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SCATTERING THEORY[1]

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Introduction. In this series of lectures we shall develop a theory of scattering for first order systems:

$$(1) \qquad u_t = Gu = \sum_{j=1}^{k} A^j \partial_j u + Bu, \qquad \partial_j u = \frac{\partial u}{\partial x_j}, \qquad u(x, 0) = f(x)$$

over R^k. Here u is an n-component vector-valued function, $A^j(x)$ and $B(x)$ are $n \times n$ matrix-valued functions depending smoothly on x but independent of t. *We impose the following conditions*:

(1) *The L_2-energy is conserved.* This means that the energy at time t, namely

$$E[u(t)] \equiv \int_{R^k} |u(t)|^2 dx$$

is constant in time. Hence with respect to the energy norm G must be skew-symmetric and this in turn requires that the A^j be Hermitian symmetric and that

$$(2) \qquad B(x) + B^*(x) = \sum_{j=1}^{k} \partial_j A^j(x).$$

In fact

$$\begin{aligned} \frac{d}{dt} E[u] &= \int (\partial_t u \cdot u + u \cdot \partial_t u) dx \\ &= \int \Big(\sum_{j=1}^{k} A^j \partial_j u \cdot u + u \cdot A^j \partial_j u + Bu \cdot u + u \cdot Bu \Big) dx \\ &= \int \sum_{j=1}^{k} \partial_j (A^j u \cdot u) dx + \int \Big(B + B^* - \sum_{j=1}^{k} \partial_j A^j \Big) u \cdot u dx. \end{aligned}$$

Integrating over all space the first term in the right vanishes; in order that $dE[u]/dt$ vanish for all smooth data u we see that the relation (2) must hold. Thus if we work in the Hilbert space H of square integrable functions $[L_2(R^k)]^n$, then we can expect that the solution operator

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$$U(t)\colon\ f \to u(t)$$

will be unitary.

(2) *The propagation speeds are nonzero.* This means that the eigenvalues of

$$G(\omega, x) \equiv \sum_{j=1}^{k} A^j(x)\omega_j$$

are $\neq 0$ for all real nonzero k-vectors ω. Otherwise said, G is an elliptic operator.

Note that when $k > 1$, the nonzero vectors of R^k lie in a connected set so that the number of positive [negative] eigenvalues is independent of ω. Since the eigenvalues for $G(-\omega)$ are just the negative of those for $G(\omega)$, it follows that the number of positive eigenvalues is the same as the number of negative eigenvalues; in particular n is even. We shall assume even for $k = 1$ that the number of positive and the number of negative eigenvalues are the same.

(3) *The coefficients are independent of x sufficiently far out.* In other words, we assume that

$$A^j(x) = A_0^j \quad \text{and} \quad B(x) = 0 \quad \text{for } |x| > \rho.$$

We shall take as the unperturbed system

$$v_t = G_0 v = \sum_{j=1}^{k} A_0^j \partial_j v, \qquad v(x, 0) = f. \tag{3}$$

(4) *The number of space variables is odd.* This assures us of Huygens' principle, at least for the unperturbed system.

(5) *The unique continuation property for G.* If $Gf = 0$ in an open set A and $f = 0$ in an open subset, then $f = 0$ in A.

Throughout, H will denote the L_2 space of square integrable vector-valued functions.
will denote the L_2 space integrable vector-valued functions.

Abstract theory. Hyperbolic problems of this sort lend themselves especially well to an abstract treatment in which certain representations play a central role. We will now sketch this abstract theory; a complete development can be found in [3]. However, in the main body of these lectures we will take a more direct approach and derive these representations from the Radon transform.

We choose two particular subspaces D_+ and D_-, the so-called *outgoing* and *incoming* subspaces defined as sets of data

$$D_+ = [f;\ [U(t)f](x) = 0 \text{ for } |x| < \rho + ct,\ t \geqq 0],$$

$$D_- = [f;\ [U(t)f](x) = 0 \text{ for } |x|<\rho - ct,\ t \leqq 0];$$

here c corresponds to the smallest velocity for the system (3). It is clear just from the definition that

(i)
$$U(t)D_+ \subset D_+ \quad \text{for } t \geqq 0,$$
$$U(t)D_- \subset D_- \quad \text{for } t \leqq 0;$$

(ii)
$$\bigcap U(t)D_\pm \equiv \{0\}.$$

Let H_0 denote the null space for G, i.e., the subspace of data f annihilated by G, and set $H' = H \ominus H_0$. It can then be proved that

(iii)
$$\bigcup U(t)D_\pm \text{ is dense in } H'.$$

The basic representation theorem is then:

THEOREM. *H' can be unitarily mapped into $L_2(-\infty, \infty; N)$, where N is an auxiliary Hilbert space, so that $D_+[D_-]$ is mapped onto $L_2(0, \infty; N)$ $[L_2(-\infty, 0; N)]$ and the action of $U(t)$ is right translation by t units.*

These representations are called the *outgoing* and *incoming translation representations*, respectively. A given $f \in H'$ can be represented by both its incoming representer k_- and its outgoing representer k_+. The mapping

$$S\colon\ k_- \rightarrow k_+$$

turns out to be the scattering operator. It is clear that S is (a) unitary, and (b) commutes with translations. It can be shown that *D_+ and D_- are orthogonal* and it follows from this that *S is causal*:

(c)
$$SL_2(-\infty, 0; N) \subset L_2(-\infty, 0; N).$$

Taking the Fourier transforms

$$\tilde{f}_\pm(\sigma) = \int e^{i\sigma s}\, k_\pm(s)ds$$

we obtain the *incoming* and *outgoing spectral representations*. The mappings $f \rightarrow \tilde{f}_\pm$ are again unitary; D_+ $[D_-]$ is mapped onto the Hardy class A_+ $[A_-]$ of square integrable functions analytic for $\operatorname{Im} s > 0$ $[\operatorname{Im} s < 0]$; the action of $U(t)$ corresponds in this case to multiplication by $e^{i\sigma t}$. The scattering operator is transformed into a unitary operator $\mathcal{S}$ which commutes with scalar multipliers and is causal: $\mathcal{S}A_- \subset A_-$. It follows from this that $\mathcal{S}$ is itself a multiplicative operator:

$$\mathcal{S}\colon \tilde{f}_-(\sigma) \rightarrow \tilde{f}_+(\sigma) = \mathcal{S}(\sigma)\tilde{f}_-(\sigma)$$

such that for each σ, $\mathcal{S}(\sigma)$ is unitary in N. Moreover, this operator-valued function is the boundary value of an operator-valued function $\mathcal{S}(z)$ which is holomorphic and of norm $\leqq 1$ in the lower half-plane.

It should be noticed that the scattering operator as defined above depends only on the group $U(t)$ and its action on D_+ and D_-. There is another object which can also be defined in terms of these three quantities,

$$Z(t) = P_+U(t)P_-, \qquad t \geqq 0,$$

where $P_\pm$ is the orthogonal projection on $D_\pm^\perp$, the orthogonal complement of $D_\pm$. This operator annihilates $D_+ \oplus D_-$ and acts like a *semigroup* of contractions on $K \equiv H' \ominus (D_+ \oplus D_-)$. The semigroup property is easily understood from the outgoing translation representation of H'. In this representation $D_+ = L_2(0, \infty; N)$ and D_- is a subspace of $L_2(-\infty, 0; N)$. Since D_- is invariant under $U(t)$, $t < 0$, its complement $D_-^\perp$ is invariant under $U(t)$, $t > 0$, so that a function representing an element of K when right translated remains in $D_-^\perp$, and if it is truncated to the right of 0, i.e., multiplied by the characteristic function of $(-\infty, 0)$, it stays in $D_+^\perp$. Hence the action of $Z(t)$ is right translation followed by truncation, obviously a semigroup action.

We have shown in [3], see also Theorem 6.4 below, that there is a close relation between $Z(t)$ and $\mathcal{S}$. Denote by B the *infinitesimal generator* of Z; the spectrum of B lies in Re $\lambda < 0$ and consists precisely of those points λ for which $\mathcal{S}(i\lambda)$ is not invertible.

This result enables one to deduce many properties of $\mathcal{S}$ by studying the semigroup Z; we summarize here some of the results which will be derived in the course of these lectures by this method:

(1) (See Theorem 6.3): $Z(2\rho)(z-B)^{-1}$ is a compact operator. This implies that $\mathcal{S}(z)$ is meromorphic in the entire z-plane.

(2) (See Theorem 7.3): If all rays associated with equation (1) tend to ∞, then for t large enough the operators $Z(t)$ are compact. It follows from this that the real parts of the eigenvalues λ_k of B tend to $-\infty$, i.e., that they can be arranged so that

$$0 \geqq \operatorname{Re}\lambda_1 \geqq \operatorname{Re}\lambda_2 \geqq \cdots \geqq \operatorname{Re}\lambda_n \rightarrow -\infty.$$

It further follows that for every f in K, $Z(t)f$ has an *asymptotic expansion* of the form

$$Z(t)f \sim \sum_{k=0}^{\infty} c_k e^{\lambda_k t}\, w_k(x),$$

valid uniformly on compact subsets of x-space. Here w_k is the eigenfunction of B corresponding to the eigenvalue of λ_k, and c_k is a constant;

if w_k is a generalized eigenfunction, c_k is a power of t.

(3) (See Theorem 7.1): Suppose that all rays associated with equation (1) tend to ∞; then for t large enough the range of $Z(t)$ is contained in the domain of B. This implies that the eigenvalues λ_k of B are contained in the region

$$\operatorname{Re}\lambda_k \leqq -a - b|\log \operatorname{Im}\lambda_k|, \qquad b > 0.$$

The values of the scattering matrix $\mathcal{S}(\sigma)$ for σ *real* are related to the asymptotic behavior of solutions of (1) for large t and *large* x; the results quoted above show a relation between the values of $\mathcal{S}(z)$ for *complex* z and the asymptotic behavior of solutions of (1) for large t and *fixed* x.

Not all results presented in these lectures concerning the location of the poles of $\mathcal{S}(z)$ are derived with the aid of the semigroup Z. The results contained in §§9–12 concerning the distribution of the imaginary zeros of the scattering matrix of the wave equation in the exterior of an obstacle are derived directly from a representation of the scattering matrix in terms of the so-called transmission coefficient of the obstacle.

PART I. SYMMETRIC HYPERBOLIC SYSTEMS

1. **Energy inequalities.** We begin by deriving certain energy inequalities which are basic to the study of symmetric hyperbolic systems. We arrange the eigenvalues of $-G(\omega, x)$ in decreasing order:

$$\begin{aligned}\tau_1(\omega, x) \geqq \cdots \geqq \tau_{n/2}(\omega, x) > 0\\ > \tau_{n/2+1}(\omega, x) \geqq \cdots \geqq \tau_n(\omega, x),\end{aligned} \tag{1.1}$$

and set

$$c_{\max} = \sup_{\omega, x} \tau_1(\omega, x) \quad \text{and} \quad c_{\min} = \inf_{\omega, x} \tau_{n/2}(\omega, x) . \tag{1.2}$$

THEOREM 1. 1. *The energy of a solution u of the system* (1) *at time $t = T$ inside the ball $\{|x| < R - c_{\max}T\}$ does not exceed the energy of u at time $t = 0$ contained inside the ball $\{|x| < R\}$.*

PROOF. It will turn out that any solution can be approximated in the energy norm by smooth solutions, so it suffices to prove the assertion for smooth u. Taking the scalar product of

$$u_t - \sum A^j \partial_j u - Bu = 0$$

by u and integrating over the truncated cone

$$C = \{|x| < R - c_{\max} t; 0 \leqq t < T\}$$

we get

$$0 = \iint_C \left[\partial_t(u\cdot u) - \sum \partial_j(A^j u\cdot u) - \left(B + B^* - \sum \partial_j A^j\right) u\cdot u \right] dx dt.$$

Because of the relation (2), the integrand is of divergence form. Applying Green's theorem we can transform this integral into an integral over the surface of C which consists of three parts: top, bottom and mantle. We get

$$\int_{|x|<R-c_{\max}T} |u(x,T)|^2 dx - \int_{|x|<R} |u(x,0)|^2 dx$$

$$= + \int_{\text{mantle}} (G(\eta, x)u - \tau u)\cdot u\, dS,$$

where $\tau = ac_{\max}$, $\eta = ax/|x|$ and $1 = a^2(c_{\max}^2 + 1)$. Since τ is at least as large as any eigenvalue of $G(\eta, x)$, the integrand in the right side is nonpositive; therefore so is the integral. This proves the assertion of the theorem.

In particular if u is identically zero in the ball $\{|x| < R\}$ at $t = 0$ it will also vanish identically in the smaller ball $\{|x| < R - c_{\max} T\}$ at time $t = T$. It follows that any signal solution of (1) does not propagate at a speed greater than $c_{\max}$. Thus a solution with initial data having support in the ball $\{|x| < R\}$ will vanish outside the cones $\{|x| < R + c_{\max}|t|\}$.

The above analysis applies as well to the unperturbed system (3). In this case, however, more is true, in fact signals *do not propagate with a speed less than* $c_{\min}$. This is a kind of Huygens' principle which we now state; its proof will be given in the next section.

THEOREM 1.2. *If $f(x) = 0$ outside the ball $\{|x| < R\}$ then the solution v of the unperturbed problem* (3) *is zero inside the cones* $\{|x| < c_{\min}|t| - R\}$.

2. **The Radon transform for the unperturbed system.** As we shall see, the Radon transform for the unperturbed system is closely related to the translation representation. We shall suppose at first that the initial data f belongs to C_0^∞ and in the process of constructing its translation representer, we will prove that a solution v to (3) exists and that the mapping

$$V(t) : f \to v(t)$$

is an isometry. A limiting procedure then extends these results to general f in H.

DEFINITION. For f in C_0^∞, s in R and a unit vector ω in S_{k-1} we set

$$m(s, \omega) = \int_{x\cdot\omega = s} f(x) dS. \tag{2.1}$$

The *Radon transform* of f is then

$$\mathcal{R}f \equiv \ell(s, \omega) = (-1)^{(k-1)/2}\, \partial_s^{k-1}\, m(s, \omega)\ . \tag{2.2}$$

We remark that $m(s, \omega)$ is an even function of (s, ω); i.e., $m(-s, -\omega) = m(s, \omega)$, that $\ell(s, \omega)$ also is even, and that

$$\int s^j\, \ell(s, \omega)ds = 0 \quad \text{for } j = 0,1,2,\cdots, k-2. \tag{2.3}$$

LEMMA 2.1. *If $f \in C_0^\infty$, then the inverse Radon transform is given by*

$$f(x) = \frac{1}{2}\int_{|\omega|=1} \ell(x\cdot\omega, \omega)d\omega. \tag{2.4}$$

If f and ℓ are so related, then f is determined by the even part of $\ell(s, \omega)$ which in turn is uniquely determined by f.

PROOF. Making use of (2.1) we can write the Fourier transform of $f(x)$ as

$$\tilde{f}(\sigma, \omega) = \int e^{i\sigma x\cdot\omega} f(x)dx = \int_{-\infty}^{\infty} e^{i\sigma s}\, m(s, \omega)ds \tag{2.5}$$

and note that $\tilde{f}(\sigma, \omega)$ is even in (σ, ω). As a consequence we can write the inverse Fourier transform as

$$\begin{aligned} f(x) &= \int_{|\omega|=1} \left[\int_0^\infty e^{-i\sigma x\cdot\omega}\tilde{f}(\sigma, \omega)\sigma^{k-1}\, d\sigma\right]d\omega \\ &= \frac{1}{2}\int_{|\omega|=1}\left[\int_{-\infty}^{\infty} e^{-i\sigma x\cdot\omega}\tilde{f}(\sigma, \omega)\sigma^{k-1}\, d\sigma\right]d\omega\,. \end{aligned} \tag{2.6}$$

For fixed ω, $\tilde{f}(\sigma, \omega)$ is by (2.5) the Fourier transform of $m(s, \omega)$. According to (2.6) the inner integral in the right is the inverse Fourier transform of $\sigma^{k-1}\tilde{f}(\sigma, \omega)$ which is simply

$$\ell(s, \omega) = i^{k-1}\, \partial_s^{k-1}\, m(s, \omega)$$

evaluated at $s = x\cdot\omega$. Inserting this in (2.6) gives (2.4).

If f and ℓ are related as in (2.4) it is easily seen by transforming ω to $-\omega$ that the odd part of ℓ does not contribute to the integral. Finally it follows from the uniqueness of the Fourier transform and the above argument that f uniquely determines the even part of ℓ.

LEMMA 2.2. *Set*

$$L(s, \omega) = \partial_s^{(k-1)/2}\, m(s, \omega).$$

Then the mapping

$$f \to \frac{1}{2} L(s, \omega)$$

is a unitary map of $L_2(R^k)$ *onto the even (odd) functions of* $L_2(R, L_2(S_{k-1}))$.

PROOF. Denote the s-Fourier transform of $\mathfrak{L}$ and L by $\tilde{\mathfrak{L}}$ and $\tilde{L}$ respectively. It follows from the proof of the above lemma that

$$\sigma^{k-1} \tilde{f}(\sigma, \omega) = \tilde{\mathfrak{L}}(\sigma, \omega).$$

On the other hand

$$\mathfrak{L}(s, \omega) = (-1)^{(k-1)/2}\, \partial_s^{(k-1)/2}\, L(s, \omega)$$

so that

$$\tilde{\mathfrak{L}}(\sigma, \omega) = (-i\sigma)^{(k-1)/2}\, \tilde{L}(\sigma, \omega) \quad \text{and} \quad (i\sigma)^{(k-1)/2} \tilde{f}(\sigma, \omega) = \tilde{L}(\sigma, \omega).$$

Hence by the Plancherel theorem

$$\|f\|^2 = \int_{|\omega|=1} \int_0^\infty |\tilde{f}(\sigma, \omega)|^2 \sigma^{k-1} d\sigma\, d\omega$$

$$= \frac{1}{2} \int_{|\omega|=1} \left[\int_{-\infty}^\infty |(i\sigma)^{(k-1)/2} \tilde{f}(\sigma, \omega)|^2 d\sigma \right] d\omega$$

$$= \frac{1}{2} \int_{|\omega|=1} \int_{-\infty}^\infty |\tilde{L}(\sigma, \omega)|^2\, d\sigma\, d\omega = \frac{1}{2} \|L\|^2 .$$

The onto property follows from the fact that the even (odd) functions with support bounded away from zero and infinity are dense among the even (odd) functions, and that for such functions $\tilde{f}(\sigma, \omega) = (i\sigma)^{(1-k)/2} \cdot \tilde{L}(\sigma, \omega)$ is the Fourier transform of data in $L_2(R^k)$.

THEOREM 2.3. *Denote by* $G_0(\omega)$ *the symbol of* G_0:

$$G_0(\omega) = \sum_{j=1}^k A_0^j \omega_j \ ;$$

arrange the eigenvalues of $-G_0(\omega)$ *in decreasing order*:

$$\tau_1(\omega) \geqq \cdots \geqq \tau_{n/2}(\omega) > 0 > \tau_{n/2+1}(\omega) \geqq \cdots \geqq \tau_n(\omega). \tag{2.7}$$

Let $r_j(\omega)$ *denote the normalized eigenvectors of* $G_0(\omega)$:

$$-\, G_0 r_j = \tau_j r_j \ .$$

If the Radon transform of the initial data $f \in C_0^\infty$ *is* $\mathfrak{L}$ *and*

(2.8) $$\ell_j(s, \omega) = \ell(s, \omega)\cdot r_j(\omega),$$

then the solution to (3) *is given by*

(2.9) $$v(x, t) = \sum_{j=1}^{n/2} \int_{|\omega|=1} \ell_j(x\cdot\omega - \tau_j(\omega)t, \omega)r_j(\omega)d\omega .$$

PROOF. Supposing v to exist we set

$$m(s, \omega; t) = \int_{x\cdot\omega=s} v(x, t)dS = \int_{R^{k-1}} v(s\omega + x', t)dx'$$

where x' in R^{k-1} is $\perp$ ω. Then

$$\partial_t m(s, \omega; t) = \int \partial_t v(s\omega + x', t)dx'.$$

Since v satisfies the differential equation we can replace $\partial_t v$ by a first order spatial operator. After a transformation of coordinates this can be written as a divergence operator in the hyperplane $x\cdot\omega = s$ plus a normal operator:

$$G_0(\omega)\,\partial_s v(s\omega+x', t).$$

The divergence part integrates out to zero and by interchanging order of integration and differentiation in the normal part we obtain

(2.10) $$\partial_t m(s,\omega; t) = G_0(\omega)\partial_s m(s, \omega, t).$$

This equation is readily solved componentwise. In fact, setting

$$m_j(s, \omega; t) = m(s, \omega; t)\cdot r_j(\omega)$$

we see that

$$\partial_t m_j(s, \omega; t) = -\tau_j(\omega)\partial_s m_j(s, \omega; t)$$

so that

$$m_j(s, \omega; t) = m_j(s-\tau_j(\omega)t, \omega)$$

where m_j is the r_jth component of

$$m(s, \omega) = \int_{x\cdot\omega=s} f(x)dS.$$

We therefore have

(2.11) $$m(s, \omega; t) = \sum_{j=1}^{n} m_j(s-\tau_j(\omega)t, \omega)r_j(\omega).$$

In order to verify that $m(s, \omega; t)$ as defined in (2.11) is even we note that

$$-G_0(-\omega)r_j(\omega) = G_0(\omega)r_j(\omega) = -\tau_j(\omega)r_j(\omega).$$

Thus the eigenvectors for $G_0(-\omega)$ and $G_0(\omega)$ are the same with corresponding eigenvalues mulitplied by (-1). Recalling the ordering (2.7) we see that

$$\tau_j(-\omega) = -\tau_{j'}(\omega) \quad \text{and} \quad r_j(-\omega) = r_{j'}(\omega).$$

where $j' = n - j + 1$. Hence

$$m_j(-s, -\omega) = m(-s, -\omega)\cdot r_j(-\omega) = m(s, \omega)r_{j'}(\omega) = m_{j'}(s, \omega) \tag{2.12}$$

and

$$m_j(-s-\tau_j(-\omega)t, -\omega) = m_{j'}(s-\tau_{j'}(\omega)t, \omega). \tag{2.13}$$

One now sees by inspection that $m(s, \omega; t)$ is indeed even in (s, ω). In fact, more is true; the relation (2.13) shows that

$$m(s, \omega; t) = \text{even part of } 2\sum_{j=1}^{n/2} m_j(s-\tau_j(\omega)t, \omega)r_j(\omega).$$

We are now essentially finished. The Radon transform of v is given by the even part of

$$2\sum_{j=1}^{n/2} \mathfrak{L}_j(s-\tau_j(\omega), \omega)r_j(\omega);$$

and hence (2.9) follows from Lemma 2.1. Finally we verify directly that (2.9) satisfies (3). In fact, writing $\mathfrak{L}'$ for $\partial_s\mathfrak{L}$ we have

$$\begin{aligned}
\partial_t v &= \sum \int -\tau_j(\omega)\mathfrak{L}_j'(x\cdot\omega-\tau_j(\omega)t, \omega)r_j(\omega)d\omega \\
&= \sum \int \mathfrak{L}'_j(x\cdot\omega-\tau_j(\omega)t, \omega)G_0(\omega)r_j(\omega)d\omega \\
&= \sum_m\sum_j A_0^m \int \omega_m \mathfrak{L}'_j(x\cdot\omega-\tau_j(\omega)t, \omega)r_j(\omega)d\omega = G_0 v
\end{aligned}$$

and

$$\begin{aligned}
v(x, 0) &= \int \text{even part of } \sum_{j=1}^{n/2} \mathfrak{L}_j(x\cdot\omega, \omega)r_j(\omega)d\omega \\
&= \frac{1}{2}\int \mathfrak{L}(x\cdot\omega, \omega)d\omega = f(x) .
\end{aligned}$$

If we proceed as in the derivation of (2.10) we see that

$$\int_{x\cdot\omega=s} G_0 f\, dS = G_0(\omega)\partial_s m(s, \omega)$$

and hence that

(2.14) $$\mathscr{R}(G_0 f) = G_0(\omega)\partial_s \mathscr{L}(s, \omega).$$

It follows from Lemma 2.2 that we can by completion extend the Radon transform to any L_2 functions f. In this case (2.9) continues to hold in the distribution sense.

We now have all of the ingredients for the translation representation of the unperturbed system. Let $C^+(\omega)$ denote the space spanned by

$$r_1(\omega), \cdots, r_{n/2}(\omega),$$

and let K be the set of all square integrable functions $k(s, \omega)$ on $(-\infty, \infty)$ with values in $C^+(\omega)$ for each ω in S_{k-1} with norm

(2.15) $$\|k\|^2 = \int_{-\infty}^{\infty} \int_{|\omega|=1} |k(s, \omega)|^2 d\omega\, ds.$$

THEOREM 2.4. *The mapping*

(2.16) $$f \text{ in } L_2(R^k) \to k_0(s, \omega) = \sum_{j=1}^{n/2} (\tau_j(\omega))^{1/2} L_j(\tau_j(\omega)s, \omega) r_j(\omega) \quad \text{in } K$$

defines the translation representation for the unperturbed system (3).

PROOF. It follows from Theorem 2.3 that under this mapping

$$V(t)f \to \sum_{j=1}^{n/2} (\tau_j(\omega))^{1/2} L_j(\tau_j(\omega)(s-t), \omega) r_j(\omega) = k_0(s-t, \omega).$$

Moreover,

$$\|k_0\|^2 = \sum_{j=1}^{n/2} \int\int |L_j(\tau_j(\omega)s, \omega)|^2 \tau_j(\omega) ds\, d\omega$$

$$= \sum_{j=1}^{n/2} \int\int |L_j(s, \omega)|^2 ds\, d\omega;$$

making use of (2.12) and Lemma 2.2 we see that

$$\|k_0\|^2 = \frac{1}{2}\|L\|^2 = \|f\|^2$$

which shows that the mapping is an isometry. In order to prove the unitary property we need only establish that the mapping is onto K. Now by Lemma 2.2 any k_0 in K has as its even [or odd depending on the parity of $(k-1)/2$] part a function corresponding to data f in $L_2(R^k)$. Under the mapping (2.16) $f \to k_0$.

We close this section with the promised *proof of Theorem* 1.2. It suffices to consider f in C^∞ with support in the ball $\{|x| < R\}$. It is clear that $m(s, \omega)$ and hence $\mathfrak{L}(s, \omega)$ will vanish outside the interval $[-R, R]$. Now

$$|x \cdot \omega - \tau_j(\omega)t| > c_{\min}|t| - |x|$$

and therefore $v(x, t)$ as given in (2.9) vanishes for $c_{\min}|t| - |x| > R$ as asserted.

3. **The spectrum of** G. We are now going to employ the Radon transform to study the point spectrum of the perturbed operator G. We denote by H_0 the null space of G.

THEOREM 3.1. (a) *The point spectrum of G consists of at most the point* 0.

(b) *H_0 is of finite dimension.*

Our proof of this theorem hinges on a Rellich type uniqueness theorem. We shall base our proof of this on properties of the Radon transform.

The next theorem gives a complete characterization of the Radon transform of functions with compact support; it is a counterpart of the Paley-Wiener theorem, which does the same for the Fourier transform, see Helgason, [2], and Ludwig, [9]; for another proof see [7].

THEOREM 3.2. *Let f be in $L_2(R^k)$, $\mathfrak{L}$ its Radon transform. Suppose that*

$$f(x) = 0 \quad \text{for } |x| > \rho;$$

then (i)

$$\mathfrak{L}(s, \omega) = 0 \quad \text{for } |s| > \rho; \tag{3.2}$$

(ii) *for every integer a*

$$\int s^a \mathfrak{L}(s, \omega) ds \tag{3.3}$$

is a polynomial in ω of degree $a - k + 1$. Conversely, if $\mathfrak{L}$ is in $\partial_s^{(k-1)/2} L_2$, and if (3.2), (3.3) *are satisfied then, the Radon inverse of $\mathfrak{L}$ is zero for $|x| > \rho$.*

REMARK. For $a < k - 1$ the conclusion is meant to say that (3.3) is zero, a fact already noted in (2.3). The assertion for $a \geqq k - 1$ is not meant to exclude the possibility that (3.3) is zero.

We show now how to use Theorem 3.2 to prove that G has no point spectrum other than 0. Suppose that for $\mu \neq 0$

$$(G - i\mu)f = 0 \tag{3.4}$$

for some square integrable f; we claim that $f = 0$. To show this we set

$$(G_0 - i\mu)f = g; \tag{3.5}$$

since G and G_0 are the same for $|x| > \rho$, it follows that $g = 0$ for $|x| > \rho$. Since G is an elliptic operator, g is square integrable.

THEOREM 3.3. *Suppose that μ is real, $\neq 0$, f in L_2, $(G_0 - i\mu)f = g$ also in L_2, and zero for $|x| > \rho$; then $f = 0$ for $|x| > \rho$.*

Using Theorem 3.3 we conclude that f satisfying (3.4) and in L_2 vanishes for $|x| > \rho$. It follows from the principle of unique continuation for G that f is zero throughout R^k; this completes the proof of the first assertion in Theorem 3.1.

We turn to the proof of Theorem 3.3; denote by $\mathfrak{L}$ and h the Radon transforms of f and g. Taking the Radon transform of (3.5) we get

$$G_0(\omega)D_s\mathfrak{L} - i\mu\mathfrak{L} = h \tag{3.6}$$

where $G_0(\omega)$ is the symbol of G_0. Since g is assumed to be zero for $|x| > \rho$, it follows from Theorem 3.2 that $h = 0$ for $|s| > \rho$; we claim the same for $\mathfrak{L}$; to this end we solve the differential equation (3.6); multiplying by the eigenvector r_m and using

$$-G_0(\omega)r_m = \tau_m r_m,$$

we get for $\mathfrak{L}_m = \mathfrak{L} \cdot r_m$ that

$$-\tau_m \partial_s \mathfrak{L}_m - i\mu\mathfrak{L}_m = h_m. \tag{3.7}$$

Using the fact that $h_m = 0$ for $|s| = \rho$ we get after integrating (3.7) that for $|s| > \rho$

$$\mathfrak{L}_m(s, \omega) = a(\omega)\exp(-is\mu/\tau_m(\omega)).$$

It follows that for $|s| > \rho$, $\partial^{(1-k)/2}\mathfrak{L}$ is a linear combination of imaginary exponentials plus a polynomial in s of degree less than $(k-1)/2$. Since by Parseval's relation this is in L_2, this can only be if $\partial^{(1-k)/2}\mathfrak{L}$, and so $\mathfrak{L}$ itself, is zero for $|s| > \rho$.

We appeal once more to the direct part of Theorem 3.2: Since $g = 0$ for $|x| > \rho$, by (3.3)

$$\int s^a h(s, \omega)ds \tag{3.8}$$

is a polynomial in ω of degree $a - k + 1$. Multiplying (3.6) by s^a and integrating we get, after integrating the first term by parts, dividing by $i\mu$ and rearranging terms, that

$$\int s^a \mathfrak{L} ds = \frac{i}{\mu} \int s^a h ds + \frac{i}{\mu} G_0(\omega) a \int s^{a-1} \mathfrak{L} ds. \tag{3.9}$$

We claim that $\int s^a \mathfrak{L} ds$ is a polynomial in ω of degree $a - k + 1$; we prove it by induction on a. For $a = k - 2$ the right side is zero by property (2.3). Suppose now we know the result for $a - 1$; we note that the first term on the right in (3.9) is a constant multiple of (3.8) and so of degree $a + k - 1$. The integral in the second term is, by induction hypothesis, of degree $a + k - 2$; it is multiplied by a linear function of ω which makes the whole second term of order $a + k - 1$ and completes the inductive proof that $\mathfrak{L}$ satisfies (3.3). We conclude then by the converse part of Theorem 3.2 that f, whose Radon transform $\mathfrak{L}$ is, vanishes for $|x| > \rho$. This completes the proof of Theorem 3.3.

N.B.: For the proof of Theorem 3.3 we only need that $G_0(\omega)$ has real spectrum for ω real; symmetry and $\tau \neq 0$ are unnecessary.

We turn now to proving part (b) of Theorem 3.1, i.e., the finite-dimensionality of the null space H_0 of G. We shall deduce this from

LEMMA 3.4. *The unit sphere of H_0 is compact. According to a classical result of F. Riesz, this implies the finite-dimensionality of H_0.*

PROOF OF LEMMA 3.4. f belongs to H_0 if it satisfies

$$Gf = 0. \tag{3.10}$$

Since G is elliptic, one can estimate the L_2 norm of f and of its first two derivatives in terms of f:

$$\|D^\alpha f\| \leqq \text{const } \|f\|, \qquad |\alpha| \leqq 2. \tag{3.11}$$

Set

$$G_0 f = g; \tag{3.12}$$

since G and G_0 have the same coefficients for $|x| > \rho$, the function g is zero for $|x| > \rho$; for $|x| < \rho$, g is a linear combination of f and its first derivatives. It follows then from estimate (3.11) and the Rellich compactness criterion that the set of g corresponding to f in the unit ball of H_0 is precompact.

We now take the Radon transform of (3.12):

$$G_0(\omega) D_s \mathfrak{L} = h.$$

Integrating with respect to s we get

$$\partial^{(1-k)/2} \mathfrak{L} = G_0^{-1}(\omega)\, \partial_s^{-(1+k)/2} h. \tag{3.13}$$

We use now the Parseval relation for f and g to conclude:

(i) Since $\partial_s^{(1-k)/2} \mathfrak{L}$ is square integrable, and since h is zero for $|s| > \rho$, it follows that $\partial_s^{(1-k)/2} \mathfrak{L}$ is zero for $|s| > \rho$.

(ii) Since the functions g form a precompact set, so do $\{\partial_s^{(1-k)/2}h\}$; therefore so do the functions $\partial_s^{(1-k)/2}\mathfrak{k}$ as given by (3.13), over the *compact* set $[-\rho, \rho] \times S^{k-1}$.

This completes the proof of Lemma 3.4.

4. **The wave operators.** As before we denote by $V(t)$ and $U(t)$ the solution operators for the free space and perturbed systems respectively. They define one-parameter groups of unitary operators on H, symbolically

$$V(t) = \exp(G_0 t) \quad \text{and} \quad U(t) = \exp(Gt).$$

Again let H_0 denote the null space of G. For convenience we take $c_{\min} = 1$.

The wave operators for these two groups are defined as

$$W_{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} U(-t)V(t) \ . \tag{4.1}$$

The main result of this section is

THEOREM 4.1. (a) *The wave operators exist,*
(b) *Range* $W_+ =$ *Range* $W_- = H \ominus H_0$.

It follows that the scattering operator

$$S = W_+^{-1} W_- \tag{4.2}$$

exists and is unitary on H. In order to prove Theorem 4.1 we require several intermediary results.

A given f in H can be represented as in §2 by

$$\sum_{j=1}^{n/2} h_j(s, \omega) r_j(\omega). \tag{4.3}$$

We define the subspaces D_+ [and D_-] to consist of all those f whose representers h_j $(j = 1, 2, \cdots, n/2)$ are supported on the positive [negative] real s-axis. It follows from Lemma 2.2 that D_+ and D_- are orthogonal and that $H = D_+ \oplus D_-$. It follows from (2.9) and $c_{\min} = 1$ that $V(t)f = 0$ for $|x| < t$ if f belongs to D_+, and for $|x| < -t$ if f belongs to D_-. We next define the subspaces D_+^ρ and D_-^ρ by

$$D_+^\rho = V(\rho)\, D_+ \ , \tag{4.4a}$$

$$D_-^\rho = V(-\rho)\, D_- \ . \tag{4.4b}$$

LEMMA 4.2. *With* D_+^ρ *defined as above and* f *in* D_+^ρ,

$$U(t)f = V(t)f \ , \qquad t \geqq 0.$$

PROOF. We need only note that, by (2.9), $V(t)f$ is zero in a truncated cone $|x| < t + \rho$; outside of this and for $t > 0$ the coefficients of G are constant. Both G and G_0 act in the same way on such data so that $U(t)f = V(t)f$ for $t \geqq 0$.

Using the above lemma we can now prove

LEMMA 4.3. *W_+ exists for all f in $V(-r)D_+^\rho$.*

PROOF. Let $f = V(-r)g$ for some g in D_+^ρ. Then

$$U(-t)V(t)f = U(-t)V(t)V(-r)g \ ,$$

or

$$U(-t)V(t)f = U(-r)U(-t+r)V(t-r)g \ .$$

Making use of Lemma 4.2 this becomes for all $t \geqq r$

$$U(-t)\ V(t)f = U(-r)g = U(-r)\ V(r)f \ ,$$

so that

$$W_+f = U(-r)\ V(r)f \ .$$

Now if f vanishes outside the ball $\{|x| < R\}$ then its Radon transform vanishes for $|s| > R$ and it follows from this and (2.9) that $V(R + \rho)f$ belongs to D_+^ρ. Since data of this sort is dense in H, and since $U(-t)V(t)$ is unitary, part (a) of Theorem 4.1 now follows from the principle of dense convergence. Part (b) which is considerably more difficult will follow from the

DENSITY LEMMA. *$\{U(-r)D_+^\rho \mid \forall r > 0\}$ is dense in $H \ominus H_0$.*

We shall show that if there exists an m in $H \ominus H_0$ orthogonal to $U(-r)D_+^\rho$ for all r then m must be zero. Now $m \perp U(-r)D_+^\rho$ is the same as $U(r)m \perp D_+^\rho$. With this in mind we shall list as lemmas some facts about $U(r)$ which we shall use in the proof of the Density Lemma.

LEMMA 4.4. *Define $X = U(-2\rho) - V(-2\rho)$ and Q_p by*

$$\begin{aligned}[Q_pf](x) &= f, \qquad |x| \leqq p \ , \\ &= 0, \qquad |x| > p \ ,\end{aligned}$$

then

$$Q_{\rho+2c\rho}\, X = X \ , \tag{4.5}$$

$$XQ_{\rho+4c\rho} = X \ ; \tag{4.6}$$

here c = maximum sound speed.

PROOF. The relation (4.5) asserts that the range of X consists of functions supported in $|x| < \rho + 2c\rho$. This is true because U and V

satisfy the same equation outside of $|x| \leqq \rho$ and because signals propagate with speed $\leqq c$. For the same reason the values of $U(-2\rho)f$ and $V(-2\rho)f$ in $|x| \leqq \rho + 2c\rho$ are not influenced by values of f at $|x| > \rho + 4c\rho$, proving (4.6). We also have the

LEMMA 4.5. *For all f in $H \ominus H_0$ and every p we have*

$$\liminf_{t\to\infty} \|Q_p U(t)f\| = 0 \ . \tag{4.7}$$

This is a very weak decay theorem whose proof will be deferred. Using these two lemmas we are now in a position to prove the Density Lemma.

Note. For every f in $U(-r)D_+^\rho$, $U(t)f$ is zero for $|x| < t - r + \rho$ so that for such f

$$\lim_{t\to\infty} \|Q_p U(t)f\| = 0 \ . \tag{4.8}$$

Combining this with the Density Lemma it follows that (4.8) holds for all f in $H \ominus H_0$. In a later section we will investigate the rate of decay in (4.8) and its uniformity with respect to f.

PROOF OF DENSITY LEMMA. Suppose $m \perp U(-r)D_+^\rho$. From the Lemma 4.5 there exist arbitrarily large T such that

$$\|Q_{\rho+4c\rho} U(T)m\| < \epsilon \ .$$

Set

$$U(T)m = a.$$

Since $m \perp U(-T)D_+^\rho$ we have $a \perp D_+^\rho$. Now $a \perp D_+^\rho$ implies $V(-\rho)a \perp V(-\rho)D_+ = D_+$ which implies $V(-\rho)a \in D_-$ so that $V(-2\rho)a \in V(-\rho)D_- = D_-^\rho$; that is

$$V(-2\rho)a \in D_-^\rho \ . \tag{4.9}$$

Now define

$$n = V(-T)a$$

and rewrite $V(-T)$ to get $n = V(2\rho - T)V(-2\rho)a$. Using (4.9) and the incoming property of D_-^ρ we deduce that

$$n(x) = 0, \qquad |x| < T - \rho \ . \tag{4.10}$$

Also

$$n = U(2\rho - T)V(-2\rho)a$$

since by Lemma 4.2 $U(t) = V(t)$ in D_-^ρ for $t < 0$. Subtracting this expression for n from

$$m = U(2\rho - T)U(-2\rho)a$$

gives

$$m - n = U(2\rho - T)Xa \ .$$

Now using the isometric property of U, property (4.6) of X and the fact that $\|X\| \leqq 2$, we get

$$\|m - n\| = \|Xa\| = \|XQ_{\rho+4c\rho}\, a\| \leqq 2\|Q_{\rho+4c\rho}\, a\| < 2\,\epsilon \ .$$

Since by (4.10) n vanishes for $|x| < T - \rho$ it follows that

$$\|Q_{T-\rho} m\| < 2\epsilon \ .$$

Since ϵ is arbitrarily small and T arbitrarily large, we see that $m \equiv 0$. Now we proceed to the

PROOF OF LEMMA 4.5. It is enough to prove (4.7) for a dense set of $f \perp H_0$. Take f in the domain of G. For such an f

$$\|GU(t)f\| = \|U(t)Gf\| = \|Gf\|$$

and

$$\|U(t)f\| = \|f\| \ ,$$

that is, $\{U(t)f\} = \{f_t\}$ is a collection of elements for which $\|f_t\|$ and $\|Gf_t\|$ are uniformly bounded. Since G is elliptic

$$\|D_x f_t\| \leqq \text{constant}$$

which means, by Rellich's compactness criterion, that $\{f_t\}$ is precompact in the L_2-norm over any bounded subset. In view of this compactness it suffices, in order to prove (4.7), to construct a sequence $\{t_N\} \rightarrow \infty$ such that

$$U(t_N)f \xrightarrow{\text{weakly}} 0 \ . \tag{4.11}$$

That means that

$$(g, U(t_N)f) \rightarrow 0 \tag{4.12}$$

for all g in H. Again it suffices to show (4.12) for a dense subset of g.

By the spectral resolution

$$(g, U(t)f) = \int e^{it\lambda} d(g, E_\lambda f) = \int e^{it\lambda} dm_g$$

i.e., $(g, U(t)f)$ is the Fourier transform $\tilde{m}_g(t)$ of the measure m_g. The total variation of m_g is $\leqq \|g\|\,\|f\|$. If m_g were absolutely continuous it would follow from the Riemann-Lebesgue lemma that

$$\lim_{t\rightarrow\infty} \tilde{m}_g(t) = 0 \ .$$

We cannot verify directly that m_g is absolutely continuous; we have shown, however, that G has no point spectrum, except possibly 0. Since f was taken to be orthogonal to the null space H_0 of G, it follows that $m_g = d(g, E(\lambda)f)$ has no point mass. Wiener has shown:

Let m be any complex measure of finite total variation, and no point mass. Then its Fourier transform tends to zero in the mean, i.e.

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{m}(t)|^2\, dt = 0 \ .$$

From this fact one can produce a sequence $t_N \to \infty$ such that (4.12) holds for a denumerable set of g. This completes the proof of the Density Lemma.

Proof of Part (b) in Theorem 4.1. It is clear from Lemma 4.2 that W_+ acts like the identity in D_+^ρ. On the other hand, by the intertwining property for W_+:

$$U(t)W_+ = W_+V(t)$$

we see that the range of W_+ contains

$$\bigcup U(t)D_+$$

and by the Density Lemma this set is dense in $H \ominus H_0$. This completes the proof of Theorem 4.1.

5. **Properties of the scattering operator.** In §4 we have proved the existence of the wave operators $W_\pm$, defined as the following limits:

$$(5.1)_+ \qquad W_+ = \operatorname*{s-lim}_{t\to\infty} U(-t)V(t) \ ,$$

$$(5.1)_- \qquad W_- = \operatorname*{s-lim}_{r\to\infty} U(r)V(-r)$$

and we have shown that the range of $W_\pm$ is $H \ominus H_0$. In what follows we shall, for simplicity, take H_0 to be zero so that $W_\pm$ are unitary.

The scattering operator S was defined as

$$(5.2) \qquad S = W_+^* W_- \ .$$

Using the definitions $(5.1)_\pm$ we can write S as the double limit

$$(5.3) \qquad S = \operatorname*{s-lim}_{t,r\to\infty} V(-t)U(r+t)V(-r).$$

The following theorem expresses the basic properties of S:

Theorem 5.1. (a) *S is unitary,*

(b) *S commutes with $V(p)$,*

(c) *S maps D_-^ρ into the orthogonal complement of D_+^ρ.*

PROOF. Since by definition (5.2) S is the product of two unitary operators, (a) follows. To prove (b) denote by $S(r, t)$ the operator on the right of (5.3). Using the group property of V we easily verify that for any p

$$S(r, t)V(p) = V(p)S(r-p, t+p);$$

letting r, t tend to infinity we get $SV(p) = V(p)S$ which proves part (b).

To prove part (c) we recall from Lemma 4.2 that $U(t) = V(t)$ on D_+^ρ for $t > 0$, and on D_-^ρ for $t < 0$. From this and definition (5.1) it follows that

$$W_- f = f \quad \text{for} \quad f \text{ in } D_-^\rho, \qquad W_+ g = g \quad \text{for} \quad g \text{ in } D_+^\rho . \tag{5.4}$$

For f and g as in (5.4) we get using (5.2) and the orthogonality of D_+^ρ and D_-^ρ that

$$(Sf, g) = (W_+^* W_- f, g) = (W_- f, W_+ g) = (f, g) = 0.$$

This proves part (c).

Spectral theory tells us that an operator S commuting with a group of unitary operators in a Hilbert space H can be studied advantageously by introducing a *spectral representation* of H for the group, i.e., one in which the action of $V(t)$ goes over into multiplication by $e^{i\sigma t}$. We shall now describe in two stages such a representation. We start with the *translation representation* for $V(t)$ constructed in Theorem 2.4; this gives a linear mapping M of H onto the space $L_2(R; N)$ of square integrable functions on R whose values lie in some auxiliary Hilbert space N. This becomes more apparent if we write k_0 of (2.16) in component form

$$k_0 \sim \{k_0^{(1)}, k_0^{(2)}, \ldots, k_0^{n/2}\}$$

where

$$k_0^{(j)} = (\tau_j(\omega))^{1/2} L_j(\tau_j(\omega)s, \omega)$$

lies in $L_2(R, N)$, $N = L_2(S_{k-1})$; that is k_0 consists of $n/2$-tuples of square integrable functions on R to N. In this case

$$\|f\| = \|k_0\| = \left[\sum_{j=1}^{n/2} \int\int |k_0^j(s, \omega)|^2 \, ds \, d\omega \right]^{1/2} .$$

We summarize what we have proved so far and point out an additional property:

THEOREM 5.2. *Denote by M the assignment $f \to k_0$ defined by Theorem 2.4; then*

(a) *M is unitary.*
(b) *M is a translation representation of H for the group* $V(t)$.
(c) *M maps D_+ onto $L_2(R_+; N)$, D_- onto $L_2(R_-; N)$.*

We have already verified the first two properties; (c) follows from the definitions of $D_\pm$.

According to (4.4), $D^\rho_\pm$ is defined as the image of $D_\pm$ under $V(\pm\rho)$. Combining (b) and (c) we get

COROLLARY 5.3. (d) *M maps D^ρ_+ onto $L_2((\rho, \infty); N)$, D^ρ_- onto $L_2((-\infty, -\rho); N)$.*

Next we study the action of S in this representation. Denote MSM^* by S_M. Combining Theorem 5.1 and Theorem 5.2 we get

THEOREM 5.4. (a) *S_M is a unitary map of $L_2(R; N)$ onto $L_2(R; N)$.*
(b) *S_M commutes with translation.*
(c) *S_M maps $L_2(-\infty, -\rho)$ into $L_2(-\infty, \rho)$.*
Using (b) *we can restate* (c) *in the following form:*
(c)$'$ *$T(-2\rho)S_M$ maps $L_2(R_-; N)$ into $L_2(R_-; N)$, where $T(a)$ denotes translation to the right by a units.*

According to a semiclassical theorem, *an operator S_M which commutes with translation is convolution:*

$$(S_M h)(s) = \int S(r)h(s-r)dr. \tag{5.5}$$

where $S(r)$ *is an operator valued* $(N \to N)$ *distribution on* R.

$T(-2\rho)S_M$ is convolution with $S(r + 2\rho)$; according to part (c)$'$ of Theorem 5.4, $T(-2\rho)S_M$ maps functions supported on R_- into functions supported on R_-. According to a semiclassical corollary, this implies that $S(r + 2\rho)$ has its support on the negative axis; thus

$$S(s) = 0 \quad \text{for } s > 2\rho \quad . \tag{5.6}$$

We shall not prove precisely these statements; a rigorous proof can be found in Chapter 2 of [3]. Rather we pass to the spectral representation, obtained from the translation representation by Fourier transformation. We denote the dual variable by σ, and the Fourier transform of $L_2(R_+, N)$ by $A_+(N)$:

$$A_+(N) = \mathcal{F}L_2(R_+, N). \tag{5.7}$$

In what follows we shall use the vector version of the Paley-Wiener theorem, which gives an intrinsic characterization of A_+:

PALEY-WIENER THEOREM. *$A_+(N)$ consists of N-valued analytic* *functions in the upper half-plane whose square integral along lines of constant imaginary part is bounded.*

The proof of the vectorial case is the same as that of the scalar case. As a consequence *multiplication by any scalar function f analytic and bounded in the upper half-plane maps* A_+ *into itself.*

We shall denote by $\mathcal{S}$ the action of the scattering operator in this spectral representation. Fourier transforming Theorem 5.4 yields

THEOREM 5.5. (a) $\mathcal{S}$ *is a unitary map of* $L_2(R, N)$ *onto* $L_2(R, N)$.

(b) $\mathcal{S}$ *commutes with multiplication by* $e^{i\sigma p}$, *p any real number.*

(c) $e^{-2i\rho\sigma}\mathcal{S}$ *maps* A_- *into* A_-.

It follows from (b) *that* $\mathcal{S}$ *commutes with multiplication by linear combination of imaginary exponentials. Since every bounded measurable scalar function can be approximated boundedly a.e. by such linear combinations, we deduce*

(b)′ $\mathcal{S}$ *commutes with multiplication by any bounded scalar function.*

THEOREM 5.6. *Every operator* $\mathcal{S}$ *which has the three properties listed in Theorem* 5.5 *is multiplication by an operator valued function* $\mathcal{S}(\sigma)$ *with the following properties*:

(a) $\mathcal{S}(\sigma)$ *is unitary for* σ *real.*

(b) $\mathcal{S}(\sigma)$ *is analytic in lower half-plane.*

(c)

$$|\mathcal{S}(z)| \leqq e^{2\rho|\mathrm{Im}\, z|} \tag{5.8}$$

where $|\cdot|$ *denotes the operator norm of* $\mathcal{S}(z){:}N \to N$.

This is a simple case of a more general theorem of Segal and Fourés: Trans. Amer. Math. Soc. 78(1955), 385-405; see also Chapter 2 of [3].

The unitary character of $\mathcal{S}(\sigma)$ for σ real can be used to continue $\mathcal{S}$ analytically to the upper half-plane:

For Im $z > 0$, $\mathcal{S}(z) = (\mathcal{S}^*(\bar{z}))^{-1}$. As this formula shows, if $\mathcal{S}(z)$ has a zero at some point z_0 in the lower half-plane (i.e., is not invertible there), its analytic continuation has a singularity at $\bar{z}_0$. In the next section we will show that the zeros of $\mathcal{S}$ in the lower half-plane are isolated; it is not hard to deduce from this that the singularities of its analytic continuation to the upper half-plane are poles. Furthermore we will explain the significance of these poles for the asymptotic description for large t of solutions $u(x, t)$ of (1).

6. **The associated semigroup.** We begin by obtaining the incoming and outgoing translation representations for the perturbed group $U(t)$ from the corresponding representation for $V(t)$, namely M: $H \to$

$L_2(R; N)$, and the wave operators $W_\pm$.

Recall that

$$MV(t) = T(t)M. \tag{6.1}$$

Hence setting

$$M_\pm = MW_\pm^* \tag{6.2}$$

and making use of the intertwining property of the wave operators, we have

$$M_\pm U(t) = MW_\pm^* U(t) = MV(t)W_\pm^* = T(t)MW_\pm^* = T(t)M_\pm. \tag{6.3}$$

The previously mentioned properties of M now imply

THEOREM 6.1. (a) M_+ *is unitary,*
(b) M_+ *is a translation representation for* $U(t)$,
(c) M_+ *maps* D_+^ρ *onto* $L_2(\rho, \infty; N)$,
(d) M_+ *maps* D_-^ρ *into* $L_2(-\infty, \rho; N)$; *in fact*

$$M_+ D_-^\rho = S_M L_2(-\infty, -\rho; N) . \tag{6.4}$$

PROOF. Property (a) follows trivially from the fact that M_+ is the product of two unitary operators; (b) from (6.3); and (c) from the fact that W_+ and hence W_+^* when restricted to D_+^ρ is the identity. To prove (d) we use the analogous property of W_-, namely that W_- when restricted to D_-^ρ is the identity. Since D_-^ρ fills out $L_2(-\infty, -\rho; N)$ in the M representation we have for f in D_-^ρ and hence for any $k = Mf$ in $L_2(-\infty, -\rho; N)$

$$M_+ f = MW_+^* W_- f = MSM^* Mf = S_M k .$$

The operator M_- has analogous properties; the representations given by M_- and M_+ are called the incoming and outgoing translation representations of $U(t)$. Finally we note that the M-translation representation of S takes the form

$$S_M = MSM^* = MW_+^* W_- M^* = M_+ M_-^* . \tag{6.5}$$

Thus S_M can be defined intrinsically in terms of the pair of translation representations M_- and M_+ without reference to wave operators.

Taking Fourier transforms we obtain the incoming and outgoing spectral representations $\mathcal{M}_-$ and $\mathcal{M}_+$, respectively, onto $L_2(R, N)$ and from Theorem 6.1 we see for instance that $\mathcal{M}_+$ maps D_+^ρ onto $e^{i\rho\sigma} A_+(N)$.

We turn now to the family of operators already introduced in §1 as a very useful tool for studying the properties of $\mathcal{S}$:

$$Z(t) = P_+^\rho U(t) P_-^\rho , \qquad t \geqq 0. \tag{6.6}$$

Here P_+^ρ and P_-^ρ denote the orthogonal projections onto $D_+^{\rho\perp}$ and $D_-^{\rho\perp}$, respectively. The operator P_-^ρ removes that component of the signal which might be coming in from very far away and which would not be scattered for a long time; the factor P_+^ρ removes all components which cannot undergo any further scattering. Thus $Z(t)$ retains all of the interesting features of $U(t)$, while at the same time neglecting data which is unessential to the scattering process.

THEOREM 6.2. (a) *For* $f \perp D_-^\rho$

$$Z(t)f = U(t)f \tag{6.7}$$

for $|x| < \rho$.

(b) $Z(t)$ *annihilates* D_+^ρ *and* D_-^ρ *and maps* H *into*

$$K^\rho = H \ominus (D_+^\rho \oplus D_-^\rho). \tag{6.8}$$

(c) $Z(t)$ *forms a one-parameter semigroup over* K^ρ.

(d) $\|Z(t)\| \leqq 1$.

PROOF. Since f is orthogonal to D_-^ρ, $P_-^\rho f = f$. Likewise, since elements of D_+^ρ vanish in $|x| \leqq \rho$, for any g, $P_+^\rho g = g$ for $|x| < \rho$; this proves part (a).

We turn to (b); clearly

$$Z(t)f = P_+^\rho U(t) P_-^\rho f$$

annihilates every f in D_-^ρ, since P_-^ρ does. For f in D_+^ρ, $P_-^\rho f = f$, $U(t)$ maps D_+^ρ into D_+^ρ, and the resulting element is annihilated by P_+^ρ.

To show that Z maps H into K^ρ we have to verify that Zf is orthogonal to both D_+^ρ and D_-^ρ. We have

$$(Zf, g) = (P_+ U P_- f, g) = (UP_- f, P_+ g); \tag{6.9}$$

if g belongs to D_+^ρ, $P_+ g = 0$ so (6.9) is zero. Suppose g is in D_-^ρ; then $P_+ g = g$ and so we have from (6.9) that

$$(Zf, g) = (UP_- f, g) = (f, P_- U^* g) . \tag{6.10}$$

Since U forms a unitary group, $U^*(t) = U(-t)$; recall that $U(-t)D_-^\rho \subset D_-^\rho$ for $t > 0$ and so $P_- U^* g = 0$ for g in D_-^ρ. This shows that (6.10) is zero for g in D_- and completes the proof of part (b).

(c) To show the semigroup property over K^ρ, we note that for f in K^ρ

$$Z(0)f = P_+ P_- f = f \quad .$$

Next we look at

$$Z(t)Z(s)f = P_+ U(t) P_- P_+ U(S) P_- f \quad .$$

Since we have shown that the range of $Z(s)$ is in K^{ρ}, we can omit P_- in the middle and write

$$Z(t)Z(s)f = P_+[U(t)\{U(S)P_-f + g\}]$$

where g is some element of D_+^{ρ}. Since for $t > 0$, $U(t)$ maps D_+ into itself we conclude that $P_+U(t)g = 0$, and so

$$Z(t)Z(s)f = P_+U(t)U(s)P_-f = Z(t+s)f$$

which is the semigroup property.

REMARK. Our proof shows that

$$Z(t)U(s)f = Z(t+s)f \quad \text{for } f \text{ in } K^{\rho};$$

we will have occasion to use this later on.

Part (d) is obvious; Z, being the product of three operators, each of which has norm 1, has itself norm $\leqq 1$.

REMARK. Since $P_\pm$ are orthogonal projections, $\|P_\pm f\| = \|f\|$ implies $P_\pm f = f$. It follows from this that $\|Z(t)f\| = \|f\|$ implies $Z(t)f = U(t)f$.

This concludes the proof of Theorem 6.2.

Since $\{Z(t),\ t \geqq 0\}$ is a semigroup of operators with norm $\leqq 1$, it has, according to the Hille-Yosida theorem, a densely defined infinitesimal generator B. Symbolically, $Z(t) = \exp Bt$. The spectrum of B is confined to the half-plane of complex numbers with nonpositive real part; every z with $\operatorname{Re} z > 0$ belongs to the resolvent set, the resolvent being given by the Laplace transform

$$(z-B)^{-1} = \int_0^\infty Z(t)e^{-zt}dt. \tag{6.11}$$

THEOREM 6.3. *The spectrum of B is a discrete point spectrum in* $\operatorname{Re} \lambda < 0$.

We shall prove that the operator $(z-B)^{-1}Z(2\rho)$ is compact; this implies that $(z-B)^{-1}Z(2\rho)$ has a discrete point spectrum, from which it follows by the known functional calculus for semigroups, see Chapter 3 of [3], that so does B.

Again we make use of the fact that $V(2\rho)$ maps $D_-^{\rho\perp}$ into D_+^{ρ}; this shows that $P_+V(2\rho)P_- = 0$. Subtracting this from the definition $P_+U(2\rho)P_- = Z(2\rho)$ we get, using the abbreviation

$$U(2\rho) - V(2\rho) = X \tag{6.12}$$

that

$$P_+XP_- = Z(2\rho) \quad . \tag{6.13}$$

We turn now to studying

$$(6.14) \qquad (z-B)^{-1}Z(2\rho) = \int_0^\infty Z(2\rho)Z(t)e^{-zt}dt\ .$$

As we have remarked in the course of proving the semigroup property of Z,

$$Z(2\rho)Z(t) = Z(2\rho)U(t) \quad \text{on } K^\rho\ .$$

Using this relation and (6.13) we can write the right side of (6.14) as

$$(6.14)' \qquad P_+XP_-\int_0^\infty U(t)e^{-zt}dt\ .$$

Since $U(t)$ maps $D_-^{\rho\perp}$ into itself for $t > 0$, P_- can be omitted. Next we use the resolvent relation for G:

$$(6.15) \qquad \int_0^\infty U(t)e^{-zt}dt = (z-G)^{-1}$$

and appeal to a domain of dependence argument according to which the value of Xf does not depend on values of f in $|x| > \rho + 4c\rho$. This can be expressed as (see (4.6)) $X = XQ_{\rho+4c\rho}$ where the projection Q_p is multiplication by the characteristic function of $|x| \leqq p$. Using the above identity for X, and (6.15), we can rewrite (6.14)′ as

$$(6.14)'' \qquad P_+XQ_{\rho+4c\rho}(z - G)^{-1}\ .$$

We claim that this operator is compact; for $(z-G)^{-1}$ maps the unit sphere in H into a set of functions $\{g\}$ with the property that $\|g\|$ and $\|Gg\|$ are uniformly bounded. Since G is an elliptic operator, it follows from standard elliptic estimates that for such a set of functions $\{g\}$, $\|\partial_x g\|$ is uniformly bounded, and so by Rellich's criterion this set of functions is precompact in the L_2 norm over the ball of radius $\rho + 4c\rho$. This shows that (6.14)″ maps the unit ball in H into a precompact set, and completes the proof of the discreteness of $\sigma(B)$.

Next we show that the spectrum of B lies in $\operatorname{Re}\lambda < 0$. For suppose that λ were an eigenvalue of B with $\operatorname{Re}\lambda = 0$; denote by f the corresponding eigenfunction. Then $Z(t)f = \exp(\lambda t)f$; this shows that $Z(t)$ preserves the norm of f. From the definition of Z as P_+UP_- it follows that if $Z(t)$ preserves the norm of f, then $Z(t)f = U(t)f$. Therefore the above relation implies that

$$U(t)f = \exp(\lambda t)f\ .$$

Differentiating with respect to t we get $Gf = \lambda f$; but according to Theorem 3.1 the only point eigenvalue of G is $\lambda = 0$ and we have assumed that $\lambda = 0$ was not an eigenvalue. This completes the proof of Theorem 6.3.

REMARK. In case $\lambda = 0$ were an eigenvalue for G we would have limited our considerations to $H' = H \ominus H_0$, where H_0 denotes the null space of G. As can be readily verified $D_\pm^\rho$ are contained in H' so that the theory goes through as before if we now take K^ρ equal to $H' \ominus (D_+^\rho \oplus D_-^\rho)$.

THEOREM 6.4. *λ is an eigenvalue of B iff $z = i\bar{\lambda}$ is a zero of the scattering matrix $\mathcal{S}^*(z)$.*

PROOF. Consider $Z_+(t) = M_+Z(t)M_+^{-1}$, i.e., the action of Z in the outgoing translation representation for U. Omitting P_-, which acts as the identity on K^ρ, we have

$$Z_+(t) = M_+P_+M_+^{-1}M_+U(t)M_+^{-1} = Q_\rho T(t) \tag{6.16}$$

where T is translation and Q_ρ is orthogonal projection onto $L_2(-\infty, \rho)$.

Next we determine the image K_+ of K^ρ under M_+ : using parts (c) and (d) of Theorem 4.1 we have

$$\begin{aligned} K_+ &= M_+(H \ominus D_+^\rho \ominus D_-^\rho) = M_+(H \ominus D_+^\rho) \ominus M_+D_-^\rho \\ &= L_2(-\infty, \rho) \ominus S_M L_2(-\infty, -\rho). \end{aligned} \tag{6.17}$$

REMARK. From (6.16) and (6.17) it is easy to devise a new proof of Theorem 6.2.

Let $h(s)$ in K_+ be an eigenfunction of B_+ with eigenvalue λ; then h is an eigenfunction of $Z_+(t)$ with eigenvalue $e^{\lambda t}$; by (6.16) this means

$$h(s-t) = e^{\lambda t}h(s), \qquad s \leqq \rho .$$

Setting $s = \rho$ we get, with $r = \rho - t$

$$\begin{aligned} h(r) &= e^{-\lambda r}n \quad \text{on} \quad (-\infty, \rho) , \\ &= 0 \qquad\quad \text{on} \quad (\rho, \infty) , \end{aligned} \tag{6.18}$$

where $n = e^{\lambda\rho}h(0)$ is some vector in N.

According to (6.17), h belongs to K_+ iff it is orthogonal to $S_M L_2(-\infty, -\rho)$:

$$0 = (h, S_M k) \tag{6.19}$$

for all k in $L_2(-\infty, -\rho)$. Using Parseval's theorem we write (6.19) as

$$(\mathcal{S}\tilde{k}, \tilde{h}) = 0 \tag{6.20}$$

where $\tilde{h}, \tilde{k}$ denote the Fourier transforms of h and k.

The Fourier transform of h can be calculated explicitly from (6.18):

$$\tilde{h}(\sigma) = \int_{-\infty}^{\infty} h(r)e^{ir\sigma}dr = -\frac{e^{-\rho(\lambda - i\sigma)}}{\lambda - i\sigma}n \quad .$$

Substituting this into (6.20) we get

$$\int_{-\infty}^{\infty} [\mathcal{S}(\sigma)\tilde{k}(\sigma),n] \frac{e^{-\rho(\bar{\lambda}+i\sigma)}}{\bar{\lambda} + i\sigma} d\sigma = 0 \ , \tag{6.21}$$

where [,] denotes the scalar product in N. According to Theorem 5.6, $\mathcal{S}(z)$ is analytic in the lower half-plane and grows there at most like $e^{2\rho|\operatorname{Im} z|}$.

$$\tilde{k}(\sigma) = \int_{-\infty}^{-\rho} k(s)\, e^{is\sigma}\, ds$$

also is analytic in the lower half-plane and its L_2 norm along $\operatorname{Im} z =$ const decreases like $O(e^{-\rho|\operatorname{Im} z|})$. It follows by the Paley-Wiener theorem that $[\mathcal{S}(\sigma)\tilde{k}(\sigma)\, n]\ e^{-i\rho\sigma}$ belongs to A_-.

Now we shift the line of integration in (6.21) from the real axis to the line $z = \sigma - iR$; since the L_2 norm along $\operatorname{Im}\, z = -R$ of $[\mathcal{S}(z)\tilde{k}(z), n]\, e^{-\rho i z}$ remains bounded, the resulting integral tends to 0 as $R \to \infty$. Since the integrand has a simple pole at $z = i\bar{\lambda}$ we conclude that the residue there must be zero:

$$[\mathcal{S}(i\bar{\lambda})\tilde{k}(i\bar{\lambda}), n] = 0 \ .$$

Finally we note that $\tilde{k}(i\bar{\lambda})$ may be any element of N, and therefore $\mathcal{S}^*(i\bar{\lambda})n = 0$. The converse assertion follows by reversing the above argument. This completes the proof of Theorem 6.4.

7. **Energy decay.** In what follows we shall investigate the *rate of decay* of the energy contained in bounded sets, and the uniformity of the decay with respect to f. These turn out to hinge on the behavior of the *bicharacteristic rays* associated with the equation (1). Rays are solutions of the Hamiltonian system of ordinary differential equations

$$\frac{d}{dt} x_i = \tau_{p_i}, \qquad \frac{d}{dt} p_i = -\tau_{x_i} \ , \tag{7.1}$$

where $\tau = \tau(x, p)$ is an eigenvalue of $A(x, p) = \Sigma A^i(x)p_i$, and where subscripts refer to partial differentiation. There are $n/2$ families of eigenvalues: consequently there are $n/2$ Hamiltonian systems (7.1).

We say that the *rays of the operator* $\partial_t - \Sigma A^i\partial_i - B$ *tend to infinity* if for each solution of (7.1) $m = 1, \ldots, n/2$, $|x(t)| \to \infty$ as $t \to \infty$. More specifically we require that there exist a time T called the *sojourn time* such that each ray which at time $t = 0$ lies *inside the sphere* $\{|x| = \rho\}$ lies for $t \geqq T$ *outside* $\{|x| = \rho\}$.

REMARK. Since outside $\{|x| = \rho\}$ the rays are straight lines propagating with speed $\geqq 1$, it follows that for any values of a and $b > \rho$, any ray which at time $t = 0$ lies inside $\{|x| = a\}$ will be outside $|x| = b$ for $t > T + a + b$.

THEOREM 7.1. *If the rays of* (1) *tend to infinity, then* (a) *for any* $f \perp H_0$

$$\|Q_\rho U(t)f\| \leqq Ae^{-dt}\|f\| \tag{7.2}$$

where d and A are positive constants independent of f.

(b) *The nonzero eigenvalues $\{\lambda_k\}$ of B can be arranged so that*

$$0 > \operatorname{Re}\lambda_1 \geqq \operatorname{Re}\lambda_2 \geqq \cdots, \tag{7.3}$$

with $\operatorname{Re}\lambda_k \to -\infty$.

(c) *The asymptotic expansion*

$$U(t)f \sim \sum_1^\infty a_k e^{\lambda_k t} w_k(x) \quad , \tag{7.4}$$

where w_k are eigenfunctions of B, holds uniformly for $|x| < \rho$ for every $f \perp H_0 + D_-^\rho$.

(d) *The eigenvalues λ of B satisfy*

$$\operatorname{Re}\lambda \leqq a - b\log|\lambda|, \qquad b > 0. \tag{7.5}$$

The proof is based on

THEOREM 7.2. *If the rays tend to ∞, $Z(t_0)$ is compact for $t_0 > T + 6c\rho + 6\rho$ where T is the sojourn time. Furthermore the range of $Z(t_0)$ lies in the domain of B, and $BZ(t_0)$ is bounded.*

We show how Theorem 7.2 implies Theorem 7.1. According to the spectral theory of semigroups, if $Z(t_0)$ is compact

$$\sigma(Z(t_0)) = \exp(t_0\sigma(B)); \tag{7.5$'$}$$

here $\sigma(B)$ denotes the spectrum of the operator B. Since $Z(t_0)$ is compact, its spectrum accumulates only at 0; it follows then from (7.5)$'$ that the real part of the spectrum of B accumulates only at $-\infty$. Since according to Theorem 6.3 the nonzero eigenvalues of B have negative real parts, this proves (7.3) and part (b).

As we have shown in the proof of Theorem 6.3, the null space of B is H_0. Since $\|Z\| \leqq 1$, it follows that 0 is a simple eigenvalue. It follows from this that the spectrum of B on $H \ominus H_0$ does not include 0; from this and (7.5)$'$ it follows that $\sigma(Z(t_0)) = \{\exp(t_0\lambda_k)\}$ where the numbers λ_k satisfy (7.3). This implies

$$|\sigma(Z(t_0))| = \exp(t_0 \mathrm{Re}\,\lambda_1) < 1 ,$$

i.e., that the spectral radius of $Z(t_0)$ is < 1.

According to the Gelfand formula for the spectral radius

$$\lim \|Z^n(t_0)\|^{1/n} = \exp(t_0 \mathrm{Re}\,\lambda_1). \tag{7.6}$$

By the semigroup property, for $t = nt_0$, $Z(t) = Z^n(t_0)$, so it follows from (7.6) that for any $\epsilon > 0$ and for t large enough

$$\|Z(t)\| < \exp(t(\mathrm{Re}\,\lambda_1 + \epsilon)). \tag{7.6$'$}$$

Since elements of D_+^ρ are zero for $|x| < \rho$, it follows, as already stated in Theorem 6.2, that $U(t)f = Z(t)f$ for $|x| \leqq \rho$ and $f \perp D_-^\rho$. Therefore $\|Q_\rho U(t)\| \leqq \|Z(t)\|$. Combining this with (7.6)$'$ we deduce (7.2).

The seemingly stronger conclusion (7.4) follows by a similar argument. We denote by $K^{(N)}$ the subspace of K^ρ consisting of all functions which are orthogonal to the eigenfunctions of B^* corresponding to the first $N-1$ eigenvalues of B. The spectral radius of $Z(t_0)$ restricted to $K^{(N)}$ is $\exp(t_0 \mathrm{Re}\,\lambda_N)$, and applying the Gelfand formula gives an estimate analogous to (7.6)$'$:

$$\|Z(t) r^{(N)}\| < \exp(t(\mathrm{Re}\,\lambda_N + \epsilon))\|r^{(N)}\| \tag{7.6$''$}$$

for every $r^{(N)}$ in $K^{(N)}$.

Denote by P_N the spectral projection onto the eigenspaces associated with $\lambda_0, \ldots, \lambda_{N-1}$; for any f, $P_N f$ is of the form

$$P_N f = \sum_0^{N-1} a_k w_k$$

and $f - P_N f = r^{(N)}$ belongs to $K^{(N)}$. So

$$\begin{aligned} Z(t)f &= Z(t)P_N f + Z(t)r^N \\ &= \sum a_k e^{\lambda_k t} w_k + Z(t) r^N. \end{aligned}$$

The estimate (7.6)$''$ proves the asymptotic character of this series, as asserted in (7.4). Further results of this kind are contained in [6].

Remark. If λ_k is an eigenvalue of index > 1, the kth term in (7.4) is an exponential polynomial rather than an exponential.

We turn next to part (d); if λ is an eigenvalue of B then $\lambda \exp(\lambda t)$ is an eigenvalue of $BZ(t)$. Denote by c the norm of $BZ(t)$; since this is a bound for the spectral radius we have $|\lambda \exp(\lambda t)| \leqq c$ which implies

$$\mathrm{Re}\,\lambda \leqq \log c - \frac{1}{t}\log|\lambda| ,$$

as asserted in (7.5).

We return now to the proof of Theorem 7.2. We write

$$Z(t) = P_+U(t)P_- = P_+U(2\rho)U(t-4\rho)U(2\rho)P_- \; . \tag{7.7}$$

It is clear that $V(2\rho)$ maps the orthogonal complement of D_-^ρ into D_+^ρ; using this fact we can subtract $V(2\rho)$ from $U(2\rho)$ on the right in (7.7) and get

$$\begin{aligned} Z(t) &= P_+[U(2\rho) - V(2\rho)]\,U(t-4\rho)[U(2\rho) - V(2\rho)]\,P_- \\ &= P_+XU(t-4\rho)XP_-, \end{aligned}$$

where $X = U(2\rho) - V(2\rho)$. We use now the identities (4.5) and (4.6) of Lemma 4.4 derived in §4 for X; we get

$$Z(t) = P_+XQ_{\rho\ +4c\rho}\ U(t-4\rho)Q_{\rho\ +2c\rho}\ XP_- \; .$$

We claim that the product in the middle $Q_aU(s)Q_b$ is compact for $s > T + a + b$; clearly this implies the compactness of $Z(t)$ for $t > T + 6c\rho + 6\rho$.

$U(s)$ can be represented as an integral operator:

$$[U(s)g]\,(y) = \int R(y, z; s)g(z)dz$$

where the kernel R is the Riemann function. It is known, see Ludwig [8], that singularities of solutions of hyperbolic equations propagate along characteristics; this implies that $R(y, z; s)$ is a C^∞ function of y, z at all pairs except when $y = x(s)$, $z = x(0)$, where $x(t)$ denotes any bicharacteristic. Since we have assumed that all bicharacteristics starting inside $|x| = b$ lie at time $> T + a + b$ outside $|x| = a$, it follows that for $s > T + a + b$ the operator has, except for jump discontinuities along $|y| = a$ and $|z| = b$, a C^∞ kernel and so is a compact operator. This completes the proof of the first part of Theorem 7.2.

To prove the second part of Theorem 7.2 we observe that in the identities (4.5) and (4.6) of Lemma 4.4 we may replace the operator Q_p by the operator Q_p^ϵ defined as multiplication by a C_0^∞ function $\zeta(x)$ which equals 1 for $|x| \leqq p$ and equals zero for $|x| > p + \epsilon$. We have analogously to (7.7)

$$Z(t) = P_+XCXP_- \tag{7.7$'$}$$

where

$$C(t) = Q^\epsilon_{\rho+4c\rho}\ U(t-4\rho)\ Q^\epsilon_{\rho+2c\rho} \quad .$$

This operator has for $t > T + 6c\rho + 6\rho$ a C_0^∞ kernel and therefore it is differentiable with respect to t in the uniform topology.

We claim that for $t > T + 6c\rho + 6\rho$ *the range of* $Z(t)$ *is contained in the domain of* B, *and* $BZ(t)$ *is a bounded operator.* To see this we write

$$BZ(t) = \lim \frac{Z(t+h) - Z(t)}{h}$$

$$= \lim P_+ X \frac{C(t+h) - C(t)}{h} XP_-$$

$$= P_+ XC_t XP_- \quad .$$

Since C_t is a bounded operator, this proves the boundedness of $BZ(t)$; thus the proof of Theorem 7.2 is complete.

Theorem 7.1 is sharp, i.e., the hypothesis concerning the rays cannot be omitted:

THEOREM 7.3. *If the bicharacteristics do not tend to* ∞, *i.e., if there are bicharacteristics which stay inside* $|x| \leqq \rho$ *for an arbitrary length of time, then* $\|Z(t)\| = 1$ *for all* t.

COROLLARY. $Z(t)$ *is not compact for any value of* t.

To prove this one would have to construct highly oscillatory solutions of (1) which follow a given ray; for particulars see Ralston, [13].

To conclude we give an example of a simple system where the bicharacteristics do not tend to ∞. Take

$$\tau(x, p) = |x|\,|p| \ ,$$

where

$$|x|^2 = \sum x_i^2, \qquad |p|^2 = \sum p_i^2 \ .$$

The ray equations (7.1) are

$$\frac{dx}{dt} = \frac{|x|}{|p|}p, \qquad \frac{dp}{dt} = -\frac{|p|}{|x|}x \ . \tag{7.8}$$

We compute now the t-derivatives of $x \cdot p$ and of $|x|^2$; using (7.8) we get

$$\frac{d}{dt} x \cdot p = \frac{dx}{dt} \cdot p + x \cdot \frac{dp}{dt} = 0 \ , \tag{7.9}$$

$$\frac{1}{2}\frac{d|x|^2}{dt} = x \cdot \frac{dx}{dt} = \frac{|x|}{|p|} x \cdot p \ . \tag{7.10}$$

If at $t = 0$ the variables x,p are chosen to be orthogonal then (7.9) shows that they remain orthogonal and (7.10) shows that $|x|$ remains constant; thus it does not tend to ∞.

On the other hand it is easy to show that if τ does not vary too fast with x, and if ρ is not too large then all rays tend to infinity. Suppose that

$$(7.11) \qquad |\tau_x| \leqq \epsilon |p| \ .$$

From the second ray equations (7.1) we have therefore

$$\left|\frac{dp}{dt}\right| = |\tau_x| \leqq \epsilon |p|$$

which implies that

$$(7.12) \qquad |p(T)| \leqq |p(0)| e^{\epsilon T} \ .$$

In what follows we shall choose

$$(7.13) \qquad p(0) = \omega, \qquad |\omega| = 1 \ .$$

Using (7.12) we get

$$(7.14) \quad |p(T) - \omega| = \left| \int_0^T \frac{dp}{dt}\, dt \right| \leqq \epsilon \int_0^T |p|\, dt \leqq (e^{\epsilon T} - 1) \ .$$

Using the first ray equation we get

$$(7.15) \qquad \omega\, dx/dt = \omega\tau_p = p\tau_p + (\omega - p)\tau_p \ .$$

Since τ is homogeneous of degree 1 in p

$$p\tau_p = \tau \quad \text{and} \quad |\tau_p| \leqq k \ .$$

Furthermore we have assumed that $\tau \geqq c_{\min}|p| \geqq |p|$.

Using these estimates on the right side in (7.15) we get

$$\omega\, dx/dt \geqq |p| - k|\omega - p| \geqq 1 - (k+1)\, |\omega - p| \ .$$

Using (7.14) we get

$$\omega\, dx/dt \geqq 1 - (k+1)(e^{\epsilon t} - 1) \ .$$

We integrate this with respect to t; since $x(0)$ lies in $|x| \leqq \rho$, $|\omega \cdot x(0)| \leqq \rho$ and we get

$$(7.16) \qquad \begin{aligned} \omega \cdot x(T) &\geqq -\rho + T - (k+1)\,[e^{\epsilon T} - 1 - \epsilon T]/\epsilon \\ &= -\rho + \frac{1}{\epsilon} a(\epsilon T) \end{aligned}$$

where

$$(7.17) \qquad a(s) = s - (k+1)(e^s - 1 - s) \ .$$

Clearly, $a(s)$ is positive for s small; this guarantees that

$$(7.17)' \qquad M = \operatorname*{Max}_{s>0} a(s)$$

is positive. If s_M is the value where the maximum is achieved, then for $T = s_M/\epsilon$ we have from (7.16) that $\omega \cdot x(T) \geqq -\rho + M/\epsilon$; if

$$\rho\epsilon < M/2 \tag{7.18}$$

we have

$$\omega \cdot x(T) \geqq -\rho + 2\rho > \rho$$

which shows that $x(T)$ has gotten out of the ball $|x| \leqq \rho$ and is on its way to ∞. This proves

THEOREM 7.4. *If* (7.11) *and* (7.18) *are satisfied, then all rays tend to* ∞.

8. **Explicit form of the spectral representation and the scattering matrix.** In §6 we have proved the existence of two unitary spectral representations $\mathcal{M}_+$ and $\mathcal{M}_-$ for the group U, which map H onto $L_2(-\infty, \infty; N)$ with the properties that $\mathcal{M}_+$ maps D_+^ρ onto $e^{i\rho\sigma}A_+(N)$, and $\mathcal{M}_-$ maps D_-^ρ onto $e^{-i\rho\sigma}A_-(N)$. In this section we shall display a fairly explicit analytic form for $\mathcal{M}_+$ and $\mathcal{M}_-$.

We start with the observation that the spectral representation $\mathcal{M}$ for the unperturbed group V, which is both incoming and outgoing, is essentially the Fourier transformation, suitably adjusted to take care of the dependence of the propagation speeds on direction. Denoting $\mathcal{M}f$ by $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{n/2})$ we have

$$\tilde{f}(\sigma, \omega) = (f, \varphi(\sigma, \omega)), \tag{8.1}$$

where φ is the $(1 \times n/2)$-matrix function whose jth column is

$$\varphi(x; \sigma, \omega, j) = \sigma^{-1/2}\left(\frac{\sigma}{2\pi\tau_j(\omega)}\right)^{k/2} \exp\left(\frac{-i\sigma x \cdot \omega}{\tau_j(\omega)}\right) r_j(\omega) \quad . \tag{8.2}$$

Observe that φ satisfies $G_0\varphi = i\sigma\varphi$, i.e., φ is a generalized eigenfunction of G_0.

Denote $\mathcal{M}_\pm f$ by $\tilde{f}_\pm$; we expect $\tilde{f}_\pm$ to be given by similar formulas:

$$\tilde{f}_\pm(\sigma, \omega) = f, \varphi_\mp(\sigma, \omega) \ , \tag{8.3$_\pm$}$$

where $\varphi_\pm$ are generalized eigenfunctions of G:

$$G\varphi_\pm = i\sigma\varphi_\pm \ . \tag{8.4}$$

We further expect $\varphi_\pm$ not to differ too much from φ:

$$\varphi_\pm = \varphi + v_\pm \ . \tag{8.5}$$

Our task now is to determine the correction terms $v_\pm$; clearly they must satisfy the eigenvalue equation (8.4) so that

$$(G - i\sigma)v_\pm = -(G - i\sigma)\varphi \ . \tag{8.6}$$

From our construction of $\mathcal{M}_\pm$ from $\mathcal{M}$ we know that $\mathcal{M}$ agrees with $\mathcal{M}_+$ on D_+^ρ, and with $\mathcal{M}_-$ on D_-^ρ; this implies that

$$(D^{\rho}_{\mp}, v_{\pm}) = 0 \; . \tag{8.7}$$

So far this discussion has been heuristic; next we sketch a rigorous proof.

THEOREM 8.1. *The spectral representation theorem:*
(A) *Equation* (8.6) *has a pair of solutions* $v_{\pm}$ *which satisfy* (8.7).
(B) *These solutions are locally* L_2.
(C) $\tilde{f}_{\pm}$, *defined by* (8.3) *where* $\varphi_{\pm}$ *is given by* (8.5), φ *by* (8.2) *and* $v_{\pm}$ *are as constructed, is an outgoing, respectively incoming spectral representation for* U.

As a first step we prove an auxiliary result on the analytic continuation of the resolvent of G.

THEOREM 8.2. *For every* g *with support in* $|x| < \rho$, $v = (G-\lambda)^{-1}g$ *can be continued analytically from the half-plane* $\operatorname{Re} \lambda > 0$ *into the complement of* $\sigma(B)$. *The analytic continuation has these properties*
(a) v *is locally* L_2,
(b) v *satisfies the differential equation*

$$(G - \lambda)v = g, \tag{8.8}$$

(c) v *is orthogonal to those elements of* D^{ρ}_{-} *which have compact support.*

PROOF. Recall that the semigroup $Z(t)$ was defined as $P_+U(t)P_-$; the resolvent of its infinitesimal generator B is given by the Laplace transform

$$(\lambda - B)^{-1} = \int_0^\infty Z(t)e^{-\lambda t}dt = P_+\int_0^\infty U(t)e^{-\lambda t}dt\, P_- = P_+(\lambda - G)^{-1}P_- \; .$$

Let f be any element of H with support in $|x| < \rho$; then $P_+f = f$, and since $P_-g = g$ we have

$$\begin{aligned}((B-\lambda)^{-1}g,f) &= (P_+(G-\lambda)^{-1}P_-g,f) = ((G-\lambda)^{-1}P_-g, P_+f) \\ &= ((G-\lambda)^{-1}g,f) \, .\end{aligned}$$

Since the left side can be continued analytically into the resolvent set of B, so can the right side. The analytic continuation is, for each $\lambda \notin \sigma(B)$, a linear functional of f, bounded by const $\|f\|$, where const $= \|(B-\lambda)^{-1}\|$. Therefore by the Riesz representation theorem for each $\lambda \notin \sigma(B)$ there exists a square integrable function v such that in $|x| < \rho$ $(G - \lambda)^{-1}g = v$. Since ρ is arbitrary, we see that v can be defined in all of R^k and is locally L_2. As a function of λ, v is weakly locally analytic outside $\sigma(B)$. This completes the proof of part (a).

Next we show that v satisfies the differential equation (8.8) in the distribution sense, i.e. that for all f in C_0^∞, $(v,(-G-\bar{\lambda})f) = (g, f)$. This is certainly true for Re $\lambda > 0$, and therefore continues to hold under analytic continuation. We complete the proof of (b) by remarking that if g is a C_0^∞ function, then it follows from elliptic theory that the distribution solution v of (8.8) is actually a C^∞ solution.

We turn to part (c); since g is zero for $|x| > \rho$, certainly g is orthogonal to D_-^ρ, and since for $t > 0$ $U(-t)$ maps D_-^ρ into D_-^ρ, $U(t)g \perp D_-^\rho$ for $t > 0$. It follows then that for Re $\lambda > 0$

$$(\lambda - G)^{-1} g = \int_0^\infty e^{-\lambda t} U(t) g \, dt$$

is $\perp D_-^\rho$, i.e., that

$$((G-\lambda)^{-1}g, f) = 0 \quad \text{for all } f \text{ in } D_-^\rho \ , \tag{8.9}$$

in particular for those with compact support. Since relation (8.9) remains true under analytic continuation, our proof of part (c) is complete.

Corollary 8.3. *It is easily seen that if g depends piecewise smoothly on some parameters then $(G-\lambda)^{-1}g = v$ also depends piecewise smoothly on these parameters in the local L_2 topology.*

The following characterization of functions satisfying property (c) is useful:

Lemma 8.4. *Suppose that v is locally L_2, and that $(v, f) = 0$ for all f in D_-^ρ with compact support. Then*

$$[V(t)v](x) = 0 \quad \textit{for } |x| < t - \rho.$$

Proof. We note first that, on account of the finiteness of signal speed, $V(t)v$ is well defined for all locally L_2 functions v, and is itself locally L_2. To show that $V(t)v$ vanishes in $|x| < t-\rho$ we have to verify that $(V(t)v, h) = 0$ for all h with support in $|x| < t-\rho$. According to the explicit formula derived for solutions of the unperturbed equation, for such h, $V(-t)h$ belongs to D_-^ρ; clearly it has compact support, and so

$$(V(t)v, h) = (v, V(-t)h) = 0 \ .$$

This completes the proof of the lemma.

We are now ready to prove the spectral representation theorem. Let $\varphi = \varphi(\sigma, \omega)$ be the generalized matrix-valued eigenfunction of G_0 defined by (8.2); since G and G_0 are the same for $|x| > \rho$, it follows that

(8.10) $$g(\sigma, \omega) = -(G - i\sigma)\varphi$$

is zero for $|x| > \rho$. Now set

(8.11) $$v_+(\sigma, \omega) = (G - i\sigma)^{-1}g(\sigma, \omega) ,$$

where the right-hand side is defined by analytic continuation from $\operatorname{Re} \lambda > 0$; according to part (a) of Theorem 8.2, v_+ has properties (A), (B) asserted in the spectral representation theorem.

Since $\varphi(\sigma, \omega)$ depends smoothly on σ and piecewise smoothly on ω so does $g(\sigma, \omega)$ and it follows from Corollary 8.3 that $v_+(\sigma, \omega) = (G - i\sigma)^{-1}g(\sigma, \omega)$ defined by analytic continuation also depends piecewise smoothly on σ, ω in the L_2 norm over every compact subset of R^k.

We define now

(8.12) $$\varphi_+ = \varphi + v_+ ;$$

clearly φ_+ is a piecewise smooth function of σ, ω in the L_2 norm over any compact set of R^h; therefore for any f with compact support,

(8.13) $$\tilde{f}_-(\sigma, \omega) = (f, \varphi_+)$$

is a piecewise smooth function of σ, ω.

We appeal now to part (b) of Theorem 8.2; since v_+ is defined by (8.11) we get, using the definition (8.10) of g that

$$(G - i\sigma)v_+(\sigma, \omega) = -(G - i\sigma)\varphi ;$$

this shows that

(8.14) $$(G - i\sigma)\varphi_+ = 0,$$

i.e., that φ_+ is indeed a generalized eigenfunction of G.

We turn now to proving part (C) i.e., that we have an incoming spectral representation. We shall show that

(i) For every f in C_0^∞

$$(Gf, \varphi_+) = i\sigma(f, \varphi_+) = i\sigma\tilde{f}_- .$$

(ii) For f in $D_-^\rho \cap C_0^\infty$, $\tilde{f}_- = \tilde{f}$.

(iii) $f \to \tilde{f}_-$ is an isometry.

(iv) The set of $\tilde{f}$ is dense in $L_2(-\infty, \infty; N)$.

Property (i) follows by integrating by parts and using (8.14):

$$((G - i\sigma)f, \varphi_+) = -(f, (G - i\sigma)\varphi_+) = 0$$

as asserted in (i).

(i)′ For every f in C_0^∞,

$$(U(t)f, \varphi_+) = e^{i\sigma t}(f, \varphi_+) = e^{i\sigma t}\tilde{f}\,; \tag{8.15}$$

to see this just differentiate with respect to t and use (i).

To prove part (ii) we have to show that (f, v_+), the difference between $\tilde{f}_+$ and $\tilde{f}$, is zero for f in $D_-^\rho \cap C_0^\infty$; but this is an immediate consequence of part (c) of Theorem 8.2. Since $f \to \tilde{f}$ is an isometry (this is the Parseval relation for the Fourier transform) it follows that for f in $D_-^\rho \cap C_0^\infty$, $f \to \tilde{f}_-$ is an isometry. Using (8.15) and the isometry of $U(t)$ we conclude that $f \to \tilde{f}_-$ is an isometry for all f in $U(t)D_-^\rho \cap C_0^\infty$. Since according to the Density Lemma the set of these elements is dense in H, property (iii) follows.

There remains to show that the mapping is onto $L_2(-\infty, \infty; N)$. This follows from the fact that the image of D_-^ρ under the unperturbed spectral representation is $e^{i\sigma\rho} A_-(N)$; therefore its image is the same under the incoming spectral representation. The image of $U(t)D_-^\rho$ is $e^{i\sigma(t+\rho)} A_-(N)$, and the union for these for all t is a dense subset of $L_2(-\infty, \infty; N)$.

Having properties (i)—(iv) for f in C_0^∞ we define $\tilde{f}$ for all f by completion. We point out that the classical Fourier transform on L_2 is also defined by completion.

The outgoing representation $f \to \tilde{f}_+$ can be treated similarly. This completes the proof of Theorem 8.1.

We turn now to deriving an explicit form for the scattering matrix. For this we need to extend the notion of the Radon transform; the generalized Radon transform turns out to be a very useful tool.

We recall that in extending the Fourier transform to tempered distributions we had to overcome both bad local behavior as well as bad behavior at infinity. In generalizing the Radon transform we have to overcome precisely these two difficulties. Local bad behavior causes no difficulty at all, if f is a distribution in R^k with compact support,

$$h(s, \omega) = \partial_s^{k-1} \int_{x\cdot\omega = s} f(x)dS \tag{8.16}$$

makes perfectly good sense as a distribution in s, ω. On the other hand if f does not tend to zero fast enough at infinity there is no immediate way to give a meaning to (8.16). To get around this difficulty we turn to the inverse $\mathcal{J}$ of the Radon transform.

$$f(x) = \mathcal{J}\, h = \int h(x\cdot\omega, \omega)d\omega\,; \tag{8.17}$$

clearly this defines f as a distribution for any distribution h, no matter how h behaves near infinity. We define the domain of the generalized Radon transform to consist of all functions f of the form (8.17) and set $\mathcal{R}f = h$.

Consider the following example: take h to be

(8.18) $$h(s, \omega) = e^{-s\lambda/\tau_j(\omega)}\,\delta_\theta(\omega) r_j(\omega)$$

where $\delta_\theta(\omega)$ is the δ function of S^{k-1} at some point θ. Using the definition (8.17) of $\mathcal{I}$ we get

(8.19) $$\varphi = \mathcal{I}\ h = e^{-(\lambda x\cdot\theta)/\tau_j(\theta)}\ r_j(\theta)\ .$$

Note that φ satisfies $G_0\varphi = \lambda\varphi$.

The trouble with this definition is that it is not at all clear which f belong to the domain of $\mathcal{R}$. We shall prove

THEOREM 8.5. *Let g be any distribution whose support lies in $|x| < \rho$, λ any complex number; then the equation*

(8.20) $$(G_0 - \lambda)v = g$$

has two solutions v_+ and v_-, one orthogonal to $D_-^\rho \cap C_0^\infty$, the other to $D_+^\rho \cap C_0^\infty$. The solutions v_+ and v_- are uniquely determined by these conditions. Both v_+ and v_- have Radon transforms which have the following properties:

(8.21) $$\mathcal{R}v_+ = 0 \quad \textit{for } s < -\rho\ ,$$
$$\mathcal{R}v_- = 0 \quad \textit{for } s > \rho\ .$$

PROOF. Anticipating the conclusion we Radon transform (8.20); denoting $\mathcal{R}v$ by h we get

(8.22) $$G_0(\omega)\partial_s h - \lambda h = \mathcal{R}g\ .$$

Since $g = 0$ for $|x| > \rho$, $\mathcal{R}g = 0$ for $|s| > \rho$; the ordinary differential equation (8.22) has two distinguished solutions h_+ and h_-, vanishing on $(-\infty, -\rho)$, respectively (ρ, ∞). Denote $\mathcal{I}h_+$ by v_+, $\mathcal{I}h_-$ by v_-; we claim that they solve (8.20); for

$$(G_0-\lambda)v = (G_0-\lambda)\mathcal{I}\ h = \mathcal{I}(G_0(\omega)\partial_s h-\lambda h) = \mathcal{I}\ \mathcal{R}\ g = g\ .$$

The relation (8.21) follows by the construction. Since

(8.23) $$V(t)v_+ = \mathcal{I}\left[\sum_1^{n/2} h_j(s - \tau_j(\omega)t, \omega)r_j\right],$$
$$[V(t)v_+](x) = 0 \quad \text{for } |x| < t - \rho\ .$$

We show next that v_+ is uniquely determined by condition (8.21), for if there were two, their difference d would satisfy both

(8.24) $$G_0 d = \lambda d$$

and

(8.25) $$V(t)d = 0 \quad \text{for } |x| < t-\rho\ .$$

But (8.24) implies that

$$\frac{d}{dt}V(t)d = V(t)G_0 d = \lambda V(t)d \ ,$$

which implies that $V(t)d = e^{\lambda t}d$. It then follows from (8.25) that $d \equiv 0$.

The properties of v_- can be deduced similarly; this completes the proof of Theorem 8.5.

COROLLARY 8.6.

$$\begin{aligned} \mathcal{R}v_+ &= \sum_1^{n/2} a_j^+(\omega)e^{-\lambda s/\tau_j(\omega)}r_j(\omega) \quad \textit{for } s > \rho , \\ \mathcal{R}v_- &= \sum_1^{n/2} a_j^-(\omega)e^{-\lambda s/\tau_j(\omega)}r_j(\omega) \quad \textit{for } s < -\rho . \end{aligned} \tag{8.26}$$

This can be deduced from the fact that both $\mathcal{R}v_+$ and $\mathcal{R}v_-$ satisfy a homogeneous differential equation for $|s| > \rho$.

COROLLARY 8.7. *Let v be a solution of*

$$(G-\lambda)v = g \ , \qquad g = 0 \quad \textit{for } |x| > \rho \tag{8.27}$$

which is orthogonal to $D_-^\rho \cap C_0^\infty$; then v has a Radon transform which satisfies

$$\begin{aligned} \mathcal{R}v &= 0 \quad \textit{for } s < -\rho \\ &= \sum b_s(\omega)e^{-\lambda s/\tau_j(\omega)}\, r_j(\omega) \quad \textit{for } s > \rho \ . \end{aligned} \tag{8.28}$$

This follows by rewriting (8.27) as

$$(G_0 - \lambda)v = g + (G_0 - G)v = g'$$

and noting that g' also is supported in $|x| \leqq \rho$. The relation (8.28) then follows from Theorem 8.5.

Combining example (8.19) and Corollary 8.6 we conclude that the jth column of $\mathcal{R}\varphi_+$ is given by

$$\begin{aligned} (\mathcal{R}\varphi_+)(s, \omega; \sigma, \theta, j) &= \sigma^{-1/2}\left(\frac{\sigma}{2\pi\tau_j(\omega)}\right)^{n/2} e^{-i\sigma s/\tau_j(\omega)}\, \delta_\theta(\omega) r_j(\omega) \\ &\qquad\qquad \text{for } s < -\rho, \\ &= \sigma^{-1/2}\left(\frac{\sigma}{2\pi\tau_j(\omega)}\right)^{n/2} e^{-i\sigma s/\tau_j(\omega)}\, \delta_\theta(\omega) r_j(\omega) \\ &\quad + \sum_{\ell=1}^{n/2} a_\ell^+(\omega; \theta, \sigma, j)e^{-i\sigma s/\tau_\ell(\omega)} r_\ell(\omega) \quad \text{for } s > \rho . \end{aligned} \tag{8.29$_+$}$$

An analogous formula holds for $\mathcal{R}\,\varphi_-$.

THEOREM 8.8. *The scattering matrix $\mathcal{S}(\sigma)$ equals the identity plus an integral operator whose kernel K is the $n/2 \times n/2$ matrix-valued function*

$$K_{\ell j}(\theta, \omega;\sigma) = \sigma^{1/2}\left(\frac{2\pi\tau_\ell(\theta)}{\sigma}\right)^{n/2} \overline{a_\ell^-}(\theta;\sigma, \omega, j)\ , \tag{8.30}$$

where a_ℓ^- is defined by (8.29)$_-$; *that is*

$$\begin{aligned}\tilde{f}_+(\sigma, \omega; j) &= \tilde{f}_-(\sigma, \omega; j)\\ &+ \sigma^{1/2}\sum_{\ell=1}^{n/2}\int_{|\theta|=1}\left(\frac{2\pi\tau_\ell(\theta)}{\sigma}\right)^{n/2}\overline{a_\ell^-}(\theta;\sigma, \omega, j)\tilde{f}_-(\sigma, \theta; \ell)d\theta.\end{aligned} \tag{8.31}$$

PROOF. In order to verify (8.31) we substitute in it the previously obtained expressions for $\tilde{f}_+$ and $\tilde{f}_-$ and replace $\overline{a_\ell^-}$ by $K_{\ell j}$; we then obtain

$$\begin{aligned}(f, \varphi_-(\sigma, \omega, j)) &= (f, \varphi_+(\sigma, \omega, j))\\ &+ \sum_{\ell=1}^{n/2}\int (f, \varphi_+(\sigma, \theta', \ell))K_{\ell j}(\theta', \omega;\sigma)d\theta'\ .\end{aligned} \tag{8.32}$$

Interchanging the order of x and θ integration, we conclude that (8.32) holds if and only if the quantity

$$\begin{aligned}\psi(x;\sigma, \omega, j) \equiv\ & \varphi_-(x;\sigma, \omega, j) - \varphi_+(x;\sigma, \omega, j)\\ &- \sum_{\ell=1}^{n/2}\int \varphi_+(x;\sigma, \theta', \ell)\bar{K}_{\ell j}(\theta', \omega;\sigma)d\theta'\end{aligned} \tag{8.33}$$

is zero.

It suffices therefore to show that $\psi \equiv 0$. We note that each term of the sum is annihilated by $(G - i\sigma)$ and therefore so is ψ. Next we show that for K defined as in (8.30) ψ is orthogonal to $D_-^\rho \cap C_0^\infty$. It will then follow from a generalization of the Rellich uniqueness theorem (see [3, Theorem 2.5 of Chapter 6]) that ψ vanishes identically.

We now verify that ψ is orthogonal to $D_-^\rho \cap C_0^\infty$ by showing that its Radon transform is zero for $s < -\rho$ as required by Corollary 8.7. The Radon transform of $\varphi_- - \varphi_+$ is the same as that of v_- for $s < -\rho$, namely

$$\mathcal{R}(\varphi_- - \varphi_+)(\sigma, \omega, j) = \sum_{\ell=1}^{n/2} a_\ell^-(\theta;\sigma, \omega, j)e^{-i\sigma s/\tau_\ell(\theta)}\, r_\ell(\theta) \quad \text{for } s < -\rho\ .$$

On the other hand the Radon transform of φ_+ is the same as that of φ for $s < -\rho$, that is

$$\mathscr{R}(\varphi_+)(\sigma, \omega, j) = \sigma^{-1/2}\left(\frac{\sigma}{2\pi\tau_j(\theta)}\right)^{n/2} e^{-i\sigma s/\tau_j(\theta)}\,\delta_\omega(\theta) r_j(\theta) \quad \text{for } s < -\rho\ .$$

Substituting this in (8.33) we find that for $s < -\rho$, $(\mathscr{R}\psi)(s, \theta; \sigma, \omega, j)$ is

$$\sum_{\ell=1}^{n/2}\left[a_\ell^-(\theta;\sigma, \omega, j) - \sigma^{-1/2}\left(\frac{\sigma}{2\pi\tau_\ell(\theta)}\right)^{n/2}\bar{K}_{\ell j}(\theta, \omega;\sigma)\right] e^{-i\sigma s/\tau_\ell(\theta)}\, r_\ell(\theta)\,,$$

which is indeed zero if K is chosen as in (8.30).

Part II. The Wave Equation

9. **Purely decaying modes for the wave equation in the exterior of an obstacle.** Next we consider the behavior of solutions of the wave equation in the exterior of an obstacle. The differences between the exterior and interior problems are so great that they tend to hide the points of similarity. Our purpose in the remaining sections of this paper is to bring out certain analogies between these two problems and to this end we first discuss the relevant, but familiar, facts about the interior problem; that is the behavior of solutions of the wave equation

(9.1) $$u_{tt} - \Delta u = 0$$

in some smoothly bounded compact domain $\mathcal{O}$ on whose boundary u is required to satisfy a boundary condition, say

(9.2) $$u = 0 \quad \text{on } \partial\mathcal{O}\ .$$

The spectral decomposition of Δ over $\mathcal{O}$ leads to the following representation of the totality of solutions of (9.1), (9.2):

(9.3) $$u(x, t) = \sum_{k=1}^{\infty}(a_k e^{i\mu_k t} + b_k e^{-i\mu_k t})v_k(x)\ ;$$

here $\{\mu_k^2\}$ are the eigenvalues of $-\Delta$ arranged in increasing order with $\mu_k > 0$ and $\{v_k\}$ are the corresponding eigenfunctions:

(9.4) $$\Delta v_k + \mu_k^2 v_k = 0, \qquad v_k(x) = 0 \quad \text{on } \partial\mathcal{O}\ .$$

For the interior problem $(-\Delta)^{-1}$ is a positive compact operator and therefore the $\{\mu_k\}$ form a sequence of positive numbers tending to ∞; each solution of the wave equation is represented in (9.3) as a superposition of harmonic motions with frequencies μ_k. These frequencies are functionals of the domain $\mathcal{O}$, and much effort has gone into studying the dependence of the set of numbers $\{\mu_k\}$ on the geometrical properties of $\mathcal{O}$. The following results described in [1] are particularly interesting mathematically and significant from the point of view of physics:

(1) $\mu_k(\mathcal{O})$ depends monotonically on $\mathcal{O}$, that is if $\mathcal{O}_1 \subset \mathcal{O}_2$ then

(9.5) $$\mu_k(\mathcal{O}_1) \geqq \mu_k(\mathcal{O}_2) \quad \text{for all } k\ .$$

(2) The asymptotic distribution of the μ_k for large k is

(9.6) $$\mu_k \sim 2\pi \left(\frac{k}{\Omega V} \right)^{1/n}$$

where n is the dimension of the x-space, Ω the volume of the n-dimensional unit ball and V the volume of $\mathcal{O}$.

(3) Among all domains $\mathcal{O}$ with given volume V, the sphere has the smallest fundamental frequency μ_1.

We turn now to the behavior of solutions of the wave equation in the exterior $\mathcal{G}$ of $\mathcal{O}$, subject to the same boundary condition (9.2). In this case $-\Delta$ has a continuous spectrum (of infinite multiplicity) extending from 0 to ∞ and one can again express all solutions of the wave equation as a superposition of harmonic motions involving all frequencies. It turns out that such a representation as it stands sheds no light on the asymptotic behavior of $u(x, t)$ for large t with x fixed.

To get some idea of what kind of asymptotic representation to look for, we first recall that the solution to the wave equation in free space of an odd number of dimensions obeys Huyghens' Principle; thus for initial data having compact support, say contained in $\{x : |x| < R\}$, the solution will vanish in the cone $\{x : |x| < t - R\}$. If an obstacle is present this is no longer true. Nevertheless if the obstacle satisfies certain geometrical conditions described below and if the space dimension is odd, then all such solutions decay exponentially for fixed x as t tends to infinity. In fact for large t such solutions behave asymptotically as follows:

(9.7) $$u(x, t) \sim \sum_{k=0}^{\infty} c_k e^{\lambda_k t} w_k(x)\ ,$$

where the numbers c_k depend on the initial data but the numbers λ_k and the functions w_k are determined solely by the obstacle $\mathcal{O}$ and are in a generalized sense eigenpairs for the operator Δ in the exterior domain. Each λ_k has a negative real part and they have been indexed so that

(9.8) $$0 > \operatorname{Re} \lambda_1 \geqq \operatorname{Re} \lambda_2 \geqq \cdots \to -\infty\ .$$

A precise meaning for (9.7) can be given in terms of the familiar semigroup of operators

$$Z(t) = P_+^\rho\, U(t) P_-^\rho, \qquad t \geqq 0 \tag{9.9}$$

defined on

$$K^\rho = H \ominus (D_-^\rho \oplus D_+^\rho) , \tag{9.10}$$

when $Z(t)$ compact for some $t > 0$. In this case for f in K^ρ one can express $Z(t)f$ asymptotically as

$$Z(t)f \sim \sum c_k e^{\lambda_k t} w_k(x) , \tag{9.11}$$

where the $\{\lambda\}$ are the eigenvalues and the $\{w_k\}$ the eigenfunctions of the infinitesimal generator B of $\{Z(t)\}$. This was proved in §7.

The parameter ρ is arbitrary; happily however the eigenvalues $\{\lambda_k\}$ do not depend on ρ, and neither do the eigenfunctions for $|x| < \rho$. In fact the $w_k(x)$ obtained for various values of ρ converge as $\rho \to \infty$ to an eigenfunction of Δ, with λ_k^2 as eigenvalue:

$$\Delta w_k = \lambda_k^2 w_k \quad \text{in } \mathcal{G} . \tag{9.12}$$

These eigenfunctions behave asymptotically like $|x|^{-1}\exp(-\lambda_k |x|)$ for large $|x|$ and therefore lie outside the Hilbert space H. They do however satisfy an outgoing radiation condition.

To connect the eventual compactness of $Z(t)$ with the geometrical properties of $\mathcal{O}$ we introduce the following notation: Consider all rays starting on the sphere of radius ρ which proceed toward the obstacle and are continued according to the law of reflection whenever they impinge on $\mathcal{O}$ until they leave the ball $\{|x| < \rho\}$. We call $\mathcal{O}$ *confining* if there are arbitrarily long rays of this kind; otherwise $\mathcal{O}$ is called *nonconfining*. Surmising that sharp signals propagate along rays we conjectured (see pp. 155–157 of [**3**]) that $Z(t)$ is eventually compact if and only if $\mathcal{O}$ is nonconfining. Ralston [**13**] has shown in an important special case of confining obstacles that $Z(t)$ is not compact for any t. In the opposite direction Ludwig and Morawetz [**10**] (see also Phillips [**12**]) have shown that if $\mathcal{O}$ is convex then $Z(t)$ is eventually compact. For star-shaped obstacles Lax, Morawetz and Phillips [**3**] have proved a related result, namely that $Z(t)$ decays in norm exponentially; i.e., $\|Z(t)\| \leqq C\exp(-\gamma t)$ with $C, \gamma > 0$.

In [**5**] we have made a start in studying the dependence of the set of 'exterior' eigenvalues $\{\lambda_k\}$ on the geometry of the obstacle $\mathcal{O}$. We have shown that the real eigenvalues, corresponding to purely decaying modes, depend monotonically on the obstacle $\mathcal{O}$, both for the Dirichlet and Neumann boundary conditions. From this we deduced, by comparison with spheres—for which the eigenvalues $\{\lambda_k\}$ can be determined as roots of special functions—upper and lower bounds for the density of the real $\{\lambda_k\}$ and upper and lower bounds for λ_1 the rate of decay of the fundamental real decaying mode.

We sketch the proofs of these statements in the next three sections; a complete discussion is given in [5].

10. The transmission coefficient. Rather than working directly with the eigenvalue problem (9.12) we found it convenient to make use of a different characterization of the eigenvalues $\{\lambda_k\}$ of the generator B of $Z(t)$, one which involves the scattering matrix. The correspondence between the scattering matrix and the eigenvalues of B has been previously stated: λ is an eigenvalue of B if and only if $i\bar{\lambda}$ is a zero of the scattering matrix $\mathcal{S}(z)$ and the degree of multiplicity is the same (see Theorem 6.4). Moreover $\mathcal{S}(z)$ is meromorphic having as its poles precisely the points $-i\lambda$ for which $i\bar{\lambda}$ is a zero of $\mathcal{S}(z)$ (see [3, Theorem 5.1 of Chapter 3]).

The scattering matrix $\mathcal{S}(z)$ is an operator on $L_2(S_2)$ and can be represented analogously to the representation given in Theorem 8.8:

$$\mathcal{S}(z) = I + K^{sc}(z), \tag{10.1}$$

where $K^{sc}(z)$ is an integral operator with kernel

$$K^{sc}(\omega, \theta; z) = \frac{iz}{2\pi} k^{sc}(\omega, -\theta; z) \ ; \tag{10.2}$$

here $k^{sc}(\omega, \theta; z)$ is the transmission coefficient (see [3, Theorem 5.4 of Chapter 5]). We shall be concerned with values of z in the lower half-plane, Im $z \leqq 0$, and for such z the transmission coefficient is determined by the solution of the reduced wave equation:

$$\begin{aligned} &z^2 v + \Delta v = 0 \quad \text{in } \mathcal{G}, \\ &v(x, \omega; z) = \exp(izx \cdot \omega) \quad \text{on } \partial\mathcal{G} \text{ (Dirichlet problem).} \end{aligned} \tag{10.3}$$

It can be shown that the asymptotic behavior of the solution v for large $|x|$ is given by

$$v(r\theta, \omega; z) = \frac{e^{-izr}}{r}\left[k^{sc}(\omega, \theta; z) + O\left(\frac{1}{r}\right)\right]$$

where θ is a unit vector and $x = r\theta$. It is known that the transmission coefficient is smooth in ω, θ and analytic in z in the lower half-plane.

We begin by stating a useful integral representation for the transmission coefficient. Since we shall be concerned with the purely imaginary zeros of $\mathcal{S}(z)$ in the lower half-plane it is convenient to work with $\sigma = iz$. In what follows σ will denote a positive real number and, again for notational convenience, we set

$$k(\omega, \theta; \sigma) = k^{sc}(\omega, \theta; -i\sigma) \tag{10.4}$$

and

$$K(\omega, \theta, \sigma) = \frac{\sigma}{2\pi} k(\omega, -\theta; \sigma) = K^{sc}(\omega, \theta, -i\sigma).$$

In this case (10.3) becomes

$$\begin{aligned} &\sigma^2 v - \Delta v = 0 \quad \text{in } \mathcal{G}, \\ &v(x, \omega; \sigma) = e^{\sigma x \cdot \omega} \quad \text{on } \partial \mathcal{G} \text{ (Dirichlet problem)}, \end{aligned} \tag{10.3'}$$

or

$$\partial v(x, \omega; \sigma)/\partial n = \partial \exp(-\sigma x \cdot \omega)/\partial n \quad \text{on } \partial \mathcal{G} \text{ (Neumann problem)};$$

and the asymptotic behavior of v is given by

$$v(r\theta, \omega; \sigma) = \frac{e^{-\sigma r}}{r} \left[k(\omega, \theta; \sigma) + O\left(\frac{1}{r}\right)\right] \quad ;$$

here n denotes the outer normal to $\partial \mathcal{G}$.

THEOREM 10.1. *If we denote by q_σ the bilinear form:*

$$q_\sigma(u, w) = \sigma^2 uw + \nabla u \cdot \nabla w \ ,$$

then the transmission coefficient is given by

$$\begin{aligned} k(\omega, \theta; \sigma) &= \frac{\alpha}{4\pi} \int_{\mathcal{O}} q_\sigma(\exp(-\sigma x \cdot \omega), \exp(-\sigma x \cdot \theta)) dx \\ &+ \frac{\alpha}{4\pi} \int_{\mathcal{G}} q_\sigma(v(x, \omega; \sigma), v(x, \theta; \sigma)) dx \end{aligned} \tag{10.5}$$

where $\alpha = 1$ for the Dirichlet problem and -1 for the Neumann problem, v is the solution of (10.3)′ *and $\mathcal{O}$ the interior, $\mathcal{G}$ the exterior of the obstacle.*

As an immediate consequence we have

COROLLARY 10.2. *For the Dirichlet and Neumann problems $\alpha k(\omega, \theta; \sigma)$ is the kernel of a symmetric nonnegative Hilbert-Schmidt operator on $L_2(S_2)$.*

Theorem 10.1 can be reformulated so as to give the following variational characterization of (10.5):

COROLLARY 10.3. *In the case of the Dirichlet problem*

$$(ka, a) = \inf \frac{1}{4\pi} \int_{R_3} [\sigma^2 B^2 + (\nabla B)^2]\, dx$$

over all smooth functions B with compact support in R_3 which are equal to $\int \exp(\sigma x \cdot \omega) a(\omega) d\omega$ *in $\mathcal{O}$. In the case of the Neumann problem*

$$-(ka, a) = \inf \frac{1}{4\pi} \int_{R_3} [\sigma^2 B^2 + (\nabla B)^2]\, dx$$

over all smooth functions B in $\mathcal{O} \cup \mathcal{G}$, vanishing near infinity, and which are equal to $\int \exp(\sigma x \cdot \omega) a(\omega) d\omega$ *in $\mathcal{O}$ and have a continuous normal derivative (but need not be continuous) across $\partial \mathcal{G}$.*

We come now to one of our main results.

THEOREM 10.4. *Denote by k_1 and k_2 the transmission coefficients for the scattering objects $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively. If $\mathcal{O}_1 \subset \mathcal{O}_2$, then considered as operators on $L_2(S_2)$*

$$\alpha k_1(\sigma) \leqq \alpha k_2(\sigma) \quad \text{for all } \sigma > 0; \tag{10.6}$$

here $\alpha = 1$ for the Dirichlet problem and $\alpha = -1$ for the Neumann problem.

It can be shown that $\alpha k(\sigma)$ is strictly positive for a spherical scatterer. Since every scatterer with a nonempty interior contains a sphere, from this and the monotonicity theorem we conclude

COROLLARY 10.5. *If the scatterer $\mathcal{O}$ has a nonempty interior, α times the transmission coefficient is the kernel of a strictly positive operator.*

11. **On the purely imaginary zeros of the scattering matrix.** We recall that a purely decaying mode of $Z(t)$ with eigenvalue $e^{-\sigma t}$ corresponds to a purely imaginary zero of the scattering matrix $\mathcal{S}(z)$ at $z = -i\sigma$ with the same degree of multiplicity. Since in the notation of the previous section $\mathcal{S}(-i\sigma) = I + K(\sigma)$, this simply means that the purely decaying modes of $Z(t)$ correspond to those positive values of σ for which -1 is an eigenvalue of $K(\sigma)$; the kernel of $K(\sigma)$ is given by

$$K(\omega, \theta; \sigma) = \frac{\sigma}{2\pi} k(\omega, -\theta; \sigma) , \tag{11.1}$$

where k is the transmission coefficient. Denoting reflection through the origin by W: $[Wa](\theta) = a(-\theta)$, the relation (11.1) can be written in operator form as

$$K(\sigma) = \frac{\sigma}{2\pi} k(\sigma) W . \tag{11.2}$$

According to Corollary 10.2 and Theorem 10.4, $\alpha k(\sigma)$ is a symmetric strictly positive Hilbert-Schmidt operator on $L_2(S_2)$; $\alpha = 1$ for the Dirichlet problem and -1 for the Neumann problem.

The presence of W complicates the problem since K need not be symmetric and even when it is symmetric it is not positive. Nevertheless the following comparison theorem for K is valid.

THEOREM 11.1. *Suppose k_1 and k_2 are compact strictly positive operators such that $0 < k_1 \leqq k_2$ and set $K_i = k_i W$. Then the eigenvalues of K_1 are real and nonzero; if they are ordered taking multiplicities into account:*

$$\nu_1^{(i)} \geqq \nu_2^{(i)} \geqq \cdots > 0 > \cdots \geqq \kappa_2^{(i)} \geqq \kappa_1^{(i)}, \qquad i = 1, 2;$$

then for all integers n

$$\nu_n^{(1)} \leqq \nu_n^{(2)} \quad \text{and} \quad \kappa_n^{(1)} \geqq \kappa_n^{(2)}. \tag{11.3}$$

We have devised two proofs for this theorem. The first is fairly direct and is accomplished by means of a symmetric operator with the same spectrum as K. The second proof gives a minimax characterization of the eigenvalues of K in terms of K itself in the setting of a Hilbert space with an indefinite metric (see [8]).

Theorem 11.1 provides us with a substantial grip on the problem of determining the purely imaginary zeros of the scattering matrix; we must find positive values of σ for which -1 is a eigenvalue of $K(\sigma)$. It therefore suffices to study the growth of the negative eigenvalues $\{\kappa_n(\sigma)\}$ of $K(\sigma)$ as a functions of σ, picking out those values of σ and n for which $\kappa_n(\sigma) = -1$. It is known that $k(\sigma)$ is analytic in σ for real $\sigma \geqq 0$; it therefore follows from the relation (11.2) that $K(\sigma)$ converges to zero as $\sigma \to 0+$. Thus the smallest purely imaginary zero of the scattering matrix comes from the smallest root σ_1 of $\kappa_1(\sigma) = -1$ and hence we obtain as an immediate consequence of Theorems 10.4 and 11.1 the following:

THEOREM 11.2. *If $\mathcal{O}_1 \subset \mathcal{O}_2$ then the smallest purely imaginary zero of $\mathcal{S}_1$ is greater than or equal to the smallest purely imaginary zero of $\mathcal{S}_2$; that is $\sigma_1^{(1)} \geqq \sigma_1^{(2)}$.*

In general the negative eigenvalues $\{\kappa_n(\sigma)\}$ are not monotone decreasing functions of σ. However the situation is comparatively simple for star-shaped obstacles.

LEMMA 11.3. *If $\mathcal{O}$ is star-shaped then the negative eigenvalues of $K(\sigma)$ are monotone decreasing functions of σ.*

COROLLARY 11.4. *If $\mathcal{O}$ is star-shaped and if for a given σ, n of the eigenvalues of $K(\sigma)$ are less than or equal to -1, then the scattering matrix has exactly n purely imaginary zeros $\{-i\sigma_k\}$ with $\sigma_k \leqq \sigma$.*

For a general obstacle it seems likely that the negative eigenvalues of $K(\sigma)$ are not monotone decreasing functions of σ. In this case the comparison theorem furnishes us with a lower bound for the number of zeros of the scattering matrix in a given interval.

THEOREM 11.5. *If $\mathcal{O} \supset \mathcal{O}_s$ where $\mathcal{O}_s$ is star-shaped and if $\mathcal{S}_s$ has n purely imaginary zeros $\{-i\sigma_k^s\}$ with $\sigma_k^s \leqq \sigma$, then $\mathcal{S}$ has at least n purely imaginary zeros $\{-i\sigma_k\}$ with $\sigma_k \leqq \sigma$.*

It is clear that a comparison of $\mathcal{O}$ with contained and containing spheres will furnish us with good estimates of the purely imaginary zeros of $\mathcal{S}$. This will be discussed in the next section.

12. **Estimates for the distribution of the purely imaginary zeros of the scattering matrix.** In Corollary 10.3, we derived the following characterization for the quadratic form associated with the transmission coefficient: For Dirichlet boundary conditions

$$(ka, a) = \inf \frac{1}{4\pi} \int_{R_3} [\sigma^2 B^2 + (\nabla B)^2]\, dx\ , \tag{12.1}$$

where B ranges over all smooth functions with compact support which are equal to A in $\mathcal{O}$; A is defined by

$$A(x) = \int_{|\omega|=1} \exp(\sigma x \cdot \omega) a(\omega) d\omega\ . \tag{12.2}$$

A similar characterization holds for the Neumann boundary condition. This variational characterization for k leads very naturally to upper and lower bounds in the operator sense.

THEOREM 12.1. *Define*

$$k_0(\omega, \theta, \sigma) = \frac{\alpha}{4\pi}(1 + \omega \cdot \theta)\sigma^2 \int_{\mathcal{O}} \exp(\sigma x \cdot (\omega + \theta)) dx\,, \tag{12.3}$$

where $\alpha = 1$ for the Dirichlet problem and -1 for the Neumann problem. Then

$$\alpha k_0(\sigma) \leqq \alpha k(\sigma) \leqq 3\alpha k_0(\sigma); \tag{12.4}$$

the first inequality holds for all $\sigma > 0$, the second for all sufficiently large σ.

Combining Theorems 11.1 and 12.1 we now have a way of obtaining the asymptotic distribution of the purely imaginary modes for a star-shaped obstacle from the eigenvalues of the associated integral operator $k_0(\sigma)W$. Despite the rather simple form of this operator, so far we have succeeded in this endeavor only for the case of the sphere; a somewhat lengthy calculation yields

PROPOSITION 12.2. *Let $C(\sigma)$ denote the number of purely imaginary zeros of the scattering matrix for a sphere of radius R which are $\leqq \sigma$ in absolute value, under either Dirichlet or Neumann boundary conditions. Then*

$$C(\sigma) \sim \tfrac{1}{2} (\sigma R/\gamma_0)^2$$

where $\gamma_0 = .66274 \ldots$.

REMARK. The exact values of the purely imaginary zeros of the scattering matrix for a sphere of radius R can of course be computed directly; this has been done by Wilcox [11] for the Dirichlet problem; σ_n for the nth mode occurs at the real zero of $K_{n+1/2}(-\sigma R)$ where $K_{n+1/2}$ is the modified Hankel function. The asymptotic expression for this zero for large n has been found by Olver [15]; it is $\sigma_n R \sim \gamma_0 n$, in agreement with our estimate. The exact value for the lowest mode for both the Dirichlet and Neumann problems is easily computed; it is $\sigma_1 = 1/R$.

We can now apply the comparison Theorems 11.2 and 11.5 to any obstacle which is bracketed between two spheres.

THEOREM 12.3. *Suppose that the obstacle $\mathcal{O}$ contains a sphere of radius R_1 and is contained in a sphere of radius R_2. Let $C(\sigma)$ denote the number of purely imaginary zeros of the scattering matrix for $\mathcal{O}$ under either Dirichlet or Neumann boundary conditions which are $< \sigma$ in absolute value. Then*

$$\liminf_{\sigma \to \infty} \frac{C(\sigma)}{\sigma^2} \geqq \frac{1}{2} \left(\frac{R_1}{\gamma_0} \right)^2 ,$$

where $\gamma_0 = .66274 \ldots$. *If in addition $\mathcal{O}$ is star-shaped, then*

$$\limsup_{\sigma \to \infty} \frac{C(\sigma)}{\sigma^2} \leqq \frac{1}{2} \left(\frac{R_2}{\gamma_0} \right)^2 .$$

We surmise that the limit

$$\lim_{\sigma \to \infty} C(\sigma)/\sigma^2$$

exists.

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