ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 1, Winter 1971

THE ABSTRACT THEORY OF SCATTERING

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1. **Introduction.** This paper deals with the construction and properties of the wave operators $W_{\pm}(H_2, H_1)$ and the scattering operator S associated with two selfadjoint operators H_1 and H_2 in a Hilbert space $\mathcal{H}$. We shall also consider the wave operators $W_{\pm}(U_2, U_1)$ for unitary operators U_1, U_2. More generally, we shall construct wave operators for two spectral measures E_1, E_2 defined on a certain measure space.

There are two main approaches to these problems, called the stationary method and the time-dependent method. The time-dependent method is more convenient for the introduction of the wave and scattering operators. However the stationary method gives more detailed results with fewer assumptions. This paper begins with a summary of the time-dependent approach. The main part of the paper presents an exposition of the stationary method.

2. **A summary of the time-dependent theory of scattering.** Let H_1 and H_2 be selfadjoint operators and consider the associated unitary groups e^{-itH_1}, e^{-itH_2}, $-\infty < t < \infty$. The limits

$$W_{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_2} e^{-itH_1} \tag{2.1}$$

are called the wave operators. Of course such limits will exist only under strong restrictions.

A specific situation, which is typical for applications and to which reference is made frequently below, is the following:

$$\mathcal{H} = L^2(R^3), \quad H_1 = -\Delta, \quad H_2 = -\Delta + V, \quad V = q(x) ,$$

where q is a real-valued measurable function. H_1 is selfadjoint under the standard interpretation of Δ [12, p. 299] and H_2 is selfadjoint under rather mild conditions on q (it suffices if $q \in L^2(R^3) + L^{\infty}(R^3)$ (vector sum)) [12, p. 302]. These operators correspond to quantum mechanical Hamiltonians for a free particle and a particle moving in the potential field $q(x)$ respectively. In this case $W_{\pm}$ exist if $q(x)$ is sufficiently small for large $|x|$ (precise conditions are given below).

(2.1) implies that

Received by the editors November 29, 1969.
AMS 1970 *subject classifications.* Primary 47A40, 35P25; Secondary 35J10.

(2.2) $\qquad e^{-itH_2}\phi \sim e^{-itH_1}\phi_\pm$ as $t \to \pm\infty$, $\qquad$ with $\phi = W_\pm \phi_\pm$,

where $A \sim B$ means $\|A - B\| \to 0$. Thus the "perturbed motion" $e^{-itH_2}\phi$ looks like a free motion $e^{-itH_1}\phi_\pm$. (2.2) implies $\|\phi\| = \|\phi_+\| = \|\phi_-\|$. The map $S : \phi_- \to \phi_+ = S\phi_-$ is the *scattering operator*. For physical reasons S should be unitary (i.e., be defined everywhere on $\mathcal{H}$ and map $\mathcal{H}$ onto $\mathcal{H}$). This implies that

(2.3) $$W_+\mathcal{H} = W_-\mathcal{H}.$$

(More details about the physical background of the wave and scattering operators are given in the paper by Dollard in this issue.)

In the preceding discussion two problems have arisen, namely to establish

1. the existence of $W_\pm$;
2. Range $W_+ =$ Range W_-.

Strong assumptions on H_1 and H_2 are necessary for these two results to be true.

EXAMPLE. To illustrate these problems, consider the following example. Let

$$H_1 = \frac{1}{i}\frac{d}{dx}, \quad H_2 = \frac{1}{i}\frac{d}{dx} + q(x),$$

where $q(x)$ is real valued and $\mathcal{H} = L^2(-\infty,\infty)$. Then

$$(e^{-itH_1}u)(x) = u(x - t).$$

If $p(x) = \int_0^x q(x')dx'$ and $W = e^{ip(x)}$ (a unitary operator of multiplication), then

$$H_2 = W^{-1}H_1W$$

and

$$e^{-itH_2} = W^{-1}e^{-itH_1}W.$$

Thus

$$\begin{aligned}(e^{itH_2}e^{-itH}\ u)(x) &= (W^{-1}e^{itH_1}We^{-itH_1})u(x)\\ &= e^{i\{p(x+t)-p(x)\}}u(x) = \exp\left(i\int_x^{x+t} q(x')dx'\right)u(x).\end{aligned}$$

Then one obtains $W_\pm = \exp[i\int_x^{\pm\infty} q(x')dx']$ (unitary operators of multiplication), assuming that $\int_x^{\pm\infty} q(x')dx'$ exist, and

$$S = W^{-1}W_- = \exp\left(-i\int_{-\infty}^{\infty} q(x')dx'\right)$$

Thus S is multiplication by a constant with absolute value 1. Since the existence of W_+ and W_- depends on the existence of

$$\int^{+\infty} q(x')dx' \text{ and } \int_{-\infty} q(x')dx' \quad \text{respectively,}$$

it is clear that one of $W_\pm$ may exist while the other does not.

REMARK. The *inverse scattering problem* is to determine H_2 given H_1 and S. More precisely, the question is the uniqueness and/or the existence of H_2 (in a certain class) for a given pair H_1, S. In the above example, the uniqueness does not hold because the function q only affects the scattering operator through its integral $\int_{-\infty}^{\infty} q(x')dx'$. (This was noted by G. Schmidt in a slightly different form.) The inverse scattering problem for a similar situation with second order operators is different and has a nice solution (Gel'fand-Levitan [5]).

The problems of the existence of $W_\pm$ and Range W_+ = Range W_- are investigated in a more general situation below by modifying the definitions of the wave operators, using the decomposition of a self-adjoint operator into its absolutely continuous and singular parts. The (generalized) wave operators are defined by

$$W_\pm = W_\pm(H_2, H_1) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_2} e^{-itH_1} P_1 ,$$

where P_1 is the orthogonal projection on $\mathcal{H}_{1,\mathrm{ac}} \subset \mathcal{H}$. Here $\mathcal{H}_{1,\mathrm{ac}}$ is the *subspace of absolute continuity* for H_1, defined by

$$\mathcal{H}_{1,\mathrm{ac}} = \{u \in \mathcal{H} \mid (E_1(\lambda)u,u) \text{ is absolutely continuous in } \lambda\},$$

where $E_1(\lambda)$ is the spectral family of H_1, $H_1 = \int \lambda dE_1(\lambda)$. $\mathcal{H}_{1,\mathrm{ac}}$ is a closed subspace of $\mathcal{H}$ and reduces H_1 [12, p. 516]. $\mathcal{H}_{2,\mathrm{ac}}$ and P_2 are defined similarly.

Because of the factor P_1 generalized wave operators $W_\pm$ are more likely to exist than the operators (2.1). Their effect is to exclude the singular part of the operator H_1.

THEOREM. *If W_+ exists then W_+ is a partial isometry with initial set $\mathcal{H}_{1,\mathrm{ac}}$ and final set $W_+\mathcal{H} \subset \mathcal{H}_{2,\mathrm{ac}}$. $W_+\mathcal{H}$ reduces H_2. We have the intertwining relation $W_+H_1 \subset H_2W_+$. In particular we have the unitary equivalence*

$$H_1|_{\mathcal{H}_{1,\mathrm{ac}}} \approx H_2|_{W_+\mathcal{H}}.$$

A similar result holds for W_- if it exists. If both W_+ and W_- exist, $S = W_+^* W_-$ commutes with H_1.

For a proof of this theorem see the paper of Dollard. Also cf. Kato [12, Chapter 10].

From the theorem one can obtain information about the spectrum of H_2. If one can prove that W_+ (W_-) exists, then H_2 contains a part that is unitarily equivalent to $H_{1,\mathrm{ac}}$, the part of H_1 in $\mathcal{H}_{1,\mathrm{ac}}$.

W_+ (W_-) is said to be *complete* if $W_\pm \mathcal{H} = \mathcal{H}_{2,\mathrm{ac}}$. If both $W_\pm$ exist and are complete then Range $W_+ = \mathcal{H}_{2,\mathrm{ac}} =$ Range W_-. Thus $S = W_+^* W_-$ is unitary in $\mathcal{H}_{1,\mathrm{ac}}$.

The following theorem gives a sufficient condition for the existence of wave operators.

THEOREM (COOK, KURODA). *Suppose there exists $\mathfrak{D} \subset \mathcal{H}_{1,\mathrm{ac}}$ which is a fundamental set in $\mathcal{H}_{1,\mathrm{ac}}$ and which has the property that if $u \in \mathfrak{D}$ there exists t_0 such that*

$$e^{-itH_1}u \in D(H_1) \cap D(H_2) \quad \text{for } t_0 \leqq t < \infty,$$

$(H_2 - H_1)e^{-itH_1}u$ is continuous in t in (t_0, ∞) and

$$\int_{t_0}^{\infty} \|(H_2 - H_1)\, e^{-itH_1}u\| dt < \infty .$$

Then $W_+(H_2, H_1)$ exists. A similar result holds for W_-.

PROOF. For $u \in \mathfrak{D}$ and $s, t \geqq t_0$

$$\frac{d}{dt} e^{itH_2}\, e^{-itH_1}\, u = ie^{itH_2}(H_2 - H_1)e^{-itH_1}u.$$

Thus $e^{itH_2}e^{-itH_1}u - e^{isH_2}e^{-isH_1}u = i\int_s^t e^{itH_2}(H_2 - H_1)e^{-itH_1}u\,dt$. Hence

$$\|e^{itH_2}e^{-itH_1}u - e^{isH_2}e^{-isH_1}u\| \leqq \int_s^t \|(H_2 - H_1)e^{-itH_1}u\| dt.$$

Since $\int_{t_0}^{\infty}\|(H_2 - H_1)e^{-itH_1}u\| dt < \infty$ the right side of the preceding inequality tends to 0 as $s,t \to +\infty$. Hence the left side also tends to 0 as $s, t \to +\infty$. Hence

$$\lim_{t \to \pm\infty} e^{itH_2}e^{-itH_1}u \quad \text{exists for } u \in \mathfrak{D}.$$

Since the operators $e^{itH_2}e^{-itH_1}$ are uniformly bounded, the above limit exists for all $u \in \mathcal{H}_{1,\mathrm{ac}}$.

EXAMPLE. Consider the example we mentioned at the beginning where

$$H_1 = -\Delta, \qquad H_2 = -\Delta + V, \qquad V = q(x),$$

and $\mathcal{H} = L^2(R^3)$. In this case $P_1 = 1$, since H_1 is absolutely continuous. For $t \neq 0$ the operators e^{-itH_1} are integral operators whose kernels are the Green's functions for the Schrödinger equation for a free particle. If $u \in L^2 \cap L^1$ then

$$(e^{-itH_1}u)(x) = \frac{1}{(4\pi i t)^{3/2}} \int_{R^3} \exp(|x-y|^2/4it)\, u(y)dy.$$

Thus

$$|(e^{-itH_1}u)(x)| \leqq \frac{1}{(4\pi t)^{3/2}} \int_{R^3} |u(y)|dy,$$

whence

$$\int_1^\infty \| (H_2 - H_1) e^{-itH_1}u\|dt \leqq \int_1^\infty \frac{\|u\|_{L^1} \|q\|_{L^2}}{(4\pi t)^{3/2}} dt$$

$$= \text{const} \int_1^\infty \frac{dt}{t^{3/2}} < \infty$$

provided $q \in L^2$. Hence in the case $q \in L^2$ the wave operator W_+ exists and in the same way W_- exists. The assumption that $q \in L^2$ also guarantees that H_2 is a well defined selfadjoint operator. The argument above uses the convergence of $\int_1^\infty dt/t^{3/2}$. Since $\int_1^\infty dt/t^\alpha$ converges for $\alpha > 1$ the above estimates can be modified, and the assumption on q can be weakened to

$$\int_{R^3} \frac{|q(x)|^2 dx}{(1+|x|)^{1-\epsilon}} < \infty \quad (\epsilon > 0).$$

A sufficient condition for this is $q \in L^2_{\text{loc}}$ and $|q| \sim c|x|^{-1-\epsilon'}$ for large $|x|$. In particular it is seen that $H_2 = -\Delta + q$ has a part which is unitarily equivalent to H_1, and hence the spectrum of H_2 contains the positive real axis.

Nothing is said about completeness in the above example. In fact, the wave operators are complete under conditions almost the same as above. For example the condition $|q(x)| \leqq c/(1+|x|)^{1+\epsilon}$ is shown below to be sufficient for completeness.

THEOREM (CHAIN RULE). *If $W_+(H_2, H_1)$ and $W_+(H_3, H_2)$ both exist then $W_+(H_3, H_1)$ exists and is equal to $W_+(H_3, H_2)W_+(H_2, H_1)$.*

PROOF. We can multiply strong limits so that

$$W_+(H_3, H_2)W_+(H_2, H_1) = \operatorname*{s-lim}_{t\to\infty} e^{itH_3} e^{-itH_2} P_2 e^{itH_2} e^{-itH_1} P_1.$$

P_2 commutes with H_2 and hence with e^{itH_2}, so

$$W_+(H_3, H_2)W_+(H_2, H_1) = \operatorname*{s-lim}_{t\to\infty} e^{itH_3} P_2 e^{-itH_1} P_1.$$

Thus it suffices to show

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_3}(1-P_2)\, e^{-itH_1} P_1 = 0.$$

Since e^{itH_3} and e^{itH_2} are unitary this is equivalent to

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_2}(1-P_2)\,e^{-itH_1}P_1 = 0.$$

Again since P_2 and e^{itH_2} commute, we must show

$$(1 - P_2)\operatorname*{s-lim}_{t\to\infty} e^{itH_2}e^{-itH_1}P_1 = 0$$

or

$$(1-P_2)W_+(H_2, H_1) = 0.$$

This is true since Range $W_+(H_2, H_1) \subset \mathcal{H}_{2,\mathrm{ac}}$.

If we use the chain rule taking $H_3 = H_1$ we get

COROLLARY. *If $W_+(H_2, H_1)$ and $W_+(H_1, H_2)$ exist then they are complete. Similarly for W_-.*

So far most of the results have not been very deep. The following theorem which gives a sufficient condition for the existence and completeness of the wave operators is more difficult.

THEOREM (BIRMAN, DE BRANGES, KATO). *$W_\pm(H_2, H_1)$ exist and are complete if $(H_2 - \zeta)^{-1} - (H_1 - \zeta)^{-1}$ belongs to the trace class for* $\operatorname{Im}\zeta \neq 0$.

This theorem has been proved using time-dependent methods. However, we shall postpone the proof until later when it will be proved using stationary methods. (Cf. M. Sh. Birman [2]; L. de Branges [4]; T. Kato [11].)

REMARK. The condition is satisfied either for all ζ with $\operatorname{Im}\zeta \neq 0$ or for no such ζ.

EXAMPLE. Consider the problem of potential scattering which we discussed earlier. We have

$$H_1 = -\Delta, \qquad H_2 = -\Delta + V, \qquad V = q(x).$$

Then the above theorem can be applied when $q \in L^1(R^3) \cap L^2(R^3)$. This assumption on q is weakened below by means of the stationary method.

Note that the above conditions concerning the existence and completeness of the wave operators are symmetric in H_1 and H_2. This symmetry would not hold without the introduction of generalized wave operators.

The invariance principle. It states:

If ϕ is real valued and piecewise monotone increasing, with a certain mild smoothness, then

$$W_\pm(H_2, H_1) = W_\pm(\phi(H_2), \phi(H_1)).$$

It would be nice if the existence of $W_\pm(H_2, H_1)$ implied the existence of $W_\pm(\phi(H_2), \phi(H_1))$ and the invariance principle. However, this has not been shown in general. If $(H_2 - \zeta)^{-1} - (H_1 - \zeta)^{-1}$ is of trace class the invariance principle does hold, and it is known to hold in many other cases.

Suppose $H_1\ H_2 \geqq 0$ in addition to $(H_2 - \zeta)^{-1} - (H_1 - \zeta)^{-1}$ belonging to the trace class. Then the choice $\phi(\lambda) = \lambda^2$ in the invariance principle gives that $W_\pm(H_2{}^2, H_1{}^2)$ exists and equals $W_\pm(H_2, H_1)$. If we take $\phi(\lambda) = -1/\lambda$ then we have

$$W_\pm(H_2, H_1) = W_\pm(-H_2{}^{-1}, -H_1{}^{-1}) = W_\mp(H_2{}^{-1}, H_1{}^{-1}).$$

3. **Formulas for the wave operators in the stationary theory.** A formal derivation of the formula for the wave operators which forms the basis of the stationary theory is given next. The definition of the wave operators in the time-dependent theory is simpler and clearer than the definition in the stationary theory, and the following argument gives a formal link between the two.

In the time-dependent theory the wave operator W_+ is defined by

$$W_+ = \operatorname*{s\text{-}lim}_{t\to\infty} e^{itH_2}e^{-itH_1}P_1 \ .$$

We first replace $\lim_{t\to\infty}$ by the Abel limit.

$$W_+ = \lim_{\epsilon\downarrow 0} 2\epsilon \int_0^\infty e^{-2\epsilon t}e^{itH_2}e^{-itH_1}P_1\, dt$$

$$= \lim_{\epsilon\downarrow 0} 2\epsilon \int_0^\infty e^{-\epsilon t+itH_2}(e^{-\epsilon t+itH_1})^*\, dt\, P_1 \ .$$

Roughly speaking the link between the two definitions of the wave operator is by means of the Fourier transform. We have

$$\frac{1}{(2\pi)^{1/2}}\int_0^\infty e^{-\epsilon t}e^{itH_2}e^{-it\lambda}\, dt = \frac{i}{(2\pi)^{1/2}}(H_2-\lambda+i\epsilon)^{-1}$$

and similarly for $e^{-\epsilon t+itH_1}$.

We apply Parseval's relation between the Fourier transforms to get

$$W_+ = \lim_{\epsilon\downarrow 0}\frac{2\epsilon}{2\pi}\int_{-\infty}^{\infty} (H_2 - \lambda + i\epsilon)^{-1}(H_1 - \lambda - i\epsilon)^{-1}\, d\lambda\, P_1 \ .$$

We let $R_k(\zeta) = (H_k - \zeta)^{-1}$ be the resolvent of H_k for $k = 1, 2$ and $\zeta = \lambda + i\epsilon$. Then

$$W_+ = \lim_{\epsilon\downarrow 0} \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} R_2(\bar{\zeta})R_1(\zeta)d\lambda\, P_1$$

$$= \lim_{\epsilon\downarrow 0} \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} R_2(\bar{\zeta})R_2(\zeta)(H_2 - \zeta)R_1(\zeta)d\lambda\, P_1.$$

If $E_2(\lambda)$ is the spectral family associated with H_2 then

$$\frac{\epsilon}{\pi} R_2(\bar{\zeta})R_2(\zeta) = \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{dE_2(\lambda')}{(\lambda' - \bar{\zeta})(\lambda' - \zeta)} = \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{dE_2(\lambda')}{(\lambda' - \lambda)^2 + \epsilon^2}$$

$$= \int_{-\infty}^{\infty} \delta_\epsilon(\lambda' - \lambda)dE_2(\lambda') = \delta_\epsilon(H_2 - \lambda)$$

where

$$\delta_\epsilon(\mu) = \frac{\epsilon}{\pi} \frac{1}{(\mu^2 + \epsilon^2)}\,.$$

If we let $G(\zeta) = (H_2 - \zeta)R_1(\zeta)$ then

$$W_+ = \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} \delta_\epsilon(H_2 - \lambda)G(\zeta)d\lambda\, P_1.$$

If we let $\epsilon \downarrow 0$ then δ_ϵ tends to the δ measure and

$$W_+ = \int_{-\infty}^{\infty} \delta(H_2 - \lambda)G(\lambda + i0)d\lambda\, P_1 = \int_{-\infty}^{\infty} \frac{dE_2(\lambda)}{d\lambda} G(\lambda + i0)d\lambda P_1\,.$$

In the corresponding formula for W_- the factor $G(\lambda + i0)$ is replaced by $G(\lambda - i0)$. Thus

$$W_\pm = \int_{-\infty}^{\infty} \frac{dE_2(\lambda)}{d\lambda} G(\lambda \pm i0)d\lambda\, P_1.$$

One should note that the derivative of the spectral measure does not exist if one considers it in the usual operator topologies, and the boundary values of G may not exist in the usual sense. However in the stationary theory below we interpret the last formula directly in order to define the operators $W_\pm$. Then we show that $W_\pm$ have certain properties that hold for the wave operators in the time-dependent theory. Under quite general assumptions we show that $W_\pm$ are partial isometries with initial set $\mathcal{H}_{1,ac}$ and final set $\mathcal{H}_{2,ac}$ and that $W_\pm$ have the intertwining property. We also show under more restrictive assumptions that $W_\pm$ are identical with the wave operators in the time-dependent theory, i.e. $e^{itH_2}e^{-itH_1}P_1 \xrightarrow[s]{} W_+$. Under the most general assumptions it may only be possible to show that the time-dependent wave operators exist as an Abel limit. We also prove the invariance principle. The assumptions will be sufficiently general to include most known results.

Another important relation which is related to the derivation above is the following:

$$G(\lambda+i0)^*\frac{dE_2(\lambda)}{d\lambda}\,G(\lambda+i0)=\frac{dE_1(\lambda)}{d\lambda}\ .$$

We establish this as follows: we have

$$R_2(\zeta)G(\zeta)=R_1(\zeta).$$

So

$$\begin{aligned}\delta_\epsilon(H_1-\lambda)&=\frac{\epsilon}{\pi}\,R_1(\bar\zeta)R_1(\zeta)=\frac{\epsilon}{\pi}R_1(\zeta)^*R_1(\zeta)\\&=\frac{\epsilon}{\pi}G(\zeta)^*R_2(\zeta)^*R_2(\zeta)G(\zeta)=G(\zeta)^*\delta_\epsilon(H_2-\lambda)G(\zeta).\end{aligned}$$

Letting $\epsilon\downarrow 0$ we get the desired relation. The same relation holds with $G(\lambda+i0)$ replaced by $G(\lambda-i0)$. From these relations we get the following formulas for W_+ and W_-

$$W_\pm=\int_{-\infty}^{\infty}\,[G(\lambda\pm i0)^*]^{-1}\frac{dE_1(\lambda)}{d\lambda}d\lambda\;P_1.$$

REMARK. Why do we define P_1 to be the projection onto $\mathcal{H}_{1,\mathrm{ac}}$ rather than $\mathcal{H}_{1,\mathrm{c}}$ (the subspace of continuity, consisting of all $u\in\mathcal{H}$ such that $(E_1(\lambda)u,\,u)$ is continuous in λ), for example? The formal properties of $W_\pm$ would not have been much different even if we used $\mathcal{H}_{1,\mathrm{c}}$. But $W_\pm$ are more likely to exist when we use $\mathcal{H}_{1,\mathrm{ac}}$ as we do. This is closely related to the fact that the absolutely continuous spectrum is rather stable under perturbation while the continuous spectrum is not.

4. The stationary theory of scattering.

1. *Spectral representations.* The rest of this paper is devoted to an exposition of a method in the stationary theory and its applications to Schrödinger operators. We restrict most of our attention to a simplified version of this method which is broad enough to include a considerable part of the applications and has shorter proofs than the general version. In order to indicate the content in the general case, some theorems are presented in two ways—a "simplified version"

and a "general version." All proofs are given for the simplified version.

In what follows the theory is developed for a pair of selfadjoint operators H_1 and H_2. Similar considerations apply as well to unitary operators U_1 and U_2. In some respects the unitary case is simpler because all operators involved are bounded. Furthermore, the selfadjoint case can be discussed in terms of the unitary case by using the Cayley transform. In this paper, however, we feel it convenient for the purpose of application to deal with the selfadjoint case directly. For a more complete treatment the reader is referred to T. Kato and S. T. Kuroda [14].

As was discussed above, the motivation for the stationary theory lies in the heuristic formula

$$W_{\pm} = \int_{-\infty}^{\infty} \frac{dE_2(\lambda)}{d\lambda} G(\lambda \pm i0) d\lambda \, P_1, \tag{4.1}$$

and the following arguments consist in interpreting the terms in the integrand correctly and constructing them as boundary values of resolvents and related quantities. Let

$$H_j = \int_{-\infty}^{\infty} \lambda dE_j(\lambda), \qquad j = 1, 2,$$

be selfadjoint operators in a Hilbert space $\mathcal{H}$. By abuse of notation, E_j is used to denote both the spectral family $\{E_j(\lambda);\ -\infty < \lambda < \infty\}$ and the spectral measure $\{E_j(\Delta);\ \Delta \subset R^1\}$ associated with H_j.

Decomposition of E_j. The spectral measures E_1 and E_2 may be decomposed as

$$E_j = E_{j,\mathrm{ac}} + E_{j,\mathrm{s}}, \qquad j = 1, 2,$$

where $E_{j,\mathrm{ac}}$ and $E_{j,\mathrm{s}}$ are characterized by the property that they are absolutely continuous and singular, respectively, with respect to the Lebesgue measure. Namely, for every $u \in \mathcal{H}$, the nonnegative measure $(E_{j,\mathrm{ac}}(\Delta)u, u)$ and $(E_{j,\mathrm{s}}(\Delta)u, u)$ are absolutely continuous and singular, respectively, with respect to the Lebesgue measure. Such a decomposition exists and is unique.

The uniqueness of the decomposition is trivial.

If $\mathcal{H}$ is separable, the existence may be inferred as follows. Let $\{u_k\}$ be a countable fundamental set of $\mathcal{H}$. For each k we apply the Lebesgue decomposition theorem to the nonnegative measure $\rho_k(\Delta) = (E(\Delta)u_k, u_k)$, $\Delta \subset R^1$, so that $\rho_k = \rho_k^{\mathrm{ac}} + \rho_k^{\mathrm{s}}$ (dropping the subscript j for the moment). Here and in what follows, a subset Δ of R^1 is always assumed to be Borel measurable and $|\Delta|$ stands for the

Lebesgue measure of Δ. Now, there exists $\Delta_k \subset R^1$, $|\Delta_k| = 0$, such that $\rho_k^s(\Delta) = \rho_k(\Delta \cap \Delta_k)$ for all Δ. Put $\Delta_0 = \bigcup_{k=1}^{\infty} \Delta_k$. Then, $|\Delta_0| = 0$. It is now easy to see that

$$E_{ac}(\Delta) \stackrel{d}{=} E(\Delta \cap (R^1 - \Delta_0))$$

and

$$E_s(\Delta) \stackrel{d}{=} E(\Delta \cap \Delta_0)$$

satisfy the requirement of the decomposition.

This shows that $E_{j,ac}(\Delta)$ and $E_{j,s}(\Delta)$ are mutually orthogonal, their ranges reduce H_j, and they commute with any bounded operator commuting with E_j.

When $\mathcal{H}$ is not separable, one may not be able to find such Δ_0. Nevertheless, the statements made in the last paragraph remain true.

The following notation is used.

$$\mathcal{H}_{j,ac} = E_{j,ac}(R^1)\mathcal{H} = \text{the subspace of absolute continuity of } H_j;$$

$$\mathcal{H}_{j,ac}(\Gamma) = E_{j,ac}(\Gamma)\mathcal{H}, \qquad \Gamma \subset R^1;$$

$$H_{j,ac} = H_j|\mathcal{H}_{j,ac}, \qquad H_{j,ac}(\Gamma) = H_j|\mathcal{H}_{j,ac}(\Gamma).$$

Localization. For later developments it will be convenient to introduce a "localization" of the problem. Let $\Gamma \subset R^1$ be a fixed Borel set. Then, $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = E_{j,ac}(\Gamma)\mathcal{H} \oplus E_{j,s}(\Gamma)\mathcal{H} \oplus E_{j,ac}(\Gamma')\mathcal{H} \oplus E_{j,s}(\Gamma')\mathcal{H}$$

and H_j is decomposed accordingly, where $\Gamma' = R^1 - \Gamma$. The localized problem is to restrict our attention to the set Γ and discuss the unitary equivalence of $H_{1,ac}(\Gamma)$ and $H_{2,ac}(\Gamma)$. Although we are most interested in the case $\Gamma = R^1$, it is convenient even in this case to have results for the localized problem.

Outline of a construction of intertwining operators. The Hilbert spaces $\mathcal{H}_{1,ac}(\Gamma)$ and $\mathcal{H}_{2,ac}(\Gamma)$ may be represented by spaces of direct integrals $\mathcal{H}_{1,ac}(\Gamma) \sim \int_\Gamma \oplus \mathcal{H}_1(\lambda)d\lambda$, $\mathcal{H}_{2,ac}(\Gamma) \sim \int_\Gamma \oplus \mathcal{H}_2(\lambda)d\lambda$ in such a way that $H_{1,ac}(\Gamma)$ and $H_{2,ac}(\Gamma)$ are transformed into multiplication by λ in the corresponding direct integral spaces. Suppose that there is given a family of unitary operators $G'(\lambda)$: $\mathcal{H}_1(\lambda) \to \mathcal{H}_2(\lambda)$, $\lambda \in \Gamma$. Then the correspondence $\{u(\lambda)\} \to \{G'(\lambda)u(\lambda)\}$ determines a (decomposable) unitary operator from $\int_\Gamma \oplus \mathcal{H}_1(\lambda)d\lambda$ onto $\int_\Gamma \oplus \mathcal{H}_2(\lambda)d\lambda$. This operator commutes with multiplication operators. Going back to the spaces $\mathcal{H}_{j,ac}(\Gamma)$, this gives a "wave" operator which intertwines $H_{1,ac}(\Gamma)$ and $H_{2,ac}(\Gamma)$. This argument is legitimate provided that

$G'(\lambda)$ satisfies a certain measurability requirement associated with the direct integral spaces $\int_\Gamma \oplus \mathcal{H}_j(\lambda)d\lambda$.

A spectral representation. For the moment we drop the subscript j. We fix $\Gamma \subset R^1$.

DEFINITION 4.0. Let $\mathcal{X}$ be a (not necessarily closed) subspace of $\mathcal{H}$. A function $f(\lambda; x, y)$: $\Gamma \times \mathcal{X} \times \mathcal{X} \to C^1$(= complex numbers) is called a *spectral form* with respect to $E(\lambda)$ if:

(i) for every $\lambda \in \Gamma$, $f(\lambda; \cdot, \cdot)$ is a nonnegative Hermitian form on $\mathcal{X} \times \mathcal{X}$;

(ii) for every $x, y \in \mathcal{X}$ we have $f(\lambda; x, y) = (d/d\lambda)(E(\lambda)x, y)$ for a.e. $\lambda \in \Gamma$ (the exceptional null set may depend on x and y).

EXAMPLE. Suppose there is a way of determining $(d/d\lambda)(E(\lambda)x, y)$ pointwise for every $\lambda \in \Gamma'$, $|\Gamma - \Gamma'| = 0$, and every $x, y \in \mathcal{X}$ (note that Γ' does not depend on x and y). Then,

$$\begin{aligned} f(\lambda; x, y) &= (d/d\lambda)(E(\lambda)x, y), && \lambda \in \Gamma', \\ &= 0, && \lambda \in \Gamma - \Gamma', \end{aligned}$$

is an example of a spectral form. For example, this is realized if $\mathcal{H} = L^2(R^1)$, $E(\lambda)$ = multiplication by $\chi_{(-\infty, \lambda)}$ (χ denotes the characteristic function), and $\mathcal{X} = L^2(R^1) \cap C(R^1)$. Another example is a finite-dimensional $\mathcal{X}$.

Now, starting with a spectral form, we construct a representation space. Fix $\lambda \in \Gamma$. Then, $f(\lambda; \cdot, \cdot)$ is a semi-inner product on $\mathcal{X}$ and induces naturally an inner product on the quotient space $\mathcal{X}/\mathcal{N}(\lambda)$, where $\mathcal{N}(\lambda) = \{x \mid f(\lambda; x, x) = 0\}$. Let $\mathcal{X}(\lambda)$ be the completion of $\mathcal{X}/\mathcal{N}(\lambda)$. Thus, $\mathcal{X}(\lambda)$ is a Hilbert space. The norm and the inner product in $\mathcal{X}(\lambda)$ are denoted by $\| \ \|_\lambda$ and $(\ , \ )_\lambda$, respectively. We have a natural map

$$J(\lambda): \mathcal{X} \to \mathcal{X}(\lambda),$$

which is the composite of two canonical homomorphisms $\mathcal{X} \to \mathcal{X}/\mathcal{N}(\lambda) \to \mathcal{X}(\lambda)$.

Let $\prod_{\lambda \in \Gamma} \mathcal{X}(\lambda)$ be the (algebraic) product space of the $\mathcal{X}(\lambda)$. We need a concept of measurability for $\{g(L)\} \in \prod_{\lambda \in \Gamma} \mathcal{X}(\lambda)$. An $\mathcal{X}$-valued simple function on Γ is a mapping $\Gamma \to \mathcal{X}$ having the form[1]

$$\sum C_k \chi_{\Delta_k}(\lambda) x_k, \qquad C_k \in C^1, \ \Delta_k \subset \Gamma, \ x_k \in \mathcal{X}.$$

DEFINITION 4.1. $\{g(\lambda)\} \in \prod_{\lambda \in \Gamma} \mathcal{X}(\lambda)$ is said to be f-measurable if there exists a sequence of $\mathcal{X}$-valued simple functions $x^{(n)}(\lambda)$ such that as $n \to \infty$

[1]NOTE ADDED IN PROOF. Here and in the sequel, coefficients C_k and C_ℓ are redundant and may be deleted.

$$\|g(\lambda) - J(\lambda)x^{(n)}(\lambda)\|_\lambda \to 0 \quad \text{for a.e. } \lambda \in \Gamma.$$

DEFINITION 4.2. $\mathfrak{M}$ is the set of all $\{g(\lambda)\} \in \prod_{\lambda \in \Gamma} \mathcal{X}(\lambda)$ such that $\{g(\lambda)\}$ is f-measurable and

$$\|g\|_{\mathfrak{M}}^2 \overset{d}{=} \int_\Gamma \|g(\lambda)\|_\lambda^2 \, d\lambda < \infty.$$

Then we have the following proposition, whose proof is straightforward and is omitted.

PROPOSITION 4.3. (i) *If g_n is f-measurable and*

$$\|g_n(\lambda) - g(\lambda)\|_\lambda \to 0 \quad \textit{a.e. as } n \to \infty,$$

then g is f-measurable.

(ii) *$\mathfrak{M}$ with the inner product*

$$(g, h)_{\mathfrak{M}} \overset{d}{=} \int_\Gamma (g(\lambda), h(\lambda))_\lambda \, d\lambda$$

is a Hilbert space.

(iii) *If $x(\lambda)$ is an $\mathcal{X}$-valued simple function, then $\{J(\lambda)x(\lambda)\}_{\lambda \in \Gamma} \in \mathfrak{M}$ and the totality of these $\{J(\lambda)x(\lambda)\}$ is dense in $\mathfrak{M}$.*

$\mathfrak{M}$ may be denoted as $\mathfrak{M} = \int_\Gamma \oplus \mathcal{X}(\lambda) d\lambda$, but we shall not use this notation.

We can now proceed to a representation theorem. What is going to be represented is not the entire H but its part in the subspace of $\mathcal{H}$ generated by the subsets $\{E_{ac}(\Delta)x \mid x \in \mathcal{X},\ \Delta \subset \Gamma\}$. Thus, we consider $u \in \mathcal{H}$ of the form

$$u = \sum_{k=1}^{r} C_k E_{ac}(\Delta_k)x_k, \qquad C_k \in C^1,\ \Delta_k \subset \Gamma,\ x_k \in \mathcal{X}. \tag{4.2}$$

Corresponding to it is the $\mathcal{X}$-valued simple function

$$\tilde{u}'(\lambda) = \sum_{k=1}^{r} C_k \chi_{\Delta_k}(\lambda) x_k.$$

Let

$$\tilde{u}(\lambda) = J(\lambda)\tilde{u}'(\lambda) = \sum_{k=1}^{r} C_k \chi_{\Delta_k}(\lambda) J(\lambda) x_k.$$

It is easily seen that $\tilde{u} = \{\tilde{u}(\lambda)\}_{\lambda \in \Gamma}$ is f-measurable. Let us compute the $\mathfrak{M}$-norm of $\tilde{u}$.

$$\|\tilde{u}(\lambda)\|_{\mathfrak{M}}^2 = \int_\Gamma \|\tilde{u}(\lambda)\|_\lambda^2 d\lambda = \sum_{k,\ell} C_k \bar{C}_\ell \int_{\Delta_k \cap \Delta_\ell} (J(\lambda)x_k,\ J(\lambda)x_\ell)_\lambda \, d\lambda.$$

Since $(J(\lambda)x_k, J(\lambda)x_\ell) = f(\lambda; x_k, x_\ell) = (d/d\lambda)(E(\lambda)x_k, x_\ell)$, as is immediate from the definition of $J(\lambda)$, the right-hand side is equal to

$$\sum_{k,\ell} C_k\bar{C}_\ell \int_{\Delta_k \cap \Delta_\ell} \frac{d}{d\lambda}(E(\lambda)x_k, x_\ell)d\lambda$$

$$= \sum_{k,\ell} C_k\bar{C}_\ell (E_{\mathrm{ac}}(\Delta_k \cap \Delta_\ell)x_k, x_\ell)$$

$$= \sum_{k,\ell} C_k\bar{C}_\ell (E_{\mathrm{ac}}(\Delta_k)x_k, E_{\mathrm{ac}}(\Delta_\ell)x_\ell) = \|u\|^2.$$

Let $\mathring{\mathcal{G}}$ be the set of all elements in $\mathcal{H}$ of the form (4.2) and $\mathcal{G}(\Gamma)$ be the closure of $\mathring{\mathcal{G}}$. Although the right-hand side of (4.2) is not uniquely determined by u, the considerations made above show that the correspondence $u \to \tilde{u}(\lambda)$ determines a well-defined, linear, and isometric mapping $\mathring{\pi} : \mathring{\mathcal{G}} \to \mathfrak{M}$. Obviously, $\{\{J(\lambda)x(\lambda)\} | x(\lambda)$ is an $\mathcal{X}$-valued simple function$\}$ is contained in the range of $\mathring{\pi}$. Since this set is dense in $\mathfrak{M}$ (cf. Proposition 4.3), $\mathring{\pi}$ extends to a unitary operator

$$\pi : \mathcal{G}(\Gamma) \to \mathfrak{M}.$$

From the construction above it is clear that $E_{\mathrm{ac}}(\Delta)$ corresponds to multiplication by χ_Δ, i.e.

$$(\pi(E_{\mathrm{ac}}(\Delta)u))(\lambda) = \chi_\Delta(\lambda)(\pi u)(\lambda), \text{ a.e.,}$$

for every $u \in \mathcal{G}(\Gamma)$. Thus, π gives a spectral representation of $\{E_{\mathrm{ac}}(\Delta)\}$ (or equivalently $\{E(\Delta)\}$) restricted to $\mathcal{G}(\Gamma)$.

The following is characterization of the space $\mathcal{G}(\Gamma)$.

Proposition 4.4. *$\mathcal{G}(\Gamma)$ is the smallest closed subspace of $\mathcal{H}$ which contains $E_{\mathrm{ac}}(\Gamma)\mathcal{X}$ and remains invariant under $E(\Delta)$ for every $\Delta \subset \Gamma$.*

2. *General theorems.* Next we apply the spectral representation theorem discussed above to prove an abstract theorem concerning the existence and completeness of wave operators. Then we will use the abstract theorem to prove the first in a series of more concrete and applicable theorems.

General assumptions. Before going on, we make a comment about the two versions of our presentation, the "simplified version" and the "general version," which were introduced above. We shall be working in a Hilbert space and shall have a subspace $\mathcal{X}$ of $\mathcal{H}$. We suppose that $\mathcal{X}$ has its own topology. In the "simplified version," it is assumed that $\mathcal{X}$ is a Banach space. In the "general version," $\mathcal{X}$ may be just a normed space or even a linear topological space. Simplification of the proof in the simplified version stems from the completeness of $\mathcal{X}$. In the general version one has to consider the completion of $\mathcal{X}$. Except for Theorem 4.5 and Theorem 4.6 where a general situation is indicated, we assume that $\mathcal{X}$ is a Banach space. However, in the first part of the discussion the topological properties of $\mathcal{X}$ play little role.

Now consider two selfadjoint operators $H_j = \int_{-\infty}^{\infty} \lambda dE_j(\lambda)$, $j = 1, 2$, in $\mathcal{H}$. Further, assume $\mathcal{X}$ is sufficiently large in the sense that the two sets

$$\left\{ \sum_{k=1}^{r} E_j(\Delta_k)x_k \mid \Delta_k \subset R^1, x_k \in \mathcal{X} \right\}, \qquad j = 1, 2, \tag{4.3}$$

are both dense in $\mathcal{H}$. (This does not restrict the generality in an essential way.) Under this assumption on $\mathcal{X}$ we have

$$\mathcal{G}_j(\Gamma) = E_{j,\mathrm{ac}}(\Gamma)\mathcal{H} = \mathcal{H}_{j,\mathrm{ac}}(\Gamma).$$

In applications the subspace $\mathcal{X}$ will frequently be dense in $\mathcal{H}$ in which case $\mathcal{X}$ is trivially seen to be sufficiently large. As usual $B(\mathcal{X}, \mathcal{Y})$ is the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ and $B(\mathcal{X}) = B(\mathcal{X}, \mathcal{X})$.

Existence of an intertwining operator.

THEOREM 4.5 (SIMPLIFIED VERSION). *Suppose that:*

(1) *for $j = 1, 2$ there exists f_j: $\Gamma \times \mathcal{X} \times \mathcal{X} \to C^1$ which is spectral with respect to E_j;*

(2) *for each $\lambda \in \Gamma$ there exists $G(\lambda) \in B(\mathcal{X})$ such that*

(a) *$G(\lambda)$ is one-to-one and onto;*

(b) *$f_1(\lambda; x, y) = f_2(\lambda; G(\lambda)x, G(\lambda)y)$ for every $x, y \in \mathcal{X}$;*

(c) *for every $x \in \mathcal{X}$, $G(\lambda)x$ and $G(\lambda)^{-1}x$ are strongly measurable as $\mathcal{X}$-valued functions of λ in Γ. Then there exists a unique $W \in B(\mathcal{H}_{1,\mathrm{ac}}(\Gamma), \mathcal{H}_{2,\mathrm{ac}}(\Gamma))$ such that*

$$(WE_{1,\mathrm{ac}}(\Delta)x, E_{2,\mathrm{ac}}(\Delta')y) = \int_{\Delta \cap \Delta'} f_1(\lambda; G(\lambda)x, y)d\lambda \tag{4.4}$$

for every $x, y \in \mathcal{X}$, Δ, $\Delta' \subset \Gamma$. This W is unitary, and

$$WH_1 = H_2W \quad \text{on } \mathcal{H}_{1,\mathrm{ac}}(\Gamma). \tag{4.5}$$

In particular $H_{1,\mathrm{ac}}(\Gamma)$ and $H_{2,\mathrm{ac}}(\Gamma)$ are unitarily equivalent via W. (Formula (4.4) corresponds to the heuristic formula (4.1).)

PROOF. Using $f_1(\lambda; x, y)$ and $f_2(\lambda; x, y)$, the spectral representations discussed above can be constructed for $E_{1,\mathrm{ac}}$ and $E_{2,\mathrm{ac}}$. All quantities introduced there will carry a subscript 1 or 2 corresponding to E_1 or E_2 respectively, e.g. $\mathcal{X}_1(\lambda)$, $\mathfrak{M}_2$, $(\cdot, \cdot)_{1\lambda}$, π_1, etc.

We have

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{G(\lambda)} & \mathcal{X} \\ {\scriptstyle J_1(\lambda)}\downarrow & & \downarrow{\scriptstyle J_2(\lambda)} \\ \mathcal{X}_1(\lambda) & & \mathcal{X}_2(\lambda) \end{array} .$$

We want to construct a unitary operator from $\mathcal{X}_1(\lambda)$ to $\mathcal{X}_2(\lambda)$ that completes the above diagram. By the assumption (2)-(b) we have

$$\begin{aligned}(J_1(\lambda)x, J_1(\lambda)y)_{1\lambda} &= f_1(\lambda; x, y)\\ &= f_2(\lambda; G(\lambda)x, G(\lambda)y)\\ &= (J_2(\lambda)G(\lambda)x, J_2(\lambda)G(\lambda)y)_{2\lambda}.\end{aligned}$$

According to this there exists an isometric operator $\tilde{G}(\lambda) : \mathcal{X}_1(\lambda) \to \mathcal{X}_2(\lambda)$ such that $J_2(\lambda)G(\lambda) = \tilde{G}(\lambda)J_1(\lambda)$. One sees that $J_2(\lambda)\mathcal{X} \subset$ Range of $\tilde{G}(\lambda)$ (note that $G(\lambda)$ is onto), and since $J_2(\lambda)\mathcal{X}$ is dense in $\mathcal{X}_2(\lambda)$ this means that $\tilde{G}(\lambda)$ is onto and hence unitary. We use the assumption (2)-(c) to show that $\{\tilde{G}(\lambda)g(\lambda)\}$ is f_2-measurable whenever $\{g(\lambda)\}$ is f_1-measurable. Therefore the mapping

$$\tilde{W} : \{g(\lambda)\} \to \{\tilde{G}(\lambda)g(\lambda)\}$$

determines an isometric operator : $\mathfrak{M}_1 \to \mathfrak{M}_2$. Similarly we see that the mapping

$$\tilde{W}' : \{h(\lambda)\} \to \{\tilde{G}(\lambda)^{-1}h(\lambda)\}$$

determines an isometric operator $\tilde{W}' : \mathfrak{M}_2 \to \mathfrak{M}_1$. One sees easily that $\tilde{W}'\tilde{W} = I$ on $\mathfrak{M}_1$. Therefore, $\tilde{W}$ is unitary. Furthermore, $\tilde{W}$ commutes with multiplication operators.

We now go back to $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$ and $\mathcal{H}_{2,\mathrm{ac}}(\Gamma)$ by means of the unitary maps π_1 and π_2. Namely, we put

$$W = \pi_2^{-1}\tilde{W}\pi_1.$$

Then it is easy to check that W satisfies all the assumptions of the theorem. In fact, W intertwines with the spectral measures since $\tilde{W}$ commutes with multiplication operators and each π_j converts $E_j(\Delta)$ into multiplication by χ_Δ. Hence, (4.5) follows. An easy verification of (4.4) is skipped, because it is not used here.

Construction of f_j and G as boundary values. Until a later stage when a generalization is discussed, it is assumed that $D(H_1) = D(H_2)$. In other words, we think of the situation

$$H_2 = H_1 + V, \quad \text{where } V \text{ is symmetric with } D(V) \supset D(H_1).$$

As before, let

$$R_j(\zeta) = (H_j - \zeta)^{-1},$$

$$\delta_\epsilon(H_j - \lambda) = \frac{1}{2\pi i}\{R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon)\} = \frac{\epsilon}{\pi} R_j(\lambda - i\epsilon) R_j(\lambda + i\epsilon)$$

and let

$$f_{j\epsilon}(\lambda; u, v) = (\delta_\epsilon(H_j - \lambda)u, v), \qquad u, v \in \mathcal{H}.$$

THEOREM 4.6 (SIMPLIFIED VERSION). *Suppose that:*

(1) *for every* $\lambda \in \Gamma$ *and* $\epsilon > 0$ *the Hermitian form* $f_{1\epsilon}(\lambda; \cdot, \cdot)$ *is continuous on* $\mathcal{X} \times \mathcal{X}$ *(with respect to the* $\mathcal{X}$*-topology) and for every* $x, y \in \mathcal{X}$ *the limit*

$$\lim_{\epsilon \downarrow 0} f_{1\epsilon}(\lambda; x, y) \stackrel{d}{=} f_1(\lambda; x, y) \quad \textit{exists};$$

(2) *the following conditions* (a), (b), *and* (c) *hold for* $j = 1, 2$;

(a) *for every* $\lambda \in \Gamma$ *and* $\epsilon > 0$ *the operator* $VR_j(\lambda + i\epsilon)$ *maps* $\mathcal{X}$ *into* $\mathcal{X}$ *and is continuous (with respect to the* $\mathcal{X}$ *-topology)*;

(b) *for every* $\lambda \in \Gamma$ *the limit*

$$\text{s-}\lim_{\epsilon \downarrow 0} VR_j(\lambda + i\epsilon) \stackrel{d}{=} \tilde{Q}_j^+(\lambda) \textit{ exists in } B(\mathcal{X});$$

(c) *for every* $\epsilon > 0$ *and every* $x \in \mathcal{X}$, $VR_j(\lambda + i\epsilon)x$ *is strongly measurable as an* $\mathcal{X}$*-valued function of* λ.

Then the following two conclusions hold:

(i) *the statements of* (1) *in the assumption also hold for* $f_{2\epsilon}$ *instead of* $f_{1\epsilon}$ *(and hence* $f_2(\lambda; x, y)$ *is defined correspondingly).*

(ii) $f_1(\lambda; x, y)$, $f_2(\lambda; x, y)$ *and* $G(\lambda) \stackrel{d}{=} 1 + \tilde{Q}_1^+(\lambda)$ *satisfy the assumptions of Theorem* 4.5.

PROOF. For $\zeta = \lambda + i\epsilon$ put

$$G(\zeta) = (H_2 - \zeta) R_1(\zeta) = 1 + VR_1(\zeta),$$
$$G'(\zeta) = (H_1 - \zeta) R_2(\zeta) = 1 - VR_2(\zeta).$$

Then

$$G(\zeta)G'(\zeta) = G'(\zeta)G(\zeta) = 1 \quad \text{on } \mathcal{H}. \tag{4.6}$$

By (2)-(a) $G(\zeta)$ and $G'(\zeta)$ map $\mathcal{X}$ into $\mathcal{X}$, and hence

$$G(\zeta)G'(\zeta) = G'(\zeta)G(\zeta) = 1 \quad \text{in } \mathcal{X}. \tag{4.7}$$

Put

$$G(\lambda) = 1 + \tilde{Q}_1^+(\lambda), \qquad G'(\lambda) = 1 - \tilde{Q}_2^+(\lambda)\ .$$

Then taking the limit in (4.7) above as $\epsilon \downarrow 0$ we get $G(\lambda)G'(\lambda) = G'(\lambda)G(\lambda) = 1$ on $\mathfrak{X}$. Thus $G(\lambda) : \mathfrak{X} \to \mathfrak{X}$ is one-to-one and onto. Since we have taken strong limits we have $G(\lambda) \in B(\mathfrak{X})$. The measurability assumption (2)-(c) of Theorem 4.5 follows from the measurability assumption (2)-(c) in this theorem. Thus, (2)-(a) and (2)-(c) of Theorem 4.5 are verified.

It was shown in §3 that

$$\delta_\epsilon(H_1 - \lambda) = G(\lambda + i\epsilon)^* \delta_\epsilon(H_2 - \lambda)G(\lambda + i\epsilon).$$

Hence,

$$\delta_\epsilon(H_2 - \lambda) = G'(\lambda + i\epsilon)^* \delta_\epsilon(H_1 - \lambda)G'(\lambda + i\epsilon)$$

where * is taken as an operator in $\mathcal{H}$. We therefore have

$$f_{2\epsilon}(\lambda; x, y) = f_{1\epsilon}(\lambda; G'(\lambda + i\epsilon)x, G'(\lambda + i\epsilon)y).$$

We let $\epsilon \downarrow 0$. Because of (1) and (2)-(a) (not that $f_{1\epsilon}(\lambda)$ is then uniformly bounded in ϵ) the right side tends to $f_1(\lambda; G'(\lambda)x, G'(\lambda)y)$. Therefore the limit of the left side exists and

$$f_2(\lambda; x, y) \overset{d}{=} \lim_{\epsilon \downarrow 0} f_{2\epsilon}(\lambda; x, y) = f_1(\lambda; G'(\lambda)x, G'(\lambda)y).$$

This proves the first assertion. Since $G'(\lambda) = G(\lambda)^{-1}$ in $\mathfrak{X}$ we have

$$f_1(\lambda; x, y) = f_2(\lambda; G(\lambda)x, G(\lambda)y)$$

for $x, y \in \mathfrak{X}$. This yields (2)-(b) of Theorem 4.5.

What is left to be checked is that f_1 and f_2 are spectral. For $j = 1$, 2 we have

$$\begin{aligned} f_j(\lambda; x, y) &= \lim_{\epsilon \downarrow 0} f_{j\epsilon}(\lambda; x, y) = \lim_{\epsilon \downarrow 0} (\delta_\epsilon(H_j - \lambda)x, y) \\ &= \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{1}{(\lambda' - \lambda)^2 + \epsilon^2} \, d(E_j(\lambda')x, y) \\ &= \frac{d}{d\lambda}(E_j(\lambda)x, y) \quad \text{for a.e. } \lambda \in \Gamma. \end{aligned}$$

This completes the proof of Theorem 4.6.

REMARK 4.7. In Theorem 4.6 an analogous statement holds if $\lambda + i\epsilon$ is replaced by $\lambda - i\epsilon$ in assumption (2). Thus, if assumption (2) holds for $\lambda - i\epsilon$ as well as $\lambda + i\epsilon$, then there exist two intertwining operators which we denote by $W_\pm$. So far, the definition of $W_\pm$ depends on the choice of $\mathfrak{X}$. However, it is shown below that $W_\pm$ can be expressed as the Abel limit as $t \to \pm\infty$ of $e^{itH_2} e^{-itH_1}$ on $\mathcal{H}_{1,\text{ac}}(\Gamma)$, and hence $W_\pm$ are essentially independent of $\mathfrak{X}$.

EXAMPLE, PERTURBATION OF RANK 1. Suppose that

$$H_2 = H_1 + cP_\phi,$$

where c is real, $\phi \in \mathcal{H}$ with $\|\phi\| = 1$, and P_ϕ is the projection on the 1-dimensional subspace determined by ϕ : $P_\phi u = (u, \phi)\phi$. We take this subspace to be $\mathcal{X}$: $\mathcal{X} = \{\alpha\phi \mid \alpha \in C^1\}$.

Let $x = \alpha\phi$ and $y = \beta\phi$ be two arbitrary elements of $\mathcal{X}$. Then,

$$(R_j(\lambda \pm i\epsilon)x, y) = \alpha\bar{\beta}\rho_j(\lambda \pm i\epsilon),$$

where

$$\rho_j(\xi) = \int_{-\infty}^{\infty} \frac{1}{\mu - \xi} d(E_j(\mu)\phi, \phi).$$

It is well known that $\rho_j(\lambda \pm i\epsilon)$ has boundary values for a.e. $\lambda \in R^1$ as $\epsilon \downarrow 0$. Let $\Gamma \subset R^1$ be such that $|R^1 - \Gamma| = 0$ and $\lim_{\epsilon\downarrow 0} \rho_j(\lambda \pm i\epsilon)$ exists for every $\lambda \in \Gamma$ and $j = 1, 2$. Then,

$$f_{1\epsilon}(\lambda; x, y) = \frac{\alpha\bar{\beta}}{2\pi i}\{\rho_1(\lambda + i\epsilon) - \rho_1(\lambda - i\epsilon)\}$$

and

$$VR_j(\lambda \pm i\epsilon)x = c(R_j(\lambda \pm i\epsilon)x, \phi)\phi = c\alpha\rho_j(\lambda \pm i\epsilon)\phi$$

both converge for $\lambda \in \Gamma$ as $\epsilon \downarrow 0$. The other assumptions of Theorem 4.6 are trivially verified because $\mathcal{X}$ is one-dimensional.

Finally, consider the question of whether or not $\mathcal{X}$ is sufficiently large. The answer is no, because the condition stated in the paragraph containing (4.3) need not be true in general. However, this does not cause any real difficulty, as we shall now show.

LEMMA 4.8. *Denoting the closed linear hull by* $\overline{\mathrm{sp}}$, *we have*

$$\overline{\mathrm{sp}}\{E_1(\Delta)\phi \mid \Delta \subset R^1\} = \overline{\mathrm{sp}}\{E_2(\Delta)\phi \mid \Delta \subset R^1\} \overset{d}{=} \mathcal{H}_0. \tag{4.8}$$

(Note that $\mathcal{H}_0$ *reduces both* H_1 *and* H_2.*) Furthermore, on* $\mathcal{H} \ominus \mathcal{H}_0$ *one has* $H_2 = H_1$.

On $\mathcal{H} \ominus \mathcal{H}_0$ nothing interesting happens. On $\mathcal{H}_0$ the space $\mathcal{X}$ is sufficiently large precisely in the sense stated in the paragraph containing (4.3). Thus, Theorem 4.6 can be applied to H_1 and H_2 in $\mathcal{H}_0$. It must be noted that, since $|R^1 - \Gamma| = 0$, there is no difference between $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$ and $\mathcal{H}_{1,\mathrm{ac}}(R^1) = \mathcal{H}_{1,\mathrm{ac}}$. Thus, Theorem 4.6 asserts that $H_{1,\mathrm{ac}}$ in $\mathcal{H}_0$ and $H_{2,\mathrm{ac}}$ in $\mathcal{H}_0$ are unitarily equivalent. But one can drop the phrase "in $\mathcal{H}_0$" from this statement, because $H_2 = H_1$ on $\mathcal{H} \ominus \mathcal{H}_0$.

For the sake of completeness we include the proof of Lemma 4.8. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the first and the second members of (4.8). It suffices to show that $\mathcal{M}_1$ reduces H_2. In fact we then have $\mathcal{M}_1 \supset \mathcal{M}_2$, because $\mathcal{X} \subset \mathcal{M}_1$ and $\mathcal{M}_2$ is the smallest subspace containing $\mathcal{X}$ and reducing H_2. The opposite inclusion $\mathcal{M}_1 \subset \mathcal{M}_2$ is proved by symmetry. To show that $\mathcal{M}_1$ reduces H_2 let P be the projection on $\mathcal{M}_1$, and take an arbitrary $u \in D(H_2) = D(H_1)$. Since $\mathcal{M}_1$ reduces H_1, we have $Pu \in D(H_1) = D(H_2)$. Furthermore, $H_2Pu = H_1Pu + c(Pu, \phi)\phi = PH_1 u + c(u, \phi)P\phi = PH_2u$. Thus, $\mathcal{M}_1$ reduces H_2.

Using the same argument we can prove the same result for perturbations of finite rank. Alternatively, one may regard a perturbation of finite rank as a succession of perturbations of rank 1 and make a step-by-step construction of the intertwining operators.

The argument in the proof can also be extended to perturbations of trace class. Namely, $H_2 = H_1 + V$, where V is of trace class. To do this, take $\mathcal{X}$ to be the range of $|V|^{1/2}$ with the norm $\|x\|_{\mathcal{X}} = \inf\{\|u\|_{\mathcal{H}}\colon |V|^{1/2}u = x\}$ (cf. the factorization method discussed below). In verifying the hypotheses of Theorem 4.6 it is necessary to show the existence in the strong operator topology of the boundary values of $\int_a^b (1/(\mu - (\lambda + i\epsilon)))\, T(\mu)d\rho(\mu)$ for a trace class valued, integrable function $T(\mu)$ and a finite Lebesgue-Stieltjes measure ρ. This is by no means evident and the proof of the existence of such boundary values constitutes a central part of the argument. As a matter of fact, it can be shown that the boundary values exist in the Hilbert-Schmidt norm (de Branges [4], Asano [1]). Therefore, Theorem 4.6 can be applied and it follows that the absolutely continuous parts of H_1 and H_2 are unitarily equivalent.

Some Remarks.

Remark 4.9. A connection with time-dependent wave operators is discussed below. The main results are as follows.

1. Assume all the assumptions of Theorem 4.6 and let W_+ be the stationary wave operator constructed in Theorem 4.5. Then, W_+ can be expressed as the strong Abel limit of $e^{itH_2}e^{-itH_1}$ on $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$ (see Theorem 6.1).

2. Assume in addition that: (i) $\mathcal{X}$ is a Hilbert space; and (ii) assumption (2) in Theorem 4.6 is satisfied for $\lambda - i\epsilon$ as well as $\lambda + i\epsilon$. Then, the time-dependent wave operators $\text{s-lim}_{t\to\pm\infty} e^{itH_2}e^{-itH_1}E_{1,\mathrm{ac}}(\Gamma)$ exist, they coincide with the stationary wave operators $W_\pm$, and hence they are complete. Furthermore, the invariance principle holds (see Theorem 6.3).

Statement 1 shows in particular that under Theorem 4.6 the stationary operator W does not depend on $\mathcal{X}$ in an essential way. Namely, if $u \in \mathcal{H}$ belongs to the initial set of W, then Wu is unique irrespective of the $\mathcal{X}$ used in the construction.

Statement 2 can be applied to perturbations of trace class, because

$\mathcal{X}$ = the range of $|V|^{1/2}$ is a Hilbert space. Thus, we recapture the theorem of Rosenblum and Kato [12, p. 540].

REMARK 4.10. The assumption that $\mathcal{X}$ is complete is sometimes too restrictive. We shall describe how Theorems 4.5 and 4.6 can be modified when $\mathcal{X}$ is not complete. $\mathcal{X}$ can be a general linear topological space, but to fix the idea let us suppose $\mathcal{X}$ is a normed space. Let $\overline{\mathcal{X}}$ be the completion of $\mathcal{X}$. (Note that in general there is no inclusion relation between $\mathcal{H}$ and $\overline{\mathcal{X}}$.)

THEOREM 4.5′ (A GENERAL VERSION). *Suppose that:*
(1) *condition* (1) *of Theorem* 4.5 *holds and* $f_j(\lambda;\cdot,\cdot)$ *is continuous on* $\mathcal{X} \times \mathcal{X}$ *for every* $\lambda \in \Gamma$;
(2) *for each* $\lambda \in \Gamma$ *there exists* $G(\lambda) \in B(\overline{\mathcal{X}})$ *satisfying* (a), (b), (c) *of Theorem* 4.5 *with* f_2 *replaced by* $\bar{f}_2$, *the continuous extension of* f_2 *to* $\overline{\mathcal{X}} \times \overline{\mathcal{X}}$. *Then, the conclusion of Theorem* 4.5 *holds with* f_2 *in* (4.5) *replaced by* $\bar{f}_2$.

THEOREM 4.6′ (A GENERAL VERSION). *Suppose that*:
(1) *condition* (1) *of Theorem* 4.6 *holds, and for every* $\lambda \in \Gamma$ *the family* $f_{1\epsilon}(\lambda;\cdot,\cdot)$ *is equicontinuous in* ϵ.
(2) *condition* (2) *of Theorem* 4.6 *holds with the following changes*: (i) *the equicontinuity of* $VR_j(\lambda + i\epsilon)$ *in* $\epsilon > 0$ *is added*; (ii) *condition* (b) *is replaced by* (b′) *for every* $x \in \mathcal{X}$ *and* $\lambda \in \Gamma$, $VR_j(\lambda + i\epsilon)x$ *is a Cauchy net in* $\mathcal{X}$ *as* $\epsilon \downarrow 0$. *Then, the conclusion of Theorem* 4.6 *holds with obvious changes.*

In applications to Schrödinger operators, where $\mathcal{H} = L^2(R^3)$, $\mathcal{X}$ may be a weighted L^2-space. In this case $\mathcal{X} \subset \mathcal{H}$ will be complete. As another possibility $\mathcal{X}$ may be $L^{6/5}(R^3) \cap L^2(R^3)$. Then $\overline{\mathcal{X}}$ is $L^{6/5}(R^3)$.

REMARK 4.11. As can be seen from the proof, the argument leading to Theorem 4.5 can be applied to a pair of spectral measures E_1 and E_2 on an arbitrary (σ-finite) measure space (Γ, B, m). In particular Γ can be a subset of the unit circle with the Lebesgue measure. Accordingly, an analogue of Theorem 4.6 holds for a pair of unitary operators U_1 and U_2, $U_j = \int_0^{2\pi} e^{i\theta} dE_j(\theta)$. Here, $\delta_\epsilon(H_j - \lambda)$ is replaced by

$$\begin{aligned}\delta_r(U_j - \theta) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta' - \theta) + r^2}\, dE_j(\theta') \\ &= \frac{1}{2\pi}(1 - r^2)(1 - re^{-i\theta}U_j)^{-1}(1 - re^{i\theta}U_j^*)^{-1} \\ &= \frac{1}{2\pi}\{(1 - re^{i\theta}U_j^*)^{-1} - (1 - r^{-1}e^{i\theta}U_j^*)^{-1}\},\quad 0 < r < 1,\end{aligned}$$

and $1 + VR_1(\lambda + i\epsilon)$ is replaced by

$$(1 - re^{i\theta}U_2^*)(1 - re^{i\theta}U_1^*)^{-1} = 1 + re^{i\theta}(U_1^* - U_2^*)(1 - re^{i\theta}U_1^*)^{-1}.$$

Furthermore, the connections with the time-dependent wave operators, including the invariance principle, which were described in Remark 4.9, also hold for unitary operators (more details are given in §6).

Applying these considerations, one can see that, if $U_2 - U_1$ belongs to the trace class, then $U_{1,\mathrm{ac}}$ and $U_{2,\mathrm{ac}}$ are unitarily equivalent, the time-dependent wave operators exist, and they are complete. Let H_j be the inverse Cayley transform of U_j. Then, "$U_2 - U_1 \in$ trace class" is equivalent to "$R_2(\zeta) - R_1(\zeta) \in$ trace class, $\operatorname{Im} \zeta \neq 0$." This proves the theorem of Birman-de Branges-Kato stated in §3.

REMARK 4.12. If $f_{j\epsilon}(\lambda; \cdot, \cdot)$ is uniformly bounded in $\{\lambda + i\epsilon \mid \lambda \in \Gamma, \epsilon > 0\}$ and $\lim_{\epsilon \downarrow 0} f_{j\epsilon}(\lambda; x, y)$ exists for all $x, y \in \mathcal{X}$, then H_j is absolutely continuous on Γ (i.e. in $E_j(\Gamma)\mathcal{H}$)—provided of course that $\mathcal{X}$ is sufficiently large. In particular, there is no singular spectrum of H_j in Γ. This is proved in a standard way by using the formula that gives the spectral measure in terms of the resolvent.

If the hypotheses of Theorem 4.6 are fulfilled and if $f_{1\epsilon}(\lambda; x, y)$ and $VR_2(\lambda + i\epsilon)$ are both uniformly bounded then H_1 and H_2 are both absolutely continuous on Γ. The fact that H_2 is absolutely continuous on Γ follows from the fact that

$$f_{2\epsilon}(\lambda; x, y) = f_{1\epsilon}(\lambda; \{1 - VR_2(\lambda + i\epsilon)\}x, \{1 - VR_2(\lambda + i\epsilon)\}y)$$

which shows that $f_{2\epsilon}(\lambda; x, y)$ is uniformly bounded.

3. *Some specific situations.*

The factorization method. In more specific situations Theorem 4.6 can be simplified. Namely, we can eliminate $\mathcal{X}$ and/or R_2 from the assumptions. Let us first describe the elimination of $\mathcal{X}$ by the factorization method.

Suppose that H_2 can be expressed as

$$H_2 = H_1 + V = H_2 + AB, \tag{4.9}$$

where $A \in B(\mathcal{H})$ and B is closed with $D(B) \supset D(H_1)$.

EXAMPLES. Suppose V is of trace class. We let $A = |V|^{1/2}$ and $B = (\operatorname{sgn} V)|V|^{1/2}$. In the case of Schrödinger operators $H_2 = -\Delta + q(x)$ we shall use the factorization $q(x) = (1/(1 + |x|)^{\alpha})q_1(x)$. Another possibility is to factor q as $q(x) = |q(x)|^{1/2}\{\operatorname{sgn} q(x)|q(x)|^{1/2}\}$. In this factorization A might not be bounded. However, the method described below can be generalized to such a case.

Returning to the general theory, let $\overline{\mathcal{X}}$ be the range of A with the norm

$$\|x\|_{\mathcal{X}} = \inf_{Au=x} \|u\|_{\mathcal{H}}.$$

$\mathcal{X}$ is a Hilbert space. (Note that A maps $\mathcal{H} \ominus \mathcal{N}(A)$ isometrically onto $\mathcal{X}$, where $\mathcal{N}(A)$ is the kernel of A.)

THEOREM 4.13. *Suppose that A and B in* (4.9) *satisfy the following conditions*:

(F-1) $\quad \text{w-}\lim_{\epsilon \downarrow 0} A^* \delta_\epsilon(H_1 - \lambda) A$ *exists for all* $\lambda \in \Gamma$;

(F-2) $\quad \text{s-}\lim_{\epsilon \downarrow 0} BR_j(\lambda + i\epsilon)A \overset{d}{=} Q_j^+(\lambda)$ *exists for all* $\lambda \in \Gamma$.

Then, all the assumptions of Theorem* 4.6 *are satisfied.

PROOF. We shall indicate how (F-2) implies that hypothesis (2) in Theorem 4.6 is satisfied. That (F-1) implies (1) in Theorem 4.6 can be seen similarly. (2)-(a) follows from the fact that $BR_j(\lambda + i\epsilon)A \in B(\mathcal{H})$. Namely, let $x \in \mathcal{X}$ be expressed as $x = Au$. Then,

$$\begin{aligned} \|VR_j(\lambda + i\epsilon)x\|_{\mathcal{X}} &= \|ABR_j(\lambda + i\epsilon)Au\|_{\mathcal{X}} \\ &\leqq \|BR_j(\lambda + i\epsilon)Au\|_{\mathcal{H}} \\ &\leqq \|BR_j(\lambda + i\epsilon)A\| \, \|u\|. \end{aligned}$$

Since this is true for every u such that $Au = x$, we get $\|VR_j(\lambda + i\epsilon)x\|_{\mathcal{X}} \leqq \|BR_j(\lambda + i\epsilon)A\| \, \|x\|_{\mathcal{X}}$. Thus (2)-(a) of Theorem 4.6 is verified. (2)-(b) follows from (F-2) by a similar estimate. (2)-(c) is an easy consequence of the continuity of $R_j(\lambda + i\epsilon)$ in λ.

Some remarks on perturbations of trace class. By means of the factorization method the problem is reduced to the investigation of the boundary values of the integral

$$\int_{-\infty}^{\infty} \frac{1}{\mu - (\lambda + i\epsilon)} d(AE_1(\lambda)A), \qquad A = |V|^{1/2}.$$

It can be shown that this integral can be written as

$$\int_{-\infty}^{\infty} \frac{1}{\mu - (\lambda + i\epsilon)} M(\lambda) d\rho(\lambda), \quad M(\lambda) \in \text{trace class}, \ M(\lambda) \geqq 0.$$

Since the imaginary part, $(T + T^*)/2i$, of this integral is nonnegative, it suffices to investigate the boundary value of $T(\zeta)$, where $T(\zeta)$ is defined and holomorphic in $\{|\zeta| < 1\}$, with its value taken in the trace class, and satisfies $T(\zeta) + T(\zeta)^* \geqq 0$. By means of the determinant theory for operators of the type $1 + T$, $T \in$ trace class, we argue as follows.

$$
\begin{aligned}
|\det\{1 + T(\zeta)\}|^2 &= \det(1 + T^*)\det(1 + T) \\
&= \det(1 + T + T^* + T^*T) \\
&\geqq \det(1 + T^*T) = \prod(1 + |\lambda_n|^2) \\
&\geqq \begin{cases} \sum|\lambda_n|^2 = \|T(\zeta)\|^2_{\mathrm{H.S.}} \\ 1, \end{cases}
\end{aligned}
$$

where $\{\lambda_n\}$ are the eigenvalues of $|T|$ and $\| \ \|_{\mathrm{H.S.}}$ denotes the Hilbert-Schmidt norm. Thus

$$
\left\| \frac{T(\xi)}{\det(1 + T(\xi))} \right\|_{\mathrm{H.S.}} \leqq 1, \qquad \left| \frac{1}{\det(1 + T(\xi))} \right| \leqq 1.
$$

Therefore, both of these functions have boundary values almost everywhere and so does $T(\zeta)$ in the Hilbert-Schmidt norm. This is due to de Branges, [4]. Asano [1] later published his result that $\int_a^b(1/(\mu - \zeta))x(\mu)d\rho(\mu)$, where $x(\mu)$ is $\mathfrak{X}$-valued, has boundary values almost everywhere if $\mathfrak{X}$ is a Hilbert space. This can also be applied here.

Next we discuss how to eliminate the assumption involving R_2. The results can be formulated either in the general scheme of Theorem 4.6 (or Theorem 4.6′) or in the factorization situation. Here we work entirely in the factorization situation, since the results are simpler. The first application to Schrödinger operators is also given.

As before, we have $H_2 = H_1 + AB$ and use the notation $Q_j(\zeta) = BR_j(\zeta)A$. In what follows we formulate the results for the upper half plane ($\mathrm{Im}\,\zeta > 0$ or $\lambda + i\epsilon$ with $\epsilon > 0$). The same results hold for the lower half plane ($\mathrm{Im}\,\zeta < 0$ or $\lambda + i\epsilon$ with $\epsilon < 0$) as well.

Small perturbations.

PROPOSITION 4.14. *Suppose that:* (a) *there exist* η, $0 \leqq \eta < 1$, *and* $\epsilon_0 > 0$ *such that* $\|Q_1(\lambda + i\epsilon)\| \leqq \eta$ *for every* $\lambda \in \Gamma$ *and* ϵ, $0 < \epsilon < \epsilon_0$; (b) *for every* $\lambda \in \Gamma$ *the limit* $\text{s-lim}_{\epsilon\downarrow 0} Q_1(\lambda + i\epsilon) = Q_1^+(\lambda)$ *exists. Then condition* (F-2) *of Theorem* 4.13 *is satisfied.*

PROOF. We first note that

$$
1 - Q_2(\zeta) = (1 + Q_1(\zeta))^{-1}, \qquad \mathrm{Im}\,\zeta > 0. \tag{4.10}
$$

In fact, this can be verified by a direct computation using the second resolvent equation. On the other hand, assumptions (a) and (b) imply $\|Q_1^+(\lambda)\| \leqq \eta < 1$. Hence, there exists $(1 + Q_1^+(\lambda))^{-1} \in B(\mathcal{H})$. Then, we see that

$$
\begin{aligned}
&\{1 - Q_2(\lambda + i\epsilon)\} - (1 + Q_1^+(\lambda))^{-1} \\
&\quad = (1 + Q_1(\lambda + i\epsilon))^{-1}\{Q_1(\lambda) - Q_1^+(\lambda + i\epsilon)\}(1 + Q_1^+(\lambda))^{-1}.
\end{aligned}
$$

On the right-hand side the norm of the first factor is majorized by $(1-\eta)^{-1}$ and the second factor converges strongly to 0 by (b). Hence, we see that $\text{s-lim}_{\epsilon\downarrow 0} Q_2(\lambda+i\epsilon) = 1-(1+Q_1^\dagger(\lambda))^{-1}$, which implies (F-2) for $j=2$. (F-2) for $j=1$ is assumed as (b).

REMARK 4.15. Suppose that (F-1) and (a), (b) above are satisfied. If in additon $A^*\delta_\epsilon(H_1-\lambda)A$ is uniformly bounded for $\lambda\in\Gamma$ and $\epsilon>0$, then H_1 and H_2 are absolutely continuous on Γ (provided that the range of A is sufficiently large).

Application to Schrödinger operators. Consider the Schrödinger operator

$$H_2 = -\Delta + q(x), \qquad H_1 = -\Delta$$

in $\mathcal{H} = L^2(R^3)$. Suppose that the potential $q(x)$ is factored as

$$q(x) = \frac{1}{(1+|x|)^\alpha}\, q_1(x). \tag{4.11}$$

We assume $\alpha > 3/2$ for simplicity. Let A and B be multiplication operators by $1/(1+|x|)^\alpha$ and $q_1(x)$, respectively. The operator $Q_1(\zeta)$, $\operatorname{Im}\zeta>0$, then has the kernel

$$k(x,y;\zeta) = \frac{1}{4\pi}\, q_1(x) \times \frac{e^{i\sqrt{(\zeta}|x-y|)}}{|x-y|} \times \frac{1}{(1+|y|)^\alpha}\,.$$

We first investigate this kernel for $\operatorname{Im}\zeta \geqq 0$ and estimate its Hilbert-Schmidt norm.

LEMMA 4.16. *We have*

$$h(x) \overset{d}{=} \int_{R^3} \frac{1}{|x-y|^2} \times \frac{1}{(1+|y|)^{2\alpha}}\,dy \leqq \frac{c}{1+|x|^2} \quad \left(\alpha > \frac{3}{2}\right).$$

PROOF. Since $h(x)$ is continuous, we need only to take care of the behavior of $h(x)$ as $|x|\to\infty$. Divide the integral into two parts:

$$\int_{R^3} = \int_{|y|<|x|/2} + \int_{|y|>|x|/2}\,.$$

$$\int_{|y|<|x|/2} \leqq \frac{4}{|x|^2}\int_{|y|<|x|/2} \frac{1}{(1+|y|)^{2\alpha}}\,dy \leqq \frac{C_1}{|x|^2}.$$

Let $|x|>1$. Then with constants c_2, c_3 independent of x we have

$$\begin{aligned}\int_{|y|>|x|/2} &\leqq c_2 \int_{|y|>|x|/2} \frac{1}{|x-y|^2\,|y|^{2\alpha}}\,dy \\ &= \frac{c_2}{|x|^{2+2\alpha-3}} \int_{|z|>1/2} \frac{1}{|e_x-z|^2\,|z|^{2\alpha}}\,dz \leqq \frac{c_3}{|x|^2},\end{aligned}$$

where $z = y/|x|$ and e_x is the unit vector in the direction of x.

This lemma shows that $h \in L^r(R^3)$ for $3/2 < r \leqq \infty$. On the other hand,

$$\int_{R^3} \int_{R^3} |k(x, y; \zeta)|^2\, dxdy \leqq \frac{1}{(4\pi)^2} \int_{R^3} |q_1(x)|^2 h(x) dx.$$

The right-hand side is finite if $|q_1|^2 \in L^s$ for a certain s, $1 \leqq s < 3$. This is satisfied if $q_1 \in L^p$, $2 \leqq p < 6$, or more generally if q_1 is the sum of such functions. This leads us to introduce the following condition on $q(x)$.

$$\text{(4.12)} \quad \begin{cases} q(x) = \dfrac{1}{(1 + |x|)^\alpha} \times q_1(x), & \text{where } q_1(x) = q_{11}(x) + q_{12}(x) \\[2ex] q_{11} \in L^2(R^3), \quad q_{12} \in L^p(R^3), & 2 \leqq p < 6, \quad \alpha > 3/2. \end{cases}$$

PROPOSITION 4.17. *Suppose that* (4.12) *is satisfied. Then, the kernel* $k(x, y; \zeta)$, $\operatorname{Im} \zeta \geqq 0$, *determines a Hilbert-Schmidt operator* $Q_1(\zeta)$ *in* $L^2(R^3)$ *and we have*: (1) $\|Q_1(\zeta)\|_{\text{H.S.}} \leqq c(\|q_{11}\|_{L^2} + \|q_{12}\|_{L^p})$, *where* c *is independent of* ζ; *and* (2) $Q_1(\zeta)$ *is a continuous function* (*with respect to the Hilbert-Schmidt norm*) *of* ζ *in* $\{\zeta | \operatorname{Im} \zeta \geqq 0\}$.

The estimate (1) follows from the discussion made above. (2) is obtained by the Lebesgue Convergence Theorem.

REMARK 4.18. Roughly speaking q_{11} takes care of the local singularities of q and q_{12} the rate of decay of q at infinity. (4.12) is satisfied if $q \in L^2$ and $|q(x)| \leqq |x|^{-(2+\epsilon)}$, $\epsilon > 0$, $|x| > R$. In particular this contains Ikebe's condition (Ikebe, [8]).

Under assumption (4.12) it is fairly easy to see that: (i) $q \in L^2$ and hence $H_2 = -\Delta + q(x)$ is selfadjoint; (ii) $D(B) \supset D(H_1)$. Thus, the factorization method is applicable. Consider $H_2^c = -\Delta + cq(x)$ where c is a real constant. Proposition 4.17 shows that the assumptions of Proposition 4.14 are satisfied provided $|c|$ is sufficiently small. Furthermore, the argument leading to Proposition 4.17 tells us that $A^*\delta_\epsilon(H_1 - \lambda)A$ is of Hilbert-Schmidt type and depends continuously on $\zeta = \lambda + i\epsilon$ up to the real axis. To see this it suffices to replace $q_1(x)$ by $1/(1 + |x|)^\alpha$ and note that $1/(1 + |x|)^\alpha \in L^p(R^3)$ for $3/2 < p < \infty$ (we deal with $A^*R_1(\lambda + i\epsilon)A$ and $A^*R_1(\lambda - i\epsilon)A$ separately). As a matter of fact we have the situation mentioned in Remark 4.15. Thus H_2 is unitarily equivalent to H_1 if $|c|$ is sufficiently small.

Proposition 4.17 asserts much more than we need to apply the small perturbation argument. What is important is the complete continuity of $Q_1(\zeta)$ and the fact that $Q_1(\zeta)$ is continuous in ζ up to the real axis with respect to the operator norm. This leads us to another way of eliminating $R_2(\zeta)$ which does not involve any smallness assumption on V.

Smooth or gentle perturbations.

THEOREM 4.19. *In addition to condition* (F-1) *with* $\Gamma = R^1$ *suppose that the following* (1) *and* (2) *are satisfied:* (1) $Q_1(\zeta)$ *is completely continuous for any* ζ, $\mathrm{Im}\,\zeta > 0$; *and* (2) *for every* $\lambda \in R^1$ *the limit* $\lim_{\epsilon \downarrow 0} Q_1(\lambda + i\epsilon) = Q_1^+(\lambda)$ *exists in the norm topology of operators in* $\mathcal{H}$ *and the convergence is uniform for* λ *lying in a compact set of* R^1. *Then there exists a closed set* $\Gamma_0 \subset R^1$ *with* $|\Gamma_0| = 0$ *such that* (F-2) *holds for* $\Gamma = R^1 - \Gamma_0$. *In particular,* $H_{1,\mathrm{ac}}$ and $H_{2,\mathrm{ac}}$ *are unitarily equivalent. Similar statements hold for the lower half plane too.*

The essential part of the proof lies in the following lemma.

LEMMA 4.20. *Let* $\mathcal{Y}$ *be a Banach space and let* $T(\zeta) : \{\zeta | \mathrm{Im}\,\zeta \geqq 0\} \to B(\mathcal{Y})$ *satisfy the following conditions* (a) – (c): (a) $T(\zeta)$ *is holomorphic in* $\{\zeta | \mathrm{Im}\,\zeta > 0\}$ *and continuous (in the norm topology) in* $\{\zeta | \mathrm{Im}\,\zeta \geqq 0\}$; (b) $T(\zeta) - 1$ *is completely continuous for every* ζ; (c)$T(\zeta)$ *has an inverse in* $B(\mathcal{Y})$ *for every* ζ, $\mathrm{Im}\,\zeta > 0$. *Let* $\Gamma_0 = \{\lambda \in R^1 | T(\lambda)$ *is not invertible*$\}$. *Then,* Γ_0 *is a closed set with* $|\Gamma_0| = 0$.

Let us take this lemma for granted for the time being. We apply it to $G_1(\zeta) = 1 + Q_1(\zeta)$. The holomorphic property and continuity in $\{\zeta | \mathrm{Im}\,\zeta > 0\}$ follow immediately from the corresponding properties of $R_1(\zeta)$. This combined with assumption (2) of the theorem yields condition (a). Condition (b) is nothing but assumption (1). Condition (c) has been verified (see (4.10)).

We now show that condition (F-2) is satisfied for Γ. Since $Q_1(\zeta)$, $\mathrm{Im}\,\zeta > 0$, is completely continuous, $G_1(\zeta)^{-1} \in B(\mathcal{H})$ once $G_1(\zeta)$ is invertible. Hence, by the continuity of the inverse operation in $B(\mathcal{H})$ one sees that $G_2(\zeta) = G_1(\zeta)^{-1}$, $\mathrm{Im}\,\zeta > 0$, extends continuously to $\{\mathrm{Im}\,\zeta \geqq 0\} - \Gamma_0$. This implies (F-2).

REMARK 4.21. In practical situations it frequently occurs that H_1 is absolutely continuous and that the assumptions of Theorem 4.19 hold. In such a case it is likely that we also have the uniform boundedness of $A^*\delta_\epsilon(H_1 - \lambda)A$ near a compact portion of R^1. Then, one sees that H_2 is absolutely continuous on $\Gamma = R^1 - \Gamma_0$. To show this it suffices to apply Remark 4.12 to an arbitrary closed interval contained in Γ (note that $G_2(\zeta)$ is uniformly bounded near such an interval as can be seen from the proof given above).

Let us return to the *Schrödinger operator* for a moment. By virtue of Proposition 4.17 it is clear that the situation described in Remark

4.21 is realized under condition (4.12). Thus, we recapture Ikebe's theorem (see Ikebe [8]) as far as the unitary equivalence of the absolutely continuous parts of H_1 and H_2 is concerned. More information is obtained in the following sections.

An indication of the proof of Lemma 4.20. Put $K(\zeta) = T(\zeta) - 1$. Let $\lambda \in \Gamma_0$. This is equivalent to assuming that -1 is in the spectrum of $K(\lambda)$. Take a sufficiently small circle C around -1 in the complex plane in such a way that -1 is the only point of the spectrum of $K(\lambda)$ lying on and inside C. Then, there exists $\eta > 0$ such that C lies in the resolvent of $K(\zeta)$ if $|\zeta - \lambda| < \eta$. Put

$$P(\zeta) = \frac{1}{2\pi i} \int_C (w - K(\zeta))^{-1} dw.$$

Then, $P(\zeta)$ is a finite-dimensional (oblique) projection and the range of $P(\zeta)$ reduces $K(\zeta)$ and hence $T(\zeta)$. Confining our attention to the range of $P(\zeta)$, we see easily that $\lambda' \in \Gamma_0$, $|\lambda' - \lambda| < \eta$, is equivalent to

$$\det T(\lambda') = 0.$$

On the other hand, $\det T(\zeta)$ is a complex function holomorphic in $\{\mathrm{Im}\,\zeta > 0\}$, continuous in $\{\mathrm{Im}\,\zeta \geqq 0\}$, and not identically zero (by condition (c)). But it is known that such a function cannot vanish on R^1 except for a set of measure zero (theorem of F. and M. Riesz). This concludes the proof.

5. **Eigenfunction expansions in abstract scattering theory.** In this section we discuss the perturbation of eigenfunction expansions using the scheme of Gel'fand-Šilov-Vilenkin. We suppose that an eigenfunction expansion is given in a concrete form for the unperturbed problem (imagine, e.g., the Fourier transform) and try to construct an expansion of a similar type for the perturbed problem.

We shall work in the situation described in Theorem 4.5. Thus we have a Banach space $\mathcal{X} \subset \mathcal{H}$. There are spectral forms $f_j : \Gamma \times \mathcal{X} \times \mathcal{X} \to C^1$, $j = 1, 2$, and a family of operators $G(\lambda) \in B(\mathcal{X})$. $G(\lambda)$ is one-to-one and onto and related to f_j by the formula

$$f_1(\lambda; x, y) = f_2(\lambda; G(\lambda)x, G(\lambda)y).$$

In addition $G(\lambda)$ and $G(\lambda)^{-1}$ are measurable in a suitable sense. Then, there exists a family of unitary operators $\tilde{G}(\lambda) : \mathcal{X}_1(\lambda) \to \mathcal{X}_2(\lambda)$, $\lambda \in \Gamma$, satisfying

$$J_2(\lambda)\, G(\lambda) = \tilde{G}(\lambda)\, J_1(\lambda).$$

This $\tilde{G}(\lambda)$ determines a (decomposable) unitary operator

$$\tilde{W} : \mathfrak{M}_1 \to \mathfrak{M}_2 \quad \text{such that} \quad \{g(\lambda)\}_{\lambda \in \Gamma} \to \{\tilde{G}(\lambda)g(\lambda)\}_{\lambda \in \Gamma}.$$

Let π_j be the unitary operator $\mathcal{H}_{j,\mathrm{ac}}(\Gamma) \to \mathfrak{M}_j$ which gives the spectral representation. Our "wave" operator W is then defined as

$$W = \pi_2^{-1} \tilde{W} \pi_1. \tag{5.0}$$

A formulation of eigenfunction expansions. We now suppose that associated with $\mathfrak{X}$ there is given another spectral representation of $H_{1,\mathrm{ac}}(\Gamma)$ in a somewhat more refined sense. Namely, we assume the following:

(5.1) There are a σ-finite measure space (Ω, Σ, ρ), a partial isometry Φ_1 of $\mathcal{H}$ onto $L^2(\rho)$ with initial set $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$, and a measurable function $\omega : \Omega \to \Gamma$ such that

$$(*) \qquad (\Phi_1 E_1(\Delta)u)(\zeta) = \chi_\Delta(\omega(\zeta))(\Phi_1 u)(\zeta), \qquad \rho - \text{a.e.} \qquad \zeta \in \Omega,$$

for each $u \in \mathcal{H}$ and $\Delta \subset R^1$. (The measurability of ω means that $\omega^{-1}(\Delta) \in \Sigma$ for every Borel set $\Delta \subset \Gamma$.)

(5.2) There is a mapping $\phi_1 : \Omega \to \mathfrak{X}^*$ such that

$$(\Phi_1 x)(\zeta) = \langle x, \phi_1(\zeta)\rangle, \qquad \rho - \text{a.e.} \qquad \zeta \in \Omega,$$

for each $x \in \mathfrak{X}$.

EXAMPLE 5.1. Consider the Schrödinger operator. The factorization scheme used above is equivalent to taking

$$\mathfrak{X} = R(A) = \{u(x) \mid (1+|x|)^\alpha u(x) \in L^2(R^3)\}, \qquad \alpha > 3/2.$$

Hence, $L^\infty(R^3) \subset \mathfrak{X}^*$. Let (Ω, Σ, ρ) be R^3 with the field of all Borel sets and the Lebesgue measure, Φ_1 the Fourier transform, and $\omega(\zeta) = |\zeta|^2$. Let $\phi_1(\zeta) \in \mathfrak{X}^*$ be determined by

$$\langle u, \phi_1(\zeta)\rangle = \frac{1}{(2\pi)^{3/2}} \int_{R^3} u(x) e^{-i\zeta \cdot x} dx.$$

Then (5.1) and (5.2) are clearly satisfied.

REMARK 5.2. Formula $(*)$ in (5.1) can be replaced by a more general one:

$$(\Phi_1 \alpha(E_1)u)(\zeta) = \alpha(\omega(\zeta))(\Phi_1 u)(\zeta), \qquad \rho - \text{a.e.} \qquad \zeta \in \Omega,$$

for each $u \in \mathcal{H}$ and Borel measurable, bounded function α.

REMARK 5.3. In the above formulation $\phi_1(\zeta)$ is an "eigenfunction" only in the sense that it is a pointwise evaluation functional for Φ_1 on the subset $\mathfrak{X}$. Under additional conditions, however, $\phi_1(\zeta)$ will look more like an eigenfunction. This will be discussed below.

Perturbations.

THEOREM 5.4. *Suppose that in addition to all the assumptions of Theorem* 4.5 *conditions* (5.1) *and* (5.2) *are satisfied. Let* W *be the "wave" operator constructed in Theorem* 4.5. *Let* $\Phi_2 \in B(\mathcal{H}, L^2(\rho))$ *be defined by*

$$\begin{aligned} \Phi_2 &= \Phi_1 W^{-1} && \text{on } \mathcal{H}_{2,\mathrm{ac}}(\Gamma), \\ &= \quad 0 && \text{on } \mathcal{H} \ominus \mathcal{H}_{2,\mathrm{ac}}(\Gamma). \end{aligned}$$

Furthermore, put

$$\phi_2(\zeta) = [G(\omega(\zeta))^*]^{-1}\phi_1(\zeta) \in \mathcal{X}^*, \qquad \zeta \in \Omega. \tag{5.3}$$

Then, (5.1) *and* (5.2) *hold true with* Φ_1, E_1, *and* ϕ_1 *replaced by* Φ_2, E_2, *and* ϕ_2.

REMARK 5.5. Let us write $G(\lambda) = 1 + \tilde{Q}(\lambda)$. Then (5.3) can be written as

$$\phi_2(\zeta) = \phi_1(\zeta) - \tilde{Q}(\omega(\zeta))^*\phi_2(\zeta).$$

In the situation of Theorem 4.6 we have two G's which are formally given as

$$G^{\pm}(\lambda) = 1 + VR_1(\lambda \pm i0).$$

Thus, the above equation formally gives

$$\phi_2(\zeta) = \phi_1(\zeta) - R_1(\omega(\zeta) \mp i0)V\phi_2(\zeta). \tag{5.4}$$

This equation is known as the *Lippmann-Schwinger equation.* Physically, one may regard ζ as the momentum and $\omega(\zeta)$ the energy of the system.

The proof of Theorem 5.4 depends on the following lemma which gives a connection between the two representations π_1 and Φ_1. Since this applies also to the system 2 once the theorem is proved, we omit the subscripts 1, 2.

LEMMA 5.6. *Let* $u \in \mathcal{H}_{\mathrm{ac}}(\Gamma)$. *Let* $\tilde{u}' : \Gamma \to \mathcal{X}$ *be a strongly measurable* $\mathcal{X}$*-valued function on* Γ *such that*:

(i) $\tilde{u} \overset{d}{=} \{J(\lambda)\tilde{u}'(\lambda)\} \in \mathfrak{M}$; *and*

(ii) $\pi u = \tilde{u}$ *as an element of* $\mathfrak{M}$.

Then

$$(\Phi u)(\zeta) = \langle \tilde{u}'(\omega(\zeta)), \phi(\zeta)\rangle, \qquad \rho - a.e. \quad \zeta \in \Omega. \tag{5.5}$$

PROOF. It can be shown that there exists a sequence of $\mathcal{X}$-valued simple functions

$$\tilde{u}_n'(\lambda) = \sum c_k \chi_{nk}(\lambda) x_{nk}, \qquad \chi_{nk} = \chi_{\bar{\Delta}_{nk}}, \qquad \Delta_{nk} \subset \Gamma, \quad x_{nk} \in \mathcal{X},$$

such that: (i) $\tilde{u}_n'(\lambda) \to \tilde{u}'(\lambda)$ a.e. in Γ; and (ii) $\tilde{u}_n \to \tilde{u}$ in $\mathfrak{M}$ where $\tilde{u}_n = \{J(\lambda)\tilde{u}_n'(\lambda)\}$. Put $u_n = \pi^{-1}\tilde{u}_n \in \mathcal{H}_{ac}(\Gamma)$. Then, the definition of π tells us that $u_n = \sum c_k E_{ac}(\Delta_{nk})x_{nk}$. Hence, by (5.1) and (5.2) we get

$$(5.6)\quad \begin{aligned}(\Phi u_n)(\zeta) &= \sum c_k \chi_{nk}(\omega(\zeta))(\Phi x_{nk})(\zeta) = \sum c_k \chi_{nk}(\omega(\zeta))\langle x_{nk}, \phi(\zeta)\rangle \\ &= \langle \tilde{u}_n'(\omega(\zeta)), \phi(\zeta)\rangle.\end{aligned}$$

Since $\tilde{u}_n \to \tilde{u}$ in $\mathfrak{M}$, the left-hand side converges to $(\Phi u)(\zeta)$ in $L^2(\rho)$. Since $\tilde{u}_n'(\lambda) \to \tilde{u}_n'(\lambda)$ a.e., the right-hand side converges to $\langle \tilde{u}'(\omega(\zeta)), \phi(\zeta)\rangle, \rho -$ a.e. in Ω. (Here, we used the fact that $\omega^{-1}(\Delta)$ is a ρ-null set whenever $\Delta \subset \Gamma$ is a Lebesgue null set. This fact is an immediate consequence of (5.1).) Hence, (5.5) is obtained by taking the limit of (5.6) along a suitable subsequence.

Proof of Theorem 5.4. (5.1) is a direct consequence of the definition of Φ_2 and the intertwining property$E_1(\Delta)W^{-1} = W^{-1}E_2(\Delta)$. Let us prove (5.2). Formula (5.0) shows that $\pi_1 W^{-1} = W^{-1}\pi_2$. Apply both sides to $E_{2,ac}(\Gamma)x$, $x \in \mathcal{X}$. Since $\pi_2 E_{2,ac}(\Gamma)x = \{J_2(\lambda)x\}$, we get

$$\pi_1 W^{-1}E_{2,ac}(\Gamma)x = \{\bar{G}(\lambda)^{-1}J_2(\lambda)x\} = \{J_1(\lambda)G(\lambda)^{-1}x\}.$$

Hence, the assumptions of Lemma 5.6 are satisfied with $u = W^{-1}E_{2,ac}(\Gamma)x$ and $\tilde{u}'(\lambda) = G(\lambda)^{-1}x$. Hence, (5.5) yields

$$(\Phi_1 W^{-1}E_{2,ac}(\Gamma)x)(\zeta) = \langle G(\omega(\zeta))^{-1}x, \phi_1(\zeta)\rangle = \langle x, \phi_2(\zeta)\rangle, \rho-\text{a.e. } \zeta \in \Omega.$$

Since $\Phi_1 W^{-1}E_{2,ac}(\Gamma)x = \Phi_2 x$, (5.2) is proved.

Generalized eigenfunctions. We show that in the special case the $\phi(\xi)$ may be interpreted as generalized eigenfunctions. The subscripts 1 and 2 are omitted.

In discussing eigenfunction expansions it is rather natural to assume that $\mathcal{X}$ is dense in $\mathcal{H}$ and the injection: $\mathcal{X} \to \mathcal{H}$ is continuous. Then, in a canonical way we have the inclusion relation $\mathcal{X} \subset \mathcal{H} \subset \mathcal{X}^*$. Suppose further that there exists a subspace $\mathcal{Y} \subset \mathcal{X}$ such that: (i) $\mathcal{Y}$ is dense in $\mathcal{H}$; (ii) $\mathcal{Y}$ is a linear topological space, and the injection : $\mathcal{Y} \to \mathcal{X}$ is continuous; and (iii) $\mathcal{Y} \subset D(H)$, $H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$, and H maps $\mathcal{Y}$ continuously into $\mathcal{X}$. Thus, canonically $\mathcal{Y} \subset \mathcal{X} \subset \mathcal{H} \subset \mathcal{X}^* \subset \mathcal{Y}^*$. Let $H^\dagger : \mathcal{X}^* \to \mathcal{Y}^*$ be the adjoint operator to H where H is considered as an operator in $B(\mathcal{Y},\mathcal{X})$. Then, (5.1) and (5.2) show that for any $y \in \mathcal{Y}$

$$\begin{aligned}\langle y, H^\dagger\phi(\zeta)\rangle &= \langle Hy, \phi(\zeta)\rangle = (\Phi Hy)(\zeta) = \omega(\zeta)(\Phi y)(\zeta) \\ &= \omega(\zeta)\langle y, \phi(\zeta)\rangle.\end{aligned}$$

Thus, $H^\dagger\phi(\zeta) \in \mathcal{X}^*$ and we have

$$H^{\dagger}\phi(\zeta) = \omega(\zeta)\phi(\zeta).$$

Then, $\phi(\zeta)$ may be interpreted as a generalized eigenfunction of H.

Application to Schrödinger operators. Let us consider the Schrödinger operator $H_2 = -\Delta + q(x)$, under the assumption (4.12). We know that the assumptions of Theorem 4.5 are satisfied with $\mathcal{X}$ being the weighted L^2:

$$\mathcal{X} = \mathcal{X}_{\alpha} \overset{d}{=} \{u \mid (1+|x|)^{\alpha}u(x) \in L^2(R^3)\},$$

where $\alpha > 3/2$. We also know that the Fourier transform can be taken as an eigenfunction expansion in the sense of (5.1) and (5.2). Thus, the previous considerations can be applied. As a result we obtain the following theorem.

THEOREM 5.7. *Assume that* (4.12) *is satisfied. Then there exists a closed null set* $\Gamma_0 \subset R^+$ *such that the following statements are true. For every* $\zeta \in R^3$ *with* $|\zeta|^2 \notin \Gamma_0$, *the integral equation*

$$\phi_{\pm}(x,\zeta) = e^{i\zeta\cdot x} - \frac{1}{4\pi}\int_{R^3} \frac{e^{i|\zeta|\,|x-y|}}{|x-y|} q(y)\phi_{\pm}(y,\zeta)dy \tag{5.7}$$

has a unique solution $\phi_{\pm}(\cdot,\zeta)$ *in* $\mathcal{X}_{\alpha}$. *For each of* $\pm$ *the family* $\{\phi_{\pm}(x,\zeta)\}$ *forms a complete orthonormal system of eigenfunctions of* $H_{2,\mathrm{ac}}$ *in the following sense.*

For every $u \in \mathcal{H}_{2,\mathrm{ac}}$

$$\hat{u}_{\pm}(\zeta) = \frac{1}{(2\pi)^{3/2}} \operatorname*{l.i.m.}_{N\to\infty} \int_{|x|<N} u(x)\overline{\phi_{\pm}(x,\zeta)}dx$$

exists in the sense of convergence in $L^2(R^3)$. *The mapping* $W_{\pm} : u \to \hat{u}_{\pm}$ *is a unitary operator from* $\mathcal{H}_{2,\mathrm{ac}}$ *onto* $L^2(R^3)$. *Let* $\{\Gamma_N\}$ *be an increasing sequence of closed subsets of* $\Gamma = R^1 - \Gamma_0$ *such that* $\Gamma_N \to \Gamma$. *Then the inversion formula*

$$u(x) = \frac{1}{(2\pi)^{3/2}} \operatorname*{l.i.m.}_{N\to\infty} \int_{\Omega_N} \hat{u}_{\pm}(\zeta)\phi_{\pm}(x,\zeta)d\zeta$$

holds for $u \in \mathcal{H}_{2,\mathrm{ac}}$, *where* $\Omega_N = \{\zeta \mid |\zeta|^2 \in \Gamma_N\}$. *If* $u \in \mathcal{H}_{2,\mathrm{ac}} \cap D(H_2)$, *then* $(H_2u)\hat{}_{\pm}(\zeta) = |\zeta|^2\hat{u}_{\pm}(\zeta)$. $\phi_{\pm}(x,\zeta)$ *satisfies*

$$-\Delta_x\phi_{\pm}(x,\zeta) + q(x)\phi_{\pm}(x,\zeta) = |\zeta|^2\phi_{\pm}(x,\zeta) \tag{5.8}$$

where Δ_x *is taken in the sense of distributions.*

The integral equation (5.7) is obtained in the same manner that (5.4) was derived. In fact it is easy to see that $(1 - G^{\pm}(\lambda))^*$ has the kernel appearing in (5.7). The inversion formula may be derived as a consequence of an abstract inversion formula. In any event its derivation is easy. To get the differential equation (5.8), we use the result of the previous subsection. Namely, it is easy to see that $\mathcal{Y} = \mathcal{S}(R^3)$ satisfies the requirement. Then, $H_2^+ : \mathcal{S}(R^3)' \to \mathcal{X}_\alpha$ can be interpreted in the distribution sense.

We might note that the unitarity of the mapping $u \to \hat{u} : \mathcal{H}_{2,\text{ac}} \to L^2(R^3)$ is obtained without the aid of time-dependent theory. The decay rate of q given by (4.12) is $O(|x|^{-(2+\epsilon)})$. This is improved below to $O(|x|^{-(1+\epsilon)})$ as far as the existence of W is concerned. (Cf. §7.)

Remark 5.8. A detailed study (an elliptic type argument) reveals that $\phi_\pm(x, \zeta)$ is a bounded, continuous function of x. Furthermore, as $|x| \to \infty$

$$
\begin{aligned}
\phi_\pm(x, \zeta) - e^{i\zeta \cdot x} &= O(|x|^{-(1+\alpha-3/p')}), \qquad \alpha p' < 3,\\
&= O(|x|^{-1}\{\log|x|\}^{1/p'}), \quad \alpha p' = 3,\\
&= O(|x|^{-1}), \qquad \alpha p' > 3,
\end{aligned}
$$

where $p'^{-1} + p^{-1} = 1$ with p appearing in (4.12). If $|q(x)|$ decays as $|x|^{-(2+\epsilon)}$, then $\phi_\pm(x, \zeta) - e^{i\zeta \cdot x}$ decays as $|x|^{-\epsilon}$ for $0 < \epsilon < 1$ and $|x|^{-1}$ for $1 < \epsilon$. (See Kuroda, [15].)

A generalization of these results to the n-dimensional case is mentioned in §7.

6. The relationship of the stationary and time-dependent theories. In this section we discuss the relation between the stationary wave operators, $W_\pm$, and those in the time-dependent theory. In general the way of constructing W in Theorem 4.5 is too general to have any relation to the time-dependent method. However, with the additional hypotheses of Theorem 4.6 the operators $W_\pm$ are the strong Abel limits of $e^{itH_2}e^{-itH_1}$ as $t \to \pm\infty$.

Theorem 6.1. *Suppose that all the assumptions of Theorem 4.6 are fulfilled, and let W_+ be the operator constructed in Theorem 4.6. Then*

$$
(6.1) \qquad W_+ = \operatorname*{s-lim}_{\epsilon \downarrow 0} 2\epsilon \int_0^\infty e^{-2\epsilon t} e^{itH_2} e^{-itH_1} dt E_{1,\text{ac}}(\Gamma) \quad on\ \mathcal{H}_{1,\text{ac}}(\Gamma).
$$

Remark 6.2. If assumption (2) of Theorem 4.6 is assumed for $\lambda - i\epsilon$ instead of $\lambda + i\epsilon$, the operator W_- satisfies

$$
W_- = \operatorname*{s-lim}_{\epsilon \downarrow 0} 2\epsilon \int_{-\infty}^0 e^{2\epsilon t} e^{itH_2} e^{-itH_1} dt E_{1,\text{ac}}(\Gamma) \quad \text{on } \mathcal{H}_{1,\text{ac}}(\Gamma).
$$

These formulas show that $W_\pm$ are independent of the choice of $\mathcal{X}$.

Under a more restrictive situation the Abel limit can be replaced by the strong limit, and, moreover, the invariance principle holds. Namely, we have the following theorem.

THEOREM 6.3. *Suppose that $\mathfrak{X}$ in Theorem 4.6 is a Hilbert space and assume all the hypotheses of Theorem 4.6. In addition suppose that assumption (2) of Theorem 4.6 is also satisfied for $\lambda - i\epsilon$ in place of $\lambda + i\epsilon$. Let $W_\pm$ be the operators constructed in Theorem 4.5 corresponding to $\lambda \pm i\epsilon$. Let ϕ be a real valued, Borel measurable function on Γ such that*

$$\text{(6.2)} \qquad \int_0^\infty \left| \int_\Gamma f(\lambda) e^{-it\phi(\lambda) - is\lambda}\, d\lambda \right|^2 ds \to 0, \qquad t \to +\infty,$$

for any $f \in L^2(\Gamma)$. Then, the strong limits in the following formula exist and we have

$$\text{(6.3)} \qquad \operatorname*{s-lim}_{t\to\pm\infty} e^{it\phi(H_2)} e^{-it\phi(H_1)} E_{1,\mathrm{ac}}(\Gamma) = W_\pm \quad \text{on } \mathcal{H}_{1,\mathrm{ac}}(\Gamma).$$

REMARK 6.4. Let us verify (6.2) for $\phi(\lambda) = \lambda$. Let f be extended to be zero outside Γ and let Ff be its Fourier transform. Then, (6.2) is equivalent to $\int_0^\infty |(Ff)(t+s)|^2 ds \to 0$. But this is certainly true. More generally, (6.3) holds if ϕ is piecewise differentiable with $\phi'(\lambda)$ piecewise continuous, locally of bounded variation, and positive. ($\phi(\lambda)$ may tend to $\pm\infty$ at a discrete set of points (Kato, [12]).) Thus, in the situation of Theorem 6.3 the (time-dependent) wave operators exist and are complete. Moreover, the invariance principle holds. For Schrödinger operators this is true under (4.12) (or (7.0)).

REMARK 6.5. If (6.2) holds with $\int_0^\infty$ replaced by $\int_{-\infty}^0$, then (6.3) holds with $\lim_{t\to\pm\infty}$ replaced by $\lim_{t\to\mp\infty}$.

We shall give a *proof of Theorem* 6.1 under more restrictive assumptions. Namely we suppose that: (i) $\mathfrak{X}$ is dense in $\mathcal{H}$; (ii) $\Gamma = R^1$; and (iii) H_1 and H_2 are absolutely continuous. For the complete proof see Kato and Kuroda [14].

First recall that

$$\delta_\epsilon(H_j - \lambda) \stackrel{d}{=} \frac{1}{2\pi i}\{R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon)\},$$

$$f_{j\epsilon}(\lambda; u, v) \stackrel{d}{=} (\delta_\epsilon(H_j - \lambda)u, v),$$

$$f_{1\epsilon}(\lambda; u, v) = f_{2\epsilon}(\lambda; G(\lambda + i\epsilon)u,\ G(\lambda + i\epsilon)v),$$

and put

$$W_\epsilon = 2\epsilon \int_0 e^{-2\epsilon t} e^{itH_2} e^{-itH_1}\, dt.$$

As is easily seen, it suffices to show that

$$(W_\epsilon x, y) \to (W_+ x, y), \quad \epsilon \downarrow 0, \quad \text{for all } x, y \in \overline{\mathfrak{X}},$$

(note that W_+ is known to be unitary). Now, the formal computation concerning the formal transition to the stationary formulas given in §3 can be carried out legitimately to give

$$(W_\epsilon x, y) = \int_{-\infty}^{\infty} (\delta_\epsilon(H_2 - \lambda)G(\lambda + i\epsilon)x, y)d\lambda$$

$$= \int_{-\infty}^{\infty} f_{2\epsilon}(\lambda; G(\lambda + i\epsilon)x, y)d\lambda.$$

The integrand converges pointwise to $f_2(\lambda; G^+(\lambda)x, y)$. So, in view of (4.4) it suffices to see that we can take the limit as $\epsilon \downarrow 0$ under the integral sign. Now

$$|f_{2\epsilon}(\lambda; G(\lambda + i\epsilon)x, y)|^2 \leqq f_{2\epsilon}(\lambda; G(\lambda + i\epsilon)x) f_{2\epsilon}(\lambda; y)$$
$$= f_{1\epsilon}(\lambda; x) f_{2\epsilon}(\lambda; y),$$

where $f_{j\epsilon}(\lambda; z) \overset{d}{=} f_{j\epsilon}(\lambda; z, z)$. Hence, for any $\Delta \subset \Gamma$

$$\left[\int_\Delta |f_{2\epsilon}(\lambda; G(\lambda + i\epsilon)x, y)|d\lambda\right]^2 \leqq \int_\Delta f_{1\epsilon}(\lambda; x)d\lambda \int_\Delta f_{2\epsilon}(\lambda; y)d\lambda.$$

On the other hand,

$$f_{j\epsilon}(\lambda; x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(\mu-\lambda)^2 + \epsilon^2}\cdot\frac{d}{d\mu}(E_j(\mu)x, x)d\mu \to \frac{d}{d\lambda}(E_1(\lambda)x, x)$$

pointwise for a.e. λ and in $L^1(R^1)$. By virtue of the Vitali convergence theorem (see e.g. N. Dunford and J. T. Schwartz, *Linear operators*, Part 1, Interscience, New York, 1958, III.6.15) it is now easy to establish the L^1-convergence of $f_{2\epsilon}(\lambda; G(\lambda + i\epsilon)x, y)$ to $f_2(\lambda; G^+(\lambda)x, y)$.

The proof for the general case is complicated because we have to replace y by $u = \alpha(H_{2,\mathrm{ac}})y$ $(\alpha \in L^2(\Gamma))$ which is not necessarily in $\mathfrak{X}$. Thus, even the pointwise convergence is not true and one has to take a subsequence.

Proof of Theorem 6.3. For simplicity suppose that $\mathfrak{X}$ is separable. We put

$$D(\lambda + i\epsilon) = \frac{1}{2\pi i}\{G(\lambda + i\epsilon) - G(\lambda - i\epsilon)\}$$

which converges strongly in $\mathfrak{X}$ to

$$D(\lambda) = \frac{1}{2\pi i}\{G^+(\lambda) - G^-(\lambda)\}.$$

Remembering $R_1(\zeta) = R_2(\zeta)G(\zeta)$, we get

$$\delta_\epsilon(H_1 - \lambda) = \delta_\epsilon(H_2 - \lambda)G(\lambda + i\epsilon) + R_2(\lambda - i\epsilon)D(\lambda + i\epsilon).$$

Hence, for any $x, y \in \mathcal{X}$ and $\alpha \in L^2(\Gamma)$ we have

$$
\begin{aligned}
(6.4)\qquad (\delta_\epsilon(H_1-\lambda)x, \alpha(H_{2,\mathrm{ac}})y) &= f_{2\epsilon}(\lambda; G(\lambda+i\epsilon)x, \alpha(H_{2,\mathrm{ac}})y) \\
&\quad + (D(\lambda+i\epsilon)x, R_2(\lambda+i\epsilon)\,\alpha(H_{2,\mathrm{ac}})y),
\end{aligned}
$$

where we regard α to be extended as 0 outside Γ. We want to let $\epsilon \downarrow 0$ in this formula.

The left-hand side converges to $(d/d\lambda)(E_1(\lambda)x, \alpha(H_{2,\mathrm{ac}})y)$.

The first term on the right-hand side converges to

$$\overline{\alpha(\lambda)} f_2(\lambda; G^+(\lambda)x, y)$$

a.e. along a certain sequence $\{\epsilon_n\}$. This follows from the following lemma.

LEMMA 6.6. *There exists a sequence* $\{\epsilon_n\}$, $\epsilon_n \downarrow 0$, *such that* $f_{2\epsilon_n}(\lambda;\ \cdot, \alpha(H_{2,\mathrm{ac}})y)$ *belongs to* $\mathcal{X}^*$ *and converges weak* in* $\mathcal{X}^*$ *to* $\overline{\alpha(\lambda)}f_2(\lambda; \cdot, y)$ *for a.e.* $\lambda \in \Gamma$.

Incidentally, this lemma plays an important role in the proof of Theorem 6.1 for the general case.

Now we note that f_2 can be expressed as $f_2(\lambda; x, y) = (F_2(\lambda)x, y)_{\mathcal{X}}$ with $F_2(\lambda) \in B(\mathcal{X})$, $F_2(\lambda) \geqq 0$. Then, the second term on the right-hand side of (6.4) is equal to

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\overline{\alpha(\mu)}}{\mu-(\lambda-i\epsilon)} \frac{d}{d\mu}(E_2(\mu)D(\lambda+i\epsilon)x, y)\,d\mu \\
&\qquad = \left(D(\lambda+i\epsilon)x, \int_{-\infty}^{\infty} \frac{\alpha(\mu)}{\mu-(\lambda+i\epsilon)} F_2(\mu)y\,d\mu\right).
\end{aligned}
$$

Take α in such a way that α has compact support in Γ and $\alpha(\mu)\|F_2(\mu)\|$ is bounded in Γ. Then, $\alpha(\cdot)F(\cdot)y \in L^2(R^1; \mathcal{X})$. Since $\mathcal{X}$ is a Hilbert space, we can use the theory of Fourier transforms to see that the limit

$$h(\lambda; y, \alpha) \overset{d}{=} \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} \frac{\alpha(\mu)}{\mu-(\lambda+i\epsilon)} F_2(\mu)y\,d\mu$$

exists in $L^2(R^1; \mathcal{X})$. Furthermore, we have

$$(6.5)\qquad \int_{-\infty}^{\infty} \|h(\lambda; y, \alpha)\|^2 d\lambda = \mathrm{const} \int_0^{\infty} ds \left\| \int_{-\infty}^{\infty} \alpha(\lambda)\, e^{-is\lambda} F(\lambda)y\,d\lambda \right\|^2.$$

By taking the limit as $\epsilon \downarrow 0$ in (6.4) we get

$$
\begin{aligned}
\frac{d}{d\lambda}(E_1(\lambda)x, \alpha(H_{2,\mathrm{ac}})y) &= \overline{\alpha(\lambda)}\ f_2(\lambda; G^+(\lambda)x, y) \\
&\quad + (D(\lambda)x, h(\lambda; y, \alpha))_{\mathcal{X}}.
\end{aligned}
$$

Let $\beta \in L^\infty(\Gamma)$ be such that $\|\beta(\lambda)D(\lambda)\|$ is bounded in Γ. Multiplying both sides by β and integrating over Γ we obtain

$$(6.6)\qquad \begin{aligned}&((W_+ - I)\beta(H_{1,\mathrm{ac}})x, \alpha(H_{2,\mathrm{ac}})y)\\ &\qquad = -\int_{-\infty}^{\infty} \beta(\lambda)(D(\lambda)x, h(\lambda; y, \alpha))_{\mathfrak{X}}\, d\lambda.\end{aligned}$$

The conditions imposed on α and β are satisfied by $e^{-it\phi}\alpha$ and $e^{-it\phi}\beta$ as well. Therefore, replacing α and β by $e^{-it\phi}\alpha$ and $e^{-it\phi}\beta$ in (6.6) and estimating the right-hand side by (6.5) we obtain

$$\begin{aligned}&\|((W_+ - I)e^{-it\phi(H_1)}\beta(H_{1,\mathrm{ac}})x, e^{-it\phi(H_2)}\alpha(H_{2,\mathrm{ac}})y)\|^2\\ &\qquad \leqq \text{const} \int_0^\infty ds \left\| \int_\Gamma \alpha(\lambda)\, e^{-it\phi(\lambda) - is\lambda} F(\lambda)y\, d\lambda \right\|^2\end{aligned}$$

(recall that α has compact support in Γ). Now, it is easy to see that (6.2) implies the same statement for $f \in L^2(\Gamma; \mathfrak{X})$. Hence, the right-hand side of the above inequality tends to 0 as $t \to +\infty$.

By the intertwining property of W_+ this means that

$$((W_+ - e^{it\phi(H_2)}e^{-it\phi(H_1)})u, v) \to 0, \quad \text{as } t \to \infty,$$

for every u, v of the form $u = \beta(H_{1,\mathrm{ac}})x$, $v = \alpha(H_{2,\mathrm{ac}})y$. However, the set of all such u (or v) forms a fundamental set in $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$ (or $\mathcal{H}_{2,\mathrm{ac}}(\Gamma)$), provided of course that $\mathfrak{X}$ is sufficiently large. Therefore $e^{it\phi(H_2)}e^{-it\phi(H_1)}$ converges weakly to W_+ on $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$. Since W_+ is known to be unitary the strong convergence follows. The proof for W_- is similar.

Invariance principle for unitary operators. Analogues of Theorems 6.1 and 6.3 hold in the case of unitary operators. (6.1) takes the form

$$W_+ = \operatorname*{s-lim}_{r \uparrow 1}\, (1 - r^2) \sum_{k=0}^{\infty} r^{2k}\, U_2^k U_1^{-k} \quad \text{on } \mathcal{H}_{1,\mathrm{ac}}(\Gamma)$$

where Γ is now a subset of the unit circle. The invariance principle also holds. (6.2) is replaced by

$$\sum_{k=0}^{\infty} \left| \int_\Gamma e^{-it\phi(\theta) - ik\theta} f(\theta)\, d(\theta) \right|^2 \to 0; \qquad t \to \infty,$$

for any $f \in L^2(\Gamma)$. Two examples of such ϕ are $\phi(\theta) = \theta$ and $\phi(\theta) = i(1 + e^{i\theta})(1 - e^{i\theta})^{-1}$. The former gives $W_+ = \text{s-lim}_{k\to\infty} U_2^k U_1^{-k}$ on $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$. The latter gives $W_+ = \text{s-lim}_{t\to\infty} e^{itH_2}e^{-itH_1}$ on $\mathcal{H}_{1,\mathrm{ac}}(\Gamma)$, provided that U_j is the Cayley transform of H_j.

7. **Applications of the abstract theory to Schrödinger operators.** In this section we consider further applications of the general theory to Schrödinger operators. In all cases we use the factorization method. The previous results which we use are summarized in the following

THEOREM 7.1. *Let $H_2 = H_1 + AB$ where H_1 is absolutely continuous, $A \in B(\mathcal{H})$ and B is closed with $D(B) \supset D(H_1) = D(H_2)$. Suppose*

1. *$A^*\{R(\lambda + i\epsilon) - R(\lambda - i\epsilon)\}A$ converges weakly as $\epsilon \downarrow 0$ for all $\lambda \in R^1$.*

2. (a) *The operators $BR_1(\lambda \pm i\epsilon)A$ are completely continuous for all $\lambda \in R^1$ and for $0 < \epsilon < \epsilon_0$;*

(b) *$BR_1(\lambda \pm i\epsilon)A$ have boundary values in the operator norm topology as $\epsilon \downarrow 0$, the convergence being uniform for λ belonging to any compact interval of R^1.*

Then there exists $\Gamma = R^1 - \Gamma_0$ where Γ_0 is closed and has measure 0 such that

1. *the singular spectrum of H_2 is contained in Γ_0;*
2. *$H_{2,\mathrm{ac}}$ is unitarily equivalent to H_1;*
3. *the time-dependent wave operators $W_\pm$ exist and are complete;*
4. *the invariance principle holds;*
5. *if there is a generalized eigenfunction expansion for H_1, then there is also a similar one for H_2.*

We now consider applications of this theorem to Schrödinger operators. We have $H_1 = -\Delta$, $H_2 = -\Delta + q(x)$ in $\mathcal{H} = L^2(R^n)$.

1. We have considered above the case $n = 3$ and $q(x) = (1/(1 + |x|)^\alpha)q_1(x)$ with $\alpha > 3/2$, and $q_1(x) \in L^2(R^3) + L^p(R^3)$ with $2 \leqq p < 6$. Under these assumptions we proved that the hypotheses of Theorem 7.1 were satisfied. The above assumption on q roughly means that $q \in L^2_{\mathrm{loc}}(R^3)$ and $q(x) = O(|x|^{-2-\epsilon})$ for $|x|$ large.

2. The results of the above example can be generalized to n dimensions. We consider now that case $n \geqq 4$; the cases $n = 1$ and 2 can be treated similarly. If we assume

$$q(x) = \frac{1}{(1+|x|)^\alpha}\, q_1(x) \quad \text{with } \alpha > \frac{n}{2},$$

and $q_1(x) \in L^{p_1}(R^n) + L^{p_2}(R^n)$ with $n/2 < p_1$, $p_2 < n$, then the hypotheses of Theorem 7.1 are satisfied. These assumptions on q correspond roughly to $q \in L^2_{\mathrm{loc}}(R^n)$ and $q(x) = (|x|^{-(n+1)/2-\epsilon})$ for $|x|$ large.

We can take the Fourier expansion to be a generalized eigenfunction expansion of H_1, and we get a similar generalized eigenfunction expansion for H_2.

The verification of the hypotheses of Theorem 7.1 is somewhat different in the case $n \geqq 4$ than it was for $n = 3$. $R_1(\zeta)$ is an integral operator with the kernel

$$R_1(x, y, \zeta) = \text{const} \frac{1}{|x-y|^{(n/2-1)}} H^{(1)}_{n/2-1}(\sqrt{\zeta}\,|x-y|)$$

where $H^{(1)}_{n/2-1}$ is the Hankel function. We want to show that the kernel $q_1(x)R_1(x, y, \zeta)/(1+|y|)^\alpha$ corresponds to a completely continuous operator. We have

$$R_1(x, y, \zeta) \sim \begin{cases} \dfrac{\text{bounded function}}{|x-y|^{n-2}}, & |x-y| \to 0, \\ \dfrac{\text{bounded function}}{|x-y|^{(n-1)/2}}, & |x-y| \to \infty. \end{cases}$$

Because of the nature of the local singularity of $R_1(x, y, \zeta)$ it does not seem possible to show that the operator $BR_1(\zeta)A$ is of Hilbert-Schmidt type. However, by using a Sobolev type argument it is still possible to show that $BR_1(\zeta)A$ is completely continuous. (For the details see Kuroda [15].)

3. We can improve the results in the above two examples with respect to the rate of decay at ∞ (cf. Kato [13]). Suppose $n \geqq 2$ and

$$(7.0) \qquad |q(x)| \leqq \frac{a}{(1+|x|)^{1+\epsilon}} \quad \text{for } x \in R^n,$$

where $\epsilon > 0$ and a is a constant. For simplicity we deal with bounded q, but the following argument can be modified to allow certain unbounded q. We write $q(x) = 1/(1+|x|)^\alpha \cdot c(x)/(1+|x|)^\alpha$ where $\alpha = (1+\epsilon)/2$ and $c(x)$ is bounded. Then $V = AB = ACA$ where $A = 1/(1+|x|)^\alpha$ and $C = c(x)$. We shall use the following

LEMMA 7.2. *Suppose $H_2 = H_1 + ACA$ where H_1 is absolutely continuous, A and C are bounded and selfadjoint, and*

1. *$A(H_1 - \zeta)^{-1}A$ is completely continuous for* $\operatorname{Im}(\zeta) \neq 0$.

2. *$AE_1(\cdot)A$ admits a locally Hölder continuous derivative. Namely, there exists $M : R^1 \to B(\mathcal{H})$ which is locally Hölder continuous in the operator norm such that*

$$(7.1) \qquad AE_1(I)A = \int_I M_1(\lambda)d\lambda$$

for every compact I.

Then the hypotheses of Theorem 7.1 *are satisfied with* $B = CA$.

PROOF. We have

$$A(H_1 - (\lambda \pm i\epsilon))^{-1}A = \int_{-\infty}^{\infty} \frac{1}{\mu - (\lambda \pm i\epsilon)} M_1(\mu)d\mu.$$

Using Privalov's theorem for vector valued functions, we see that the Hölder continuity of $M_1(\lambda)$ implies that the boundary values, $K^{\pm}(\lambda)$, of the above integral exist as $\epsilon \downarrow 0$, and they are Hölder continuous. Hence, we have $A\{R_1(\lambda+i\epsilon) - R_1(\lambda-i\epsilon)\}A \to K^+(\lambda) - K^-(\lambda)$ and $BR_1(\lambda \pm i\epsilon)A \to CK^{\pm}(\lambda)$ as $\epsilon \downarrow 0$, both converging in the operator norm. The complete continuity of $BR_1(\zeta)A = CAR_1(\zeta)A$ follows from assumption 1.

Now we apply this lemma.

THEOREM 7.3. *Suppose* $|q(x)| \leqq \text{const } (1 + |x|)^{-(1+\epsilon)}$. *Then the hypotheses of the lemma are satisfied with* $A = (1 + |x|)^{-(1+\epsilon)/2}$ *and*

$$C = (1+|x|)^{1+\epsilon}\, q(x).$$

PROOF. We first note that the complete continuity of $A(H_1-\zeta)^{-1}A$ follows easily since $q \to 0$ as $|x| \to \infty$. We shall prove condition 2 of Lemma 7.2 in the case $n = 3$; the proof is almost the same in the general case. We use spherical coordinates.

$$\mathcal{H} = \sum_{\ell,m} \oplus \mathcal{H}_{\ell m}, \text{ where } \ell = 0, 1, \ldots \text{ and } m = 0, \pm 1, \ldots, \pm \ell,$$

and $\mathcal{H}_{\ell m} \simeq L^2(0, \infty)$. If $u \in \mathcal{H}$ then $u(x) = \Sigma_{\ell m}\ (1/|x|)\ u_{\ell m}(|x|) \cdot Y_{\ell m}(\omega)$ where $u_{\ell m} \in L^2(0, \infty)$ and $Y_{\ell m}$ are the spherical harmonics. We have

$$\int_{R^3} |u(x)|^2 dx = \sum_{\ell,m} \int_0^\infty |u_{\ell m}(r)|^2 dr.$$

Each $\mathcal{H}_{\ell m}$ reduces H_1, and H_1 corresponds to $-d^2/dr^2 + \ell(\ell+1)/r^2$ in $\mathcal{H}_{\ell m}$. H_1 in $\mathcal{H}_{\ell m}$ has the generalized eigenfunction

$$j_{\ell m}(r, \lambda) = \left(\frac{r}{2}\right)^{1/2} J_{\ell+1/2}(\lambda^{1/2} r) \quad \text{for } \lambda > 0$$

where the $J_{\ell+1/2}$ are the Bessel functions.

Let

$$f_{\ell m}(r, \lambda) = \frac{\text{const}}{(1+r)^{\alpha}}\, j_{\ell m}(r, \lambda).$$

$f_{\ell m}(\cdot, \lambda) \in L^2(0, \infty)$ since $\alpha > 1/2$ and $j_{\ell m}(r, \lambda)$ is bounded. Hence, we can define an operator $M_{1,\ell m}(\lambda)$ in $\mathcal{H}_{\ell m}$ by

$$M_{1,\ell m}(\lambda) = f_{\ell m}(\cdot, \lambda) \otimes f_{\ell m}(\cdot, \lambda),$$

i.e.

$$(M_{1,\ell m}(\lambda)u)(r) = (u, f_{\ell m}(\cdot, \lambda)) f_{\ell m}(r, \lambda).$$

$M_{1,\ell m}$ is an operator of rank 1 whose range is spanned by $f_{\ell m}(r, \lambda)$. On the other hand $A\{R_1(\lambda+i\epsilon) - R_1(\lambda-i\epsilon)\}A$ is reduced by $\mathcal{H}_{\ell m}$. Restricting ourselves to $\mathcal{H}_{\ell m}$ and letting $\epsilon \downarrow 0$, we have

(7.2) $$A\{R_1(\lambda+i\epsilon) - R_1(\lambda-i\epsilon)\}A \to M_{1,\ell m}(\lambda)$$

at least weakly.

It now suffices to show that for every compact interval I,

(7.3) $$\|M_{1,\ell m}(\lambda)\| \leqq c_I \quad \text{for } \lambda \in I,$$

(7.4.) $$\|M_{1,\ell m}(\lambda) - M_{1,\ell m}(\mu)\| \leqq c_I|\lambda-\mu|^\theta \quad \text{for } \lambda, \mu \in I,$$

where c_I is independent of ℓ and m. Then by (7.3) we see that $M_1(\lambda) = \sum_{\ell,m} \oplus M_{1,\ell m}$ is bounded, and by (7.4) we see that $M_1(\lambda)$ is Hölder continuous in the operator norm with exponent θ. (7.1) is an immediate consequence of (7.2). In order to show (7.3) and (7.4) it suffices to show

(7.5) $$\|f_{\ell m}(\cdot, \lambda)\| \leqq c_I,$$

(7.6) $$\|f_{\ell m}(\cdot, \lambda) - f_{\ell m}(\cdot, \mu)\| \leqq c_I|\lambda-\mu|^\theta$$

for $\lambda, \mu \in I$, where c_1 is independent of ℓ, m.

The proof of (7.5) and (7.6) involves a detailed analysis concerning Bessel functions and cannot be given here (see Kuroda [15]). We have not been able to establish the eigenfunction expansion by distorted plane waves $\phi_2(x, \zeta) = e^{i\zeta x} + \cdots$ if $|q(x)| = O(|x|^{-(1+\epsilon)})$ with $0 < \epsilon \leqq 1$. This is because for $0 < \epsilon \leqq 1$ the space $\mathfrak{X}^*$ is not big enough to contain bounded functions such as $e^{i\xi\cdot x}$. The eigenfunction expansions in terms of spherical waves will be discussed elsewhere.

We can weaken the assumptions on q so that q may be unbounded. The conclusions remain true if $|q(x)| \leqq q^*(|x|)$ where $q^*(r) \in L^2(0, R)$ and $q^*(r) \leqq a/r^{1+\epsilon}$ for $r \geqq R$. For more details see the work of Kato [13].

4. We can make an improvement in the assumptions on q with regard to local singularities. Suppose $n = 3$ and $q \in L^{3/2}(R^3)$. Then we factor $q = |q(x)|^{1/2}\{\text{sgn } q(x)|q(x)|^{1/2}\}$, and an argument similar to the first example can be made based on the fact that

$$\int\int |q(x)| \frac{1}{|x-y|^2} |q(y)| \, dxdy < \infty.$$

This integral is finite by Sobolev's inequality. This suggests that conclusions similar to those in the previous examples would hold for $q \in L^{3/2}(R^3)$. Under this assumption, however, H_2 cannot be defined in general as $H_1 + V$. To get the correct definition of H_2 we use the theory of quadratic forms. Then, a modified form of the previous argument can be applied. For $n > 3$ we have to assume $q \in L^p(R^n) \cap L^q(R^n)$, $p > n/2$, $q < n/2$. For details see Kato and Kurado [14].

5. We now consider two additional questions concerning the singular spectrum of H_2. By Theorem 7.1 we know that the singular spectrum of H_2 is contained in a closed set of measure 0. We would like to know

1. Does H_2 have any singular continuous spectrum?
2. Does $(0, \infty)$ consist only of absolutely continuous spectrum?

Note that question 2 is stronger than question 1, because under any of the previous assumptions on q, the spectrum of H_2 is discrete in $(-\infty, 0)$ due to the stability of the essential spectrum under relatively compact perturbations.

A general principle for answering question 2 is to show that $1 + Q^{\pm}(\lambda)$, $\lambda > 0$, is not invertible if and only if $-\Delta u + qu = \lambda u$ has a nontrivial solution in a certain class of functions, say $L^2(R^3)$. Then, one uses a theorem of Kato [10] to show that such u does not exist. Kato [13] showed that if $|q(x)| \leqq (1 + |x|)^{-(1+\epsilon)}$ with $\epsilon > 1/4$ then the answer to question 2 is yes. We shall omit the details.

6. We shall consider problems where the perturbation includes second order terms. Let $H_1 = -\Delta$ and

$$\begin{aligned}(H_2u)(x) &= \sum_{j,k} (i\partial_j + b_j(x))a_{jk}(x)(i\partial_k + b_k(x))u(x) + c(x)u(x)\\ &= -\Delta u(x) + \sum_{j,k} (a_{jk}(x) - \delta_{jk})D_j D_k u(x)\\ &\quad + \sum_j \beta_j(x)D_j u(x) + \gamma(x)u(x),\end{aligned}$$

where

$$\begin{aligned}D_i = i\partial_j,\quad \beta_j(x) &= \sum_k (D_k a_{jk}(x) + 2a_{jk}(x)b_k(x)),\\ \gamma(x) &= \sum_{j,k}(D_j(a_{jk}(x)b_k(x)) + a_{jk}(x)b_j(x)b_k(x)) + c(x),\\ V &= \sum_{j,k} (a_{jk} - \delta_{jk})D_jD_k + \sum_j \beta_j D_j + \gamma\end{aligned}$$

is considered as the perturbation. In order for the wave operators to exist we should have

$$|a_{jk}(x) - \delta_{jk}|,\quad |\beta_j(x)|,\quad |\gamma(x)| \to 0 \quad \text{as } |x| \to \infty.$$

In some sense the a_{jk}, b_j and γ are assumed to be suitably nice so that $D(H_2) = D(H_1)$. For convenience we assume $\mathcal{H} = L^2(R^3)$, but some results mentioned below hold for R^n.

Case 1. $a_{jk} = \delta_{jk}$. Recently Ushijima [17] treated this case in the same way that the operator $H_2 = -\Delta + q$ was handled in 1 and 2. V is factored as $V = AB$ where

$$A = 1/(1+|x|)^{\alpha} \quad \text{and} \quad B = (1+|x|)^{\alpha} V.$$

B is a first order differential operator. Therefore, if we compute the kernel of $BR_1(\zeta)A$ by formal differentiation, its local singularity will be of the type $1/|x-y|^2$. Therefore, there is still room to apply a Sobolev-type argument. To accomplish this one has to assume that the coefficients of B tend to zero sufficiently rapidly. We omit the detailed result.

Case 2. $a_{jk} \neq \delta_{jk}$. If we use the same factorization as in the case where $a_{jk} = \delta_{jk}$ then the operator B will be a second order differential operator and the kernel of $BR_1(\zeta)A$ obtained by formal differentiation has the local singularity of the type $1/|x-y|^3$. Thus, the best one can hope is that this is a singular integral operator.

Recently, Ikebe and Toyoshi [9] treated this case by a method based on perturbations of trace class. Because of the high singularity of the kernel of $BR_1(\zeta)A$, one cannot hope to prove that $R_1(\zeta) - R_2(\zeta) \in$ trace class to apply the theorem of Birman et al. mentioned in §3. However, a generalization of that theorem gives that if $R_1(\zeta)^2 - R_2(\zeta)^2 \in$ trace class, Im $\zeta \neq 0$, then the same conclusion holds.

Under the assumption that $a_{jk} - \delta_{jk}, \beta_j, \gamma$ all belong to $L^1(R^3)$ (plus other minor conditions) Ikebe and Toyoshi showed that $R_I(\zeta)^2 - R_2(\zeta)^2 \in$ trace class. In particular, the wave operators exist and are complete under this condition. Using Cook's method they also showed the existence of the wave operators under a weaker assumption. The main assumptions are

$$\int_{R^3} (1+|x|)^{-1+\epsilon} |f(x)|^2 dx < \infty, \; f = a_{jk} - \delta_{jk},\, \beta_j, \text{ or } \gamma, \qquad \epsilon > 0.$$

The method developed above also seems to be applicable to this problem if a small modification is made. Although this has not yet been fully investigated, we would like to mention something to indicate a possibility. In the factorization situation a key role was played by the behavior near the real axis of the operator valued function $\tilde{G}(\zeta) = 1 + BR_1(\zeta)A$. If we use the resolvent equation here, we get

$$\begin{aligned} \tilde{G}(\zeta) &= 1 + BR_1(i)A + (\zeta - i)\, BR_1(i)R_1(\zeta)A \\ &= K(1 + (\zeta - i)K^{-1}BR_1(i)R_1(\zeta)A), \end{aligned}$$

where $K = 1 + BR_1(i)A$. K is invertible in $B(\mathcal{H})$. Therefore, in order to apply Lemma 4.20 to show the invertibility of $\tilde{G}^+(\lambda)$ it suffices to establish the complete continuity etc. of $BR_1(i)R_1(\zeta)A$. Now, we have a product of the resolvent. Therefore, the local singularity will become weaker and room will be recovered to apply a Sobolev type argument.

Our tentative result is that the wave operators will be complete if, roughly, $|a_{jk}(x) - \delta_{jk}| = O(|x|^{-(2+\epsilon)})$, $|x| \to \infty$, $\epsilon > 0$, etc. This will be discussed elsewhere.

ACKNOWLEDGMENT. In preparing this paper we owe much to the notes taken by Mr. Frank J. Massey III. We are also grateful to him for giving us help in preparing the manuscript.

REFERENCES. Most of the material presented in §§2 and 3 can be found in the book of T. Kato [12]. References to original papers can be found there. P. A. Rejto and J. S. Howland have also developed stationary methods for gentle or smooth perturbations (see their papers in the following list). Besides these, the following list contains only those papers which were quoted in the notes.

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