JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 27, Number 2, Summer 2015

SOLVABILITY OF A GENERAL NONLINEAR INTEGRAL EQUATION IN L^1 SPACES BY MEANS OF A MEASURE OF WEAK NONCOMPACTNESS

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Communicated by Jürgen Appell

ABSTRACT. This paper is concerned with existence results for a quite general nonlinear functional integral equation in L^1 spaces. For this purpose, making use of the De Blasi measure of weak noncompactness, we first establish a new fixed point theorem of the nonautonomous superposition operators. After that, our theorem is applied to prove the solvability of the mentioned nonlinear functional integral equation.

1. Introduction. In the present paper, we are concerned with the solvability of the following quite general nonlinear functional integral equation

$$x(t) = f\Big(t, x(t), \int_\Omega k(t,s)u(s,x(s))\,ds\Big), \quad t \in \Omega, \tag{1.1}$$

in $L^1(\Omega, X)$, the space of Lebesgue integrable functions on a measurable subset Ω of $\mathbb{R}^n$ with values in X. Here, $f : \Omega \times X \times X \to X$ and $u : \Omega \times X \to Y$ are given nonlinear functions, while X and Y are two finite dimensional Banach spaces. The kernel k is measurable on $\Omega \times \Omega$ such that, for each $t \in \Omega$, the function $s \mapsto k(t,s)$ belongs to L^∞, and the Hammerstein integral operator K, generated by k, is continuous from $L^1(\Omega, Y)$ into $L^1(\Omega, X)$.

Note that equation (1.1) may be written in the form

$$x = F(x, Ax), \tag{1.2}$$

2010 AMS *Mathematics subject classification.* Primary 47H08, 47H10, 47H30.
Keywords and phrases. Superposition operators, measure of weak noncompactness, fixed points, nonlinear integral equations, Banach algebras.
Received by the editors on April 14, 2014, and in revised form on December 2, 2014.
DOI:10.1216/JIE-2015-27-2-273

where $F : L^1(\Omega, X) \times L^1(\Omega, X) \to L^1(\Omega, X)$ and $A : \mathcal{D} \subseteq L^1(\Omega, X) \to L^1(\Omega, X)$ are two given operators. Our goals in this paper are to establish a new fixed point theorem for the solvability of equation (1.2), and to study under what conditions equation (1.1) is solvable in $L^1(\Omega, X)$ by applying our new theorem.

The organization of this paper is as follows. In Section 2, we gather some notions and preliminary facts, which will be needed in our current study, including the concepts and properties of the measure of weak noncompactness. In Section 3, we establish a fixed point theorem for equation (1.2) by means of the measure of weak noncompactness. In Section 4, we prove the existence of integrable solutions for equation (1.1) by applying our new theorem.

2. Preliminaries. Let E be a Banach space. From now on, we denote by $\mathcal{B}(E)$ the collection of all nonempty bounded subsets of E, and $\mathcal{W}(E)$ is the subset of $\mathcal{B}(E)$ consisting of all relatively weakly compact subsets of E. Denote by $\mathcal{U}_r$ the closed ball in E centered at 0 with radius r. In what follows, we accept the following definition [**3**].

Definition 2.1. Let M, M_1 and M_2 be in $\mathcal{B}(E)$. A function $\mu : \mathcal{B}(E) \to \mathbb{R}^+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

(i) The family $\ker(\mu) := \{M \in \mathcal{B}(E) : \mu(M) = 0\}$ is nonempty and $\ker(\mu) \subseteq \mathcal{W}(E)$;

(ii) $M_1 \subseteq M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$;

(iii) $\mu(\overline{\text{co}}(M)) = \mu(M)$, where $\overline{\text{co}}(M)$ refers to the closed convex hull of M;

(iv) $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$, for $\lambda \in [0,1]$;

(v) if $(M_n)_{n=1}^{\infty}$ is a decreasing sequence of nonempty, bounded and weakly closed subsets of X with $\lim_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker(\mu)$ described in (i) is called the kernel of the measure of weak noncompactness μ. Note that the intersection set M_∞ from (v) belongs to $\ker(\mu)$ since $\mu(M_\infty) \leq \mu(M_n)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(M_n) = 0$.

Definition 2.2. Let E_1 and E_2 be two Banach spaces, and let $\mathcal{D}$ be a subset of E_1. An operator $T : \mathcal{D} \to E_2$ is said to be ws-compact if it is continuous and, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ which is weakly convergent in E_1, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence; in addition, T is said to be ww-compact if it is continuous and, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ which is weakly convergent in E_1, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence.

Remark 2.3. A continuous operator is ws-compact if and only if it maps relatively weakly compact sets into relatively strongly compact ones; and it is ww-compact if and only if it maps relatively weakly compact sets into relatively weakly compact ones, since the weak compactness of the sets in a Banach space is equivalent with their weakly sequential compactness by the Eberlein-Šmulian theorem (see [**10,** V.6.1, Theorem, page 430]).

The first important example of a measure of weak noncompactness has been defined by De Blasi [**7**] as follows:

$$\omega(M) = \inf\{r > 0 : \exists W \in \mathcal{W}(E) \text{ such that } M \subseteq W + \mathcal{U}_r\}.$$

The De Blasi measure of weak noncompactness has some interesting properties. It plays a significant role in nonlinear analysis and has some applications.

Nevertheless, it is rather difficult to express the De Blasi measure of weak noncompactness with the help of a convenient formula in a concrete Banach space. Such a formula is known in the case of the space of L^1. In [**1**], Appell and De Pascale give to ω the following simple form

$$\omega(M) = \limsup_{\varepsilon\to 0} \left\{ \sup_{x\in M} \left[\int_D \|x(t)\|_X dt : D \subseteq \Omega, \text{meas}\,(D) \le \varepsilon \right] \right\}, \tag{2.1}$$

for all bounded subsets M of $L^1(\Omega, X)$, where X is a finite dimensional Banach space, $\Omega \subseteq \mathbb{R}^n$ and $\text{meas}\,(\cdot)$ denotes the Lebesgue measure. Throughout the sequel, we shall use the De Blasi measure of weak noncompactness ω.

Now, let us assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain. Consider a function $f : \Omega \times X \to Y$, where X and Y are two separable Banach spaces. We say that f is a Carатheódory function if: (i) for any fixed $x \in X$, the map $f(\cdot, x)$ is measurable from Ω to Y; (ii) for almost every $t \in \Omega$, the map $f(t, \cdot)$ is continuous from X to Y.

Let $m(\Omega, X)$ be the set of all measurable functions $\psi : \Omega \to X$. If f is a Carathéodory function, then f defines a mapping $\mathcal{N}_f : m(\Omega, X) \to m(\Omega, Y)$ by $\mathcal{N}_f \psi(t) = f(t, \psi(t))$. This mapping is said to be the superposition operator (or Nemytskii operator) associated to f. For the theory concerning superposition operators the reader may consult Appell and Zabrejko **[2]**. The following result due to Lucchetti and Patrone **[12]** is an extension to separable Banach spaces of the remarkable theorem due to Krasnosel'skii about superposition operators for scalar valued functions **[13]** (see also **[14]**).

Lemma 2.4. *Let X, Y be two separable Banach spaces, and let Ω be a domain in $\mathbb{R}^n$. If f is a Carathéodory function, then the superposition operator $\mathcal{N}_f$ maps $L^1(\Omega, X)$ into $L^1(\Omega, Y)$, if and only if there exist a function $a \in L^1_+(\Omega)$ and a constant $b > 0$ such that*

$$\|f(t, x)\|_Y \leq a(t) + b\|x\|_X,$$

where $L^1_+(\Omega)$ denotes the positive cone of the space $a \in L^1(\Omega)$.

In this case, the operator $\mathcal{N}_f$ is continuous and bounded, in the sense that bounded sets in $L^1(\Omega, X)$ are mapped into bounded sets of $L^1(\Omega, Y)$. For a given operator $A : \mathcal{D} \subseteq L^1(\Omega, X) \to L^1(\Omega, X)$, the composite operator $\mathcal{N}_f \circ A : \mathcal{D} \to L^1(\Omega, Y)$, defined by

$$\psi(t) \mapsto f(t, A\psi(t)), \quad \text{for all } \psi \in L^1(\Omega, X),$$

is called the nonautonomous type superposition operator. Now we see that the solutions of equation (1.2) are just the fixed points of the nonautonomous type superposition operator $\mathcal{N}_F \circ A$ on $\mathcal{D}$.

3. The family of φ-contractions and the fixed point theorem.

Definition 3.1. Let $\mathcal{D}$ be a nonempty subset of a Banach space E, and let $F : E \times E \to E$ be an operator. The family of operators $\{F(\cdot, y) : y \in \mathcal{D}\}$ is said to be φ-contractive (or nonlinear contractive), if there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$

such that the inequality

$$\|F(x_1, y) - F(x_2, y)\| \leq \varphi(\|x_1 - x_2\|), \quad \text{for all } x_1, x_2 \in X,$$

holds for all $y \in \mathcal{D}$, where φ satisfies $\varphi(r) < r$ for $r > 0$.

The φ-contractiveness of a family of operators plays an important role in the following results.

Lemma 3.2. *Let $\mathcal{D}$ be a nonempty subset of a Banach space E. If $F : E \times E \to E$ is continuous, and the family $\{F(\cdot, y) : y \in \mathcal{D}\}$ is φ-contractive, then there exists a continuous map $J : \mathcal{D} \to E$ such that $Jy = F(Jy, y)$ for any $y \in \mathcal{D}$.*

Proof. For an arbitrary fixed $y \in \mathcal{D}$, the mapping $F(\cdot, y)$ defined by $x \mapsto F(x, y)$ is a nonlinear contraction and maps E into itself, so it has a unique fixed point by [**6,** Theorem 1]. Let us denote by $J : \mathcal{D} \to E$ the map which assigns to each $y \in \mathcal{D}$ the unique point Jy in E such that $Jy = F(Jy, y)$. Thus, the map J is well defined.

Consider a sequence $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{D}$ converging to some $y_0 \in \mathcal{D}$, we have

$$\begin{aligned}
&\|Jy_n - Jy_0\| = \|F(Jy_n, y_n) - F(Jy_0, y_0)\| \\
&\leq \|F(Jy_n, y_n) - F(Jy_0, y_n)\| + \|F(Jy_0, y_n) - F(Jy_0, y_0)\| \\
&\leq \varphi(\|Jy_n - Jy_0\|) + \|F(Jy_0, y_n) - F(Jy_0, y_0)\|,
\end{aligned}$$

which implies

$$\|Jy_n - Jy_0\| - \varphi(\|Jy_n - Jy_0\|) \leq \|F(Jy_0, y_n) - F(Jy_0, y_0)\|.$$

Let $r_n := \|Jy_n - Jy_0\|$. From the continuity of F, we obtain that $r_n - \varphi(r_n) \to 0$ as $n \to \infty$. The property of φ shows that $r_n \to 0$, that is, $Jy_n \to Jy_0$. Thus, the map J is continuous on $\mathcal{D}$. $\square$

Theorem 3.3. *Let E be a Banach space, and let M be a nonempty subset of E. Suppose that the two continuous operators $A : M \to E$ and $F : E \times E \to E$ satisfy*:

(i) *A is ws-compact*;
(ii) *the family $\{F(\cdot, y) : y \in \overline{A(M)}\}$ is φ-contractive*;

(iii) *there exists a nonempty, weakly compact and convex subset $\mathcal{P}$ of M such that*

$$x = F(x, Az) \Longrightarrow x \in \mathcal{P}, \quad \text{for all } z \in \mathcal{P}.$$

Then there is a point x in M such that $x = F(x, Ax)$.

Proof. Let us denote by $J : \overline{A(M)} \to E$ the map which assigns to each $y \in \overline{A(M)}$ the unique point Jy in E such that $Jy = F(Jy, y)$. From Lemma 3.2, the map J is well defined and continuous on $\overline{A(M)}$.

For any $z \in \mathcal{P}$, by assumption (iii) we infer that there is $x = (J \circ A)z \in \mathcal{P}$ such that $x = F(x, Az)$. This shows that $(J \circ A)(\mathcal{P}) \subseteq \mathcal{P}$.

Moreover, $(J \circ A)(\mathcal{P})$ is relatively strongly compact since $(\mathcal{P})$ is weakly compact, A is ws-compact and J is continuous on $A(\mathcal{P})$. Now, applying the Schauder fixed point theorem, we conclude that $J \circ A$ has at least one fixed point $x \in \mathcal{P} \subseteq M$ such that $(J \circ A)x = x$, which implies that

$$F(x, Ax) = F((J \circ A)x, Ax) = (J \circ A)x = x.$$

This completes the proof. □

Remark 3.4. The above result includes a general form of some fixed point theorems involving several operators, such as $F_1 x := Ax + Bx$ in Banach space (see [**15**]), or $F_2 x := AxBx + Cx$ in Banach algebras etc.

Dhage [**8**] gives F_2 a version of strong topology; Ben Amar et al. [**5**] gives it a version of sequentially weak continuity. However, as far as the author knows, there is still not a version of ws-compactness about F_2 in the previous literature.

4. A general nonlinear integral equation in L^1 space. In this section we mainly consider equation (1.1). Solutions to it will be sought in $L^1(\Omega, X)$, endowed with the standard norm $\|\cdot\| := \int_\Omega \|x(t)\|_X dt$.

We will assume that the functions involved in equation (1.1) satisfy the following conditions:

$(\mathcal{H}_1)$ $u : \Omega \times X \to Y$ is a Carathéodory function, and there exist a function $a \in L^1_+(\Omega)$ and a constant $b > 0$ such that $\|u(t, x)\|_Y \leq a(t) + b\|x\|_X$.

$(\mathcal{H}_2)$ The function $k : \Omega\times\Omega \to \mathcal{L}(Y,X)$ is strongly measurable where $\mathcal{L}(Y,X)$ refers to the space of linear operators from Y to X.

$(\mathcal{H}_3)$ For each $t \in \Omega$, the function

$$\rho(t) : \Omega \longrightarrow \mathcal{L}(Y,X), \qquad s \longmapsto \rho(t)(s) := k(t,s)$$

belongs to $L^\infty(\Omega, \mathcal{L}(Y,X))$; and the function

$$\rho : \Omega \longrightarrow \mathcal{L}(Y,X), \qquad t \longmapsto \rho(t)$$

belongs to $L^1(\Omega, L^\infty(\Omega, \mathcal{L}(Y,X)))$ which is denoted by $L^1(\Omega, L^\infty)$ for short.

$(\mathcal{H}_4)$ $f : \Omega \times X \times X$ is a Carathéodory function, and there exist a function $g \in L^1_+(\Omega)$ and two positive numbers α, β such that

$$\|f(t,x(t),y(t))\|_X \le g(t) + \alpha\|x(t)\|_X + \beta\|y(t)\|_X,$$

for any $x, y \in L^1(\Omega, X)$.

$(\mathcal{H}_5)$ $\alpha + b\beta\|K\| + \|g\| \le 1$ if $g \ne 0$; otherwise, $a + b\beta\|K\| < 1$, where $\|K\|$ denotes the norm of the linear operator K generated by the function k.

$(\mathcal{H}_6)$ There exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(r) < r$ for $r > 0$ such that

$$\int_\Omega \|f(t,x_1(t),y(t)) - f(t,x_2(t),y(t))\|_X dt \le \varphi(\|x_1 - x_2\|),$$

for any $x_1, x_2 \in L^1(\Omega, X)$ whenever $y(t) = \int_\Omega k(t,s)u(s,z(s))\,ds$ with $z \in \mathcal{U}_{r_0}$, where r_0 satisfies

$$r_0 \ge \frac{\|g\| + \beta\|K\|\|a\|}{1 - \alpha - b\beta\|K\|}.$$

First notice that equation (1.1) may be written in the abstract form

$$x = F(x, Ax),$$

where F is the superposition operator associated to f (i.e., $F = \mathcal{N}_f$):

$$F : L^1(\Omega,X) \times L^1(\Omega,X) \longrightarrow L^1(\Omega,X)),$$

$$(x,y) \longmapsto F(x,y) : \Omega \longrightarrow X; \quad F(x,y)(t) = f(t,x(t),y(t));$$

and $A := K \circ \mathcal{N}_u$ appears as the composition of the superposition operator associated to u with the linear operator defined by

$$K : L^1(\Omega, Y) \longrightarrow L^1(\Omega, X),$$

$$\psi \longmapsto K\psi : \Omega \longrightarrow X; \qquad K\psi(t) = \int_\Omega k(t,s)\psi(s)\,ds.$$

Our aim is now to prove that the nonautonomous type superposition operator $\mathcal{N}_f \circ A$ has a fixed point in $L^1(\Omega, X)$. Before starting to prove the solvability of equation (1.1), we give some remarks.

Remark 4.1. (i) It should be noted that assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ lead to the estimate

$$\left\| \int_\Omega k(t,s)\phi(s)\,ds \right\|_X \leq \|\rho(t)\|_{L^1(\Omega,\mathcal{L}(Y,X))} \cdot \|\phi\|_{L^1(\Omega,Y)},$$

and so

$$\|K\phi\|_{L^1(\Omega,X)} = \int_\Omega \left\| \int_\Omega k(t,s)\phi(s)\,ds \right\|_X dt \leq \|\rho\|_{L^1(\Omega,L^\infty)} \cdot \|\phi\|_{L^1(\Omega,Y)},$$

for any $\phi \in L^1(\Omega, Y)$. This shows that the linear operator K is continuous, hence weakly continuous, from $L^1(\Omega, Y)$ into $L^1(\Omega, X)$ and that $\|K\| \leq \|\rho\|_{L^1(\Omega,L^\infty)}$.

(ii) Assumption $(\mathcal{H}_1)$ shows that the superposition operator $\mathcal{N}_u$ is continuous and maps bounded sets of $L^1(\Omega, X)$ into bounded sets of $L^1(\Omega, Y)$ by Lemma 2.4.

(iii) Considering in the space $X \times X$ the norm $\alpha\|x\|_X + \beta\|y\|_X$ for the product topology and using Lemma 2.4, we can see that the assumption $(\mathcal{H}_4)$ implies that the superposition operator $\mathcal{N}_f$ is continuous and maps bounded sets of $L^1(\Omega, X) \times L^1(\Omega, X)$ into bounded sets of $L^1(\Omega, X)$.

Now we are in a position to state our main result.

Theorem 4.2. *Let X and Y be two finite dimensional Banach spaces, and let Ω be a bounded domain of $\mathbb{R}^n$. Assume that the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_6)$ are satisfied, then the equation*

$$x(t) = f\left(t, x(t), \int_\Omega k(t,s)u(s,x(s)\,ds\right), \quad t \in \Omega,$$

i.e., equation (1.1) *has at least a solution* $x \in L^1(\Omega, X)$.

Proof. Define the operators A and F as follows:

$$(Az)(t) := \int_\Omega k(t,s)u(s,z(s))\,ds, \quad F(x,y)(t) := f(t,x(t),y(t)).$$

Thus, the solutions of equation $x = F(x, Ax)$ satisfy equation (1.1). We shall point out that the assumptions of Theorem 3.3 are all fulfilled. The proof is divided into several steps.

(i) By Remark 4.1, the operators $A : L^1(\Omega, X) \to L^1(\Omega, X)$ and $F : L^1(\Omega, X) \times L^1(\Omega, X) \to L^1(\Omega, X)$ are well defined and continuous.

Let S be a bounded subset of $L^1(\Omega, X)$, and let $M > 0$ be such that $\|\psi\|_{L^1(\Omega,X)} \leq M$ for all $\psi \in S$. We have:

$$\begin{aligned} \|A\psi(t)\|_X &\leq \|\rho(t)\|_{L^\infty(\Omega,\mathcal{L}(X,Y))}\|\mathcal{N}_u\psi\|_{L^1(\Omega,Y)} \\ &\leq \|\rho(t)\|_{L^\infty(\Omega,\mathcal{L}(X,Y))}(\|a\| + b\|\psi\|_{L^1(\Omega,X)}) \\ &\leq \|\rho(t)\|_{L^\infty(\Omega,\mathcal{L}(X,Y))}(\|a\| + bM). \end{aligned} \tag{4.1}$$

Now we check that A is ws-compact. To this end, let $(\psi_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of $L^1(\Omega, X)$. Since $\mathcal{N}_u$ is ww-compact by **[11,** Lemma 3.2], the sequence $(\mathcal{N}_u\psi_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in $L^1(\Omega, Y)$, say $(\mathcal{N}_u\psi_{n_k})_{k\in\mathbb{N}}$. Let η be the weak limit of $(\mathcal{N}_u\psi_{n_k})_{k\in\mathbb{N}}$. Accordingly, bearing in mind the boundedness of the function $k(t,\cdot) = \rho(t)$, we get

$$(A\psi_{n_k})(t) = \int_\Omega k(t,s)u(s,\psi_{n_k}(s))\,ds \longrightarrow \int_\Omega k(t,s)\eta(s))\,ds. \tag{4.2}$$

Thus, (4.1) along with (4.2) allow us to apply the dominated convergence theorem to conclude that the sequence $(A\psi_{n_k})_{k\in\mathbb{N}}$ converges in $L^1(\Omega, X)$. So, the operator A is ws-compact, and assumption (i) of Theorem 3.3 is fulfilled.

(ii) For any $y \in \overline{A(\mathcal{U}_{r_0})}$, there exists a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_{r_0}$ such that $\lim_{n\to\infty} Az_n = y$. Thus, by the continuity of F and $(\mathcal{H}_6)$, we obtain that:

$$\begin{aligned}
&\|F(x_1,y)-F(x_2,y)\| \\
&= \lim_{n\to\infty} \|F(x_1,Az_n)-F(x_2,Az_n)\| \\
&= \lim_{n\to\infty} \int_\Omega \|f(t,x_1(t),Az_n(t))-f(t,x_2(t),Az_n(t))\|_X \, dt \\
&\le \varphi(\|x_1-x_2\|), \quad \text{for all } x_1,x_2\in L^1(\Omega,X).
\end{aligned}$$

This shows that assumption (ii) of Theorem 3.3 is fulfilled.

(iii) If there exists $x \in L^1(\Omega, X)$ such that $x(t) = f(t, x(t), Az(t))$ for $z \in U_{r_0}$, then by $(\mathcal{H}_4)$ we have:

$$\begin{aligned}
\|f(t,x(t),Az(t))\|_X &\le g(t)+\alpha\|x(t)\|_X+\beta\|Az(t)\|_X \\
&\le g(t)+\alpha\|x(t)\|_X+\beta\|K\|\|\mathcal{N}_u z(t)\|_Y \\
&\le g(t)+\alpha\|x(t)\|_X+\beta\|K\|\left(a(t)+b\|z(t)\|_X\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x(t)\| &= \int_\Omega \|f(t,x(t),Az(t))\|_X \\
&\le \|g\|+\alpha\|x\|+\beta\|K\|(\|a\|+b\|z\|),
\end{aligned}$$

which implies

$$\begin{aligned}
\|x\| &\le (1-\alpha)^{-1}\left(\|g\|+\beta\|K\|(\|a\|+b\|z\|)\right) \\
&\le (1-\alpha)^{-1}\left(\|g\|+\beta\|K\|\|a\|+b\beta\|K\|r_0\right) \\
&\le r_0,
\end{aligned}$$

since $\|g\| + \beta\|K\|\|a\| \le r_0(1 - \alpha - b\beta\|K\|)$ by $(\mathcal{H}_6)$. Thus, we obtain that $x \in U_{r_0}$.

(iv) Let $\mathcal{P}_0 := \mathcal{U}_{r_0}$, and let

$$\begin{aligned}
\mathcal{P}_n := \overline{\mathrm{co}}\Big\{&x\in L^1(\Omega,X) : x(t) \\
&= f\left(t, x(t), \int_\Omega k(t,s)u(s,z(s))\,ds\right), \quad z\in\mathcal{P}_{n-1}\Big\}.
\end{aligned}$$

Then, $\mathcal{P}_n$ $(n = 0, 1, 2, \ldots)$ are all nonempty closed convex sets, and therefore they are weakly closed. Moreover, we have $P_1 \subseteq U_{r_0} = P_0$

from step (iii), and by the induction, we may infer that $P_n \subseteq P_{n-1}$ for all $n \in \mathbb{N}$.

On the other hand, for each $\varepsilon > 0$ and a nonempty measurable subset $D \subseteq \Omega$ such that $\operatorname{meas}(D) \leq \varepsilon$, we know that for any $z \in \mathcal{P}_{n-1}$ and $x \in \mathcal{P}_n$, if

$$x(t) = f\left(t, x(t), \int_\Omega k(t,s)u(s,z(s))\,ds\right),$$

then

$$\begin{aligned}\int_D \|x(t)\|\,dt &= \int_D \left\| f\left(t, x(t), \int_\Omega k(t,s)u(s,z(s))\,ds\right)\right\|_X dt \\ &\leq \int_D \|g(t)\|\,dt + \alpha \int_D \|x(t)\|\,dt \\ &\quad + \beta\|K\|\left(\int_D a(t)\,dt + b\int_D \|z(t)\|\,dt\right),\end{aligned}$$

which implies that

$$\int_D \|x(t)\|\,dt \leq \frac{\int_D \|g(t)\|\,dt + \beta\|K\|\left(\int_D a(t)\,dt + b\int_D \|z(t)\|\,dt\right)}{(1-\alpha)}.$$

Taking into account the fact that the set consisting of one element is weakly compact, formula (2.1) leads to

$$\limsup_{\varepsilon\to 0}\left\{\int_D \|g(t)\|_X dt : \operatorname{meas}(D) \leq \varepsilon\right\} = 0,$$

and

$$\limsup_{\varepsilon\to 0}\left\{\int_D \|a(t)\|_X dt : \operatorname{meas}(D) \leq \varepsilon\right\} = 0.$$

As a result,

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left\{\int_D \|x(t)\|_X dt : \operatorname{meas}(D) \leq \varepsilon\right\} \\ &\leq (1-\alpha)^{-1}b\beta\|K\| \limsup_{\varepsilon\to 0}\left\{\sup_{z\in P_{n-1}}\int_D \|z(t)\|_X dt : \operatorname{meas}(D) \leq \varepsilon\right\},\end{aligned}$$

which implies that $\omega(\mathcal{P}_n) \leq \lambda\omega(\mathcal{P}_{n-1})$ from (2.1), where $\lambda := (1-\alpha)^{-1}b\beta\|K\| < 1$ by $(\mathcal{H}_6)$.

Further, from $\omega(\mathcal{P}_n) \leq \lambda\omega(\mathcal{P}_{n-1}) \leq \cdots \leq \lambda^n\omega(\mathcal{P}_0)$ for $n \in \mathbb{N}$, we obtain that $\lim_{n\to\infty}\omega(\mathcal{P}_n) = 0$. Setting $\mathcal{P} = \bigcap_{n=0}^{\infty}\mathcal{P}_{n-1}$, by Definition 2.1, we see that $\mathcal{P}$ is nonempty and weakly compact. Moreover, we infer that, for any $z \in \mathcal{P}$ if $x = F(x, Az)$ holds, then $x \in \mathcal{P}$. Now assumption (iii) of Theorem 3.3 is fulfilled, and we accomplish the proof. □

On the L^1 space, there are some Hammerstein type nonlinear integral equations such as the following:

$$x(t) = g(t, x(t)) + \lambda \int_\Omega k(t,s)u(s,x(s))\,ds, \quad t \in \Omega, \tag{4.3}$$

$$x(t) = f_1(t, x(t)) + f_2(t,x(t)) \int_\Omega k(t,s)u(s,x(s))\,ds, \quad t \in \Omega. \tag{4.4}$$

Their solvability has been discussed respectively in the previous literature under different assumptions (see, for instance, [**4, 5, 8, 9, 11, 15**]). Now, it should be seen that we may investigate such equations by means of Theorem 4.2.

Example 4.3. Consider the following nonlinear integral equation:

$$\begin{aligned}\psi(t) = t^3 - &\frac{\arctan\psi(t)}{4+t} \\ &+ \frac{t^2}{2+|\psi(t)|}\sin\left(\int_0^1 (t-s)\sqrt{t^2+\psi^2(s)}\,ds\right), \quad t\in[0,1].\end{aligned} \tag{4.5}$$

In order to show that such an equation admits a solution in $L^1([0,1],\mathbb{R})$, we are going to check that the conditions of Theorem 4.2 are satisfied. In this case, $\Omega := [0,1]$ and $X = Y := \mathbb{R}$.

Define the functions as the following:

$$\begin{aligned}
&u : [0,1]\times\mathbb{R} \to \mathbb{R}, && u(t,x) = \sqrt{t^2+x^2};\\
&k : [0,1]\times[0,1] \to \mathbb{R}, && k(t,s) = t-s;\\
&f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}, && f(t,x,y) = t^3 - \frac{\arctan x}{4+t} + \frac{t^2}{2+|x|}\sin y.
\end{aligned}$$

It is obvious that u and f are all Carathéodory functions, and we easily check that $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied. By simple reasoning, we obtain that $\|K\| = 1/2$.

Taking $a(t) = t$ and $b = 1$, we have

$$|u(t,x)| = \sqrt{t^2 + x^2} \le t + |x| = a(t) + b|x|.$$

So, u satisfies $(\mathcal{H}_1)$.

We have

$$\begin{aligned}|f(t,x,y)| &= \left| t^3 - \frac{\arctan x}{4+t} + \frac{t^2}{2+|x|}\sin y \right| \\ &\le t^3 + \frac{1}{4}|x| + \frac{1}{2}|y|.\end{aligned}$$

Taking $g(t) = t^3$, $\alpha = 1/4$ and $\beta = 1/2$, it follows that f satisfies $(\mathcal{H}_4)$.

Now, we obtain that

$$\alpha + b\beta\|K\| + \|g\| = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < 1,$$

which shows that $(\mathcal{H}_5)$ is satisfied.

From the inequality,

$$\begin{aligned}&|f(t,x_1(t),y(t)) - f(t,x_2(t),y(t))| \\ &\le \frac{1}{4+t}|\arctan x_1(t) - \arctan x_2(t)| + \left|\frac{t^2 \sin y(t)}{2+|x_1(t)|} - \frac{t^2 \sin y(t)}{2+|x_2(t)|}\right|, \\ &\le \frac{1}{4+t}|x_1(t) - x_2(t)| + \frac{|x_1(t) - x_2(t)|}{4 + 2|x_1(t)| + 2|x_2(t)| + |x_1(t)x_2(t)|} \\ &\le \frac{1}{2}|x_1(t) - x_2(t)|,\end{aligned}$$

it follows that

$$\int_0^1 |f(t,x_1(t),y(t)) - f(t,x_2(t),y(t))|\,dt \le \frac{1}{2}\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in \mathbb{R},$$

for all $y \in \mathbb{R}$. So $(\mathcal{H}_6)$ holds for $\varphi(r) := \frac{1}{2}r$.

Since the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_6)$ are all satisfied, we apply Theorem 4.2 to derive the existence of a solution to equation (4.5).

Remark 4.4. Equation (4.5) is a particular case of the model integral equation (1.1). It is not included in equations (4.3) and (4.4), which implies that equation (1.1) is a new model integral equation.

Acknowledgments. The author is grateful to Prof. Jürgen Appell and the two anonymous referees for a careful reading of the manuscript. The remarks motivated the author to make a lot of valuable improvements.

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