Schwarz inequality,

$$(80) \qquad |F_\Delta(\sigma f)| \leqq \|w_\Delta\|_{(a,b)} \|\sigma f\|_{(a,b)}$$

where

$$(81) \qquad w_\Delta(t) = w_{(\mu_1', \mu_2')}(t) - w_{(\mu_1, \mu_2)}(t).$$

By essentially the same procedure as was used to obtain (59), it follows that $\|w_\Delta(t)\|_{(a,b)} \leqq \|u\|_\Delta/\sqrt{\epsilon}$, hence

$$(82) \qquad \|w_\Delta(t)\| \leqq \|u\|_\Delta/\sqrt{\epsilon}.$$

The vector u belongs to H_ρ, which implies that $\|u\|_\Delta \to 0$ as (μ_1, μ_2) (hence also (μ_2', μ_2')) tends to $(-\infty, \infty)$, i.e., $\{w_{(\mu_1, \mu_2)}(t)\}$ is a Cauchy sequence converging in the norm of $L^2(-\infty, \infty)$ to a limit function. Mean convergence implies weak convergence, hence the limit function is $\overline{s(t)}$, or, (78) converges in the mean to s, and Theorem 2 is proved.

We conclude with the remark that the limit matrix ρ is independent of the choice of the sequence of finite intervals $[a_m, b_m]$ for which $\rho_m \to \rho$, in the following sense: any ρ' which is obtained from a sequence $[a_m', b_m']$ coincides with ρ except possibly on the set, at most countable, where ρ is discontinuous. Proof is given in [8].

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University of Colorado, Colorado Springs, Colorado 80907