ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 2, Spring 1971

# A NECESSARY AND SUFFICIENT CONDITION FOR THE OSCILLATION OF SOME LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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1. **Introduction.** The ordinary differential equation

$$y'' + p(t)y = 0 \tag{1.1}$$

is called oscillatory if solutions have an infinite number of zeros in $[a, \infty)$ and disconjugate if no solution has more than one zero in $[a, \infty)$. The problem of oscillation for (1.1) has a long history, which can be surveyed by referring to [2]. In [1] we proved that if the integral of $p$ is not "too negative"—a notion which we make precise in §3—then (1.1) is disconjugate, if and only if, the integral inequality

$$\nu(t) \geqq P(t) + \int_t^\infty \nu^2(s)ds \tag{1.2}$$

has a solution $\nu \in C(a, \infty)$. Here, $P$ is an averaged integral of $p$, which means

$$P(t) = \int_t^\infty p(s)ds \tag{1.3}$$

if the integral in (1.3) exists as an improper integral. We also showed in [1] that (1.1) is disconjugate, if and only if, an integral inequality of the form

$$\nu(t) \geqq Q(t) + \int_t^\infty E(t, s)\nu^2(s)ds \tag{1.4}$$

has a solution $\nu \in C(a, \infty)$. Here, $Q$ and $E$ are nonnegative functions depending upon $P$.

In §2 of this note we give a necessary and sufficient condition in terms of nonnegative $P$ and $Q$ for the existence of a solution to (1.4). In §3 we apply this result to the oscillation problem for (1.1) to obtain a necessary and sufficient condition for the disconjugacy of (1.1). Since (1.2) is of the same general form as (1.4), we also formulate a necessary and sufficient condition for the disconjugacy of (1.1), by using (1.2), provided $P \geqq 0$.

Received by the editors September 22, 1969 and, in revised form, October 21, 1969.

*AMS* 1970 *subject classifications*. Primary 34C10.

2. **Analysis of the integral equation.** Assume that $Q(t)$ and $E(t, s)$ are nonnegative continuous functions on $[a, \infty)$ and $[a, \infty) \times [a, \infty)$, respectively.

Let

$$
(2.1) \qquad
\begin{aligned}
&Q_0(t) = Q(t), \qquad Q_1(t) = \int_t^\infty E(t, s) Q^2(s) ds, \\
&Q_n(t) = \int_t^\infty E(t, s) Q_{n-1}(s) \left[ Q_{n-1}(s) + 2 \sum_{k=0}^{n-2} Q_k(s) \right] ds, \\
&\hspace{20em} n = 2, 3, \cdots.
\end{aligned}
$$

Thus,

$$0 \leqq Q_n \leqq \infty, \qquad n = 0, 1, \cdots.$$

THEOREM 2.1. *Inequality* (1.4) *has a solution* $\nu \in C(a, \infty)$, *if and only if*,

$$(2.2) \qquad \sum_{k=0}^{\infty} Q_k(t) < \infty, \qquad a < t < \infty.$$

PROOF. Assume that (1.4) has a solution $\nu$. Since $E(t, s)\nu^2(s) \geqq 0$, clearly

$$\nu(t) \geqq Q(t) = Q_0(t), \qquad a < t < \infty.$$

It follows inductively that

$$\nu(t) \geqq \sum_{k=0}^{n} Q_k(t), \qquad n = 0, 1, \cdots, \qquad a < t < \infty.$$

This is because

$$\nu \geqq \sum_{k=0}^{n} Q_k$$

implies

$$\nu^2 \geqq \left( \sum_{k=0}^{n} Q_k \right)^2$$

by the nonnegativity of $Q_k$. Hence,

$$\nu(t) \geqq Q(t) + \int_t^\infty E(t, s) \left[ \sum_{k=0}^{n} Q_k(s) \right]^2 ds$$

$$= Q(t) + \int_t^\infty E(t, s) \left\{ \left[ \sum_{k=0}^{n-1} Q_k(s) \right]^2 + Q_n^2(s) + 2Q_n(s) \sum_{k=0}^{n-1} Q_k(s) \right\} ds$$

$$= Q_0(t) + \int_t^\infty E(t, s) \left[ \sum_{k=0}^{n-1} Q_k(s) \right]^2 ds + Q_{n+1}(t)$$

$$= \sum_{k=0}^{n+1} Q_k(t) .$$

Hence, the series $\sum_{k=0}^{\infty} Q_k(t)$ is bounded above and must converge.

Conversely, suppose that (2.2) holds. Then $Q_k(t)$, $k = 0, 1, \cdots$, exists and is nonnegative for $a < t < \infty$. Consider the successive approximations

$$\nu_0(t) = Q(t) ,$$

(2.3)

$$\nu_n(t) = Q(t) + \int_t^\infty E(t, s)\, \nu_{n-1}^2(s) ds, \qquad n = 1, \cdots .$$

We will show by induction that $\nu_n(t)$ exists for $a < t < \infty$ and

$$\nu_n(t) = \sum_{k=0}^{n} Q_k(t), \qquad n = 0, 1, \cdots . \tag{2.4}$$

Clearly (2.4) holds for $n = 0$. Suppose (2.4) holds for $n = 0, \cdots, m$. Then $\nu_{m+1}(t)$ is defined for $a < t < \infty$, because of (2.2) and (2.4), and

$$\nu_{m+1}(t) - \nu_m(t) = \int_t^\infty E(t, s) [\nu_m^2(s) - \nu_{m-1}^2(s)]\, ds$$

$$= \int_t^\infty E(t, s) [\nu_m(s) - \nu_{m-1}(s)]\, [\nu_m(s) + \nu_{m-1}(s)]\, ds$$

$$= \int_t^\infty E(t, s) Q_m(s) \left[ \sum_{k=0}^{m} Q_k(s) + \sum_{k=0}^{m-1} Q_k(s) \right] ds$$

$$= \int_t^\infty E(t, s) Q_m(s) \left[ Q_m(s) + 2 \sum_{k=0}^{m-1} Q_k(s) \right] ds$$

$$= Q_{m+1}(t) ,$$

which establishes (2.4) for $n = m + 1$. Hence, (2.4) holds for all values of $n$ by the induction principle.

Since the series $\sum_{k=0}^{\infty} Q_k$ converges monotonically, we conclude that

$$\begin{aligned}\nu(t) = \sum_{k=0}^{\infty} Q_k(t) &= Q(t) + \lim_{n\to\infty} \int_t^{\infty} E(t, s)\nu_n^2(s)ds \\ &= Q(t) + \int_t^{\infty} E(t, s)\nu^2(s)ds\,,\end{aligned}$$

and so (2.4) is satisfied by the function $\nu(t)$.

In the latter case, we have actually proven the stronger result that (2.1) implies that the integral equation

$$\nu(t) = Q(t) + \int_t^{\infty} E(t, s)\nu^2(s)ds \tag{2.5}$$

has a solution $\nu \in C(a, \infty)$.

3. **On the oscillation of $y'' + py = 0$.** Let

$$\mathfrak{F} = \left\{f : f \text{ measurable on } [a, \infty), f \geqq 0, \int^{\infty} f(s)ds = \infty \right\},$$

$$\mathfrak{F}_0 = \left\{f \in \mathfrak{F}: \lim_{t\to\infty} \left[\int^t f^2(s)ds\right] \Big/ \left[\int^t f(s)ds\right]^2 = 0 \right\},$$

$$\mathfrak{F}_1 = \left\{f \in \mathfrak{F}: \limsup_{t\to\infty} \left(\int^t f(s)ds \int_t^{\infty} f(s) \left[\int^s f^2(\tau)d\tau\right]^{-1} ds\right) > 0\right\},$$

$$A_{fp} \equiv A(s, t) \equiv \int_t^s \left[f(\tau) \int_t^{\tau} p(\sigma)d\sigma\right] d\tau \Big/ \int_t^s f(\tau)d\tau.$$

We say that a function $p \in C[a, \infty)$ has an *averaged integral* $P \equiv P_f$ *with respect to* $\mathfrak{F}$ if there exists $f \in \mathfrak{F}$ such that, for each $t \in [a, \infty)$, $\lim A_{fp}(s, t)$, as $s \to \infty$, exists in $[-\infty, \infty]$ and

$$P(t) = \lim_{s\to\infty} A_{fp}(s, t)\,. \tag{3.1}$$

For a discussion of averaged integrals and their relationship to oscillation see §§4 and 5 of [2]. Briefly, if the limit in (3.1) exists for one value of $t$, say $t = b$, it exists for all values of $t$ and

$$P_f(t) = P_f(b) - \int_b^t p(s)ds, \qquad a \leqq b,\ t < \infty\,.$$

This is easy to show from the definition of $P_f$ and the fact that $\int^{\infty} f = \infty$.

THEOREM 3.1. *Assume that* $\exists g \in \mathfrak{F}_1$ *such that*

$$\liminf_{t\to\infty} A_{gp}(t, a) > -\infty \,. \tag{3.2}$$

*Then,* $y'' + py = 0$ *is disconjugate on* $[a, \infty)$, *if and only if* $\exists f \in \mathfrak{F}_0$ *such that*

$$P(t) \equiv \lim_{s\to\infty} A_{fp}(s, t), \qquad a \leqq t < \infty \,, \tag{3.3}$$

*exists and is finite and*

$$\sum_{k=0}^{\infty} Q_k(t) < \infty \,, \qquad a < t < \infty \,,$$

*where* $Q_k(t)$ *is defined by* (2.1) *with*

$$\begin{aligned} E(t, s) &= \exp\left(2 \int_t^\infty P(\tau)d\tau\right), \\ Q(t) &= \int_t^\infty P^2(s)E(t, s)ds \,. \end{aligned} \tag{3.4}$$

THEOREM 3.2. *Assume* $\exists f \in \mathfrak{F}_0$ *such that*

$$P(t) \equiv \lim_{s\to\infty} A_{fp}(s, t), \qquad a \leqq t < \infty \,, \tag{3.6}$$

*exists with*

$$0 \leqq P(t) < \infty \,, \qquad a \leqq t < \infty \,. \tag{3.7}$$

*Then,* $y'' + py = 0$ *is disconjugate, if and only if,*

$$\sum_{k=0}^{\infty} Q_k(t) < \infty \,, \qquad a < t < \infty \,,$$

*where* $Q_k(t)$ *is defined by* (2.1) *with* $E \equiv 1$ *and* $Q \equiv P$.

COROLLARY 3.1. *Assume that*

$$P(t) = \lim_{s\to\infty} \int_t^s p(\tau)d\tau \tag{3.8}$$

*exists for* $a \leqq t < \infty$ *and* $P \geqq 0$. *Let*

$$P_0(t) = P(t), \qquad P_1(t) = \int_t^\infty P^2(s)ds \,,$$

$$P_n(t) = \int_t^\infty P_{n-1}(s)\left[\, P_{n-1}(s) + 2\sum_{k=0}^{n-2} P_k(s)\right]ds, \quad n = 2, \cdots .$$

*Then, the equation $y'' + py = 0$ is oscillatory, if and only if, there exists $n \geqq 0$ such that $P_n(t) = \infty$ or $\sum_{n=0}^{\infty} P_n(t) = \infty$, $a \leqq t < \infty$.*

COROLLARY 3.2 (WILLETT [1]). *Assume that $p$ has a finite averaged integral $P$ with respect to $\mathfrak{F}_0$, and let $E$ and $Q$ be given by* (3.4) *and* (3.5) *for this $P$. If*

$$(3.9) \qquad Q < \infty \quad \text{and} \quad \int_t^{\infty} Q^2(s)E(t, s)ds \leqq Q(t)/4, \qquad a \leqq t < \infty,$$

*then $y'' + py = 0$ is disconjugate. Conversely, if*

$$(3.10) \qquad \begin{gathered} Q = \infty, \quad \text{or} \\ \int_t^{\infty} Q^2(s)E(t, s)ds \geqq (1 + \epsilon)Q(t)/4, \qquad a \leqq t < \infty, \end{gathered}$$

*for some constant $\epsilon > 0$, then $y'' + py = 0$ is oscillatory.*

PROOF OF THEOREM 3.1. Suppose that $y'' + py = 0$ is disconjugate on $[a, \infty)$. Then $\exists$ a positive solution $y$ on $(a, \infty)$, and $\nu = y'y^{-1}$ satisfies

$$(3.11) \qquad \nu' = -p - \nu^2.$$

Furthermore, (3.2) implies by Lemmas 2.1 and 2.2 of [1, p. 181] that

$$\int^{\infty} \nu^2 < \infty$$

and $A_f(s, t)$ approaches a finite limit, as $s \to \infty$, for each $f \in \mathfrak{F}_0$ and all $t \in (a, \infty)$. As a matter of fact (cf. (2.12), p. 184 of [1]),

$$\lim_{s \to \infty} A_f(s, t) = \nu(t) - \int_t^{\infty} \nu^2(s)ds.$$

Thus, if $f$ is as in (3.3), we have that

$$(3.12) \qquad \nu(t) = P(t) + \int_t^{\infty} \nu^2(s)ds, \qquad a < t < \infty.$$

Let

$$u(t) = \int_t^{\infty} \nu^2(s)ds,$$

so that $u'(t) = -\nu^2(t)$. Substituting (3.12) for $\nu^2(t)$, multiplying by $E(t, s)$, and integrating, we conclude

$$u(t) = u(t_1)E(t, t_1) + \int_t^{t_1} P^2(s)E(t, s)ds$$

$$+ \int_t^{t_1} u^2(s)E(t, s)ds .$$

Hence,

$$\begin{aligned} u(t) &\geqq \int_t^\infty P^2(s)E(t, s)ds + \int_t^\infty u^2(s)E(t, s)ds \\ &= Q(t) + \int_t^\infty u^2(s)E(t, s)ds . \end{aligned} \tag{3.13}$$

Theorem 2.1 now implies that

$$\sum_{k=0}^{\infty} Q_k(t) < \infty , \qquad a < t < \infty . \tag{3.14}$$

Conversely, suppose that (3.14) holds, where the $Q_k$ are defined by (2.1) with $Q$, $P$, and $E$ given as in Theorem 3.1. Then Theorem 2.1 implies that (3.13) has a solution $u \in C(a, \infty)$. It is now easy to show

$$\nu(t) = P(t) + Q(t) + \int_t^\infty E(t, s)u^2(s)ds$$

satisfies

$$\nu' \leqq -p - \nu^2 ,$$

which is a well-known sufficient condition for disconjugacy. Actually Theorem 2.1 implies that (3.13) with equality has a solution for $u$. For such a solution, $w = P + u$ satisfies $w' = -p - w^2$. This was shown in [1, p. 186].

PROOF OF THEOREM 3.2. The proof of this theorem is similar to the previous proof except that we can stop and start with (3.12) instead of (3.13).

PROOF OF COROLLARY 3.1. Corollary 3.1 follows directly from Theorem 3.2, since the existence of the limit in (3.8) implies

$$\lim_{s \to \infty} \int_t^s P(\tau)d\tau = \lim_{s \to \infty} A_{fp}(s, t), \qquad a \leqq t < \infty .$$

PROOF OF COROLLARY 3.2. The key ideas in the proof of Corollary 3.2 remain the same as the direct proof given in [1, p. 185]. The only difference is that Theorem 3.1 can be used here to reduce some of the initial steps.

Assume (3.9) holds and that $Q_n$ are defined by (2.1). One can show by induction that the numbers $a_n$ given inductively by

$$a_0 = 1, \qquad a_n = 1 + a_{n-1}^2/4, \qquad n = 1, 2, \cdots,$$

satisfy

$$\sum_{k=0}^{n} Q_k \leqq a_n Q, \qquad n = 0, 1, \cdots,$$

and

$$a_n \leqq 2, \qquad n = 0, 1, \cdots.$$

Hence, $\sum_{k=0}^{n} Q_k$ is an increasing sequence bounded above by $2Q$. This means that it converges and $y'' + py = 0$ is disconjugate by Theorem 3.1.

Now, assume that (3.10) holds and $Q_n$ are again defined by (2.1). Then we can show by induction that the numbers $a_n$ given inductively by

$$a_0 = 1, \qquad a_n = 1 + (1 + \epsilon)a_{n-1}^2/4, \qquad n = 1, 2, \cdots,$$

satisfy

$$\sum_{k=0}^{n} Q_k \geqq a_n Q, \qquad n = 0, 1, \cdots,$$

and

$$a_n \uparrow \infty \quad \text{as} \quad n \uparrow \infty .$$

Hence,

$$\sum_{k=0}^{\infty} Q_k = \infty ,$$

and Theorem 3.1 implies that $y'' + py = 0$ is oscillatory.

It is worth remarking that a term such as $P_n(t)$ in Corollary 3.1 satisfies $P_n(t) = \infty$, if and only if, $\bar{P}_n(t) = \infty$, where $\bar{P}_n$ is defined inductively as follows:

$$\bar{P}_0(t) = P(t), \qquad \bar{P}_n(t) = \int_t^{\infty} \left[\sum_{k=0}^{n-1} P_k(s)\right]^2 ds, \qquad n = 1, 2, \cdots.$$

Of course, if $\bar{P}_n(t) < \infty$ for all values of $n$, the convergence of $\sum_{n=0}^{\infty} \bar{P}_n(t)$ is not related to the convergence of $\sum_{n=0}^{\infty} P_n(t)$. As a matter of fact, $\sum_{n=0}^{\infty} \bar{P}_n(t)$ always diverges if $P \not\equiv 0$.

## References

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